Quartic Curves and Their Bitangents

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Control, Optimization, and Functional Analysis: Synergies and Perspectives

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Three representations of a quartic curve

We consider smooth curves in $\mathbb{P}^2$ defined by ternary quartics

$$f(x, y, z) = c_{400}x^4 + c_{310}x^3y + c_{301}x^3z + \cdots + c_{004}z^4,$$

whose 15 coefficients $c_{ijk}$ lie in the field $\mathbb{Q}$ of rational numbers.

Our paper gives exact algorithms for computing, over the real numbers $\mathbb{R}$ whenever possible, the two alternate representations

$$f(x, y, z) = \det(xA + yB + zC),$$

where $A, B, C$ are symmetric $4 \times 4$-matrices, and

$$f(x, y, z) = q_1(x, y, z)^2 + q_2(x, y, z)^2 + q_3(x, y, z)^2,$$

where the $q_i(x, y, z)$ are quadratic forms.
Example: The Edge Quartic

\[ 25 \cdot (x^4 + y^4 + z^4) - 34 \cdot (x^2y^2 + x^2z^2 + y^2z^2) \]

\[ = \det \begin{pmatrix}
0 & x + 2y & 2x + z & y - 2z \\
x + 2y & 0 & y + 2z & -2x + z \\
2x + z & y + 2z & 0 & x - 2y \\
y - 2z & -2x + z & x - 2y & 0
\end{pmatrix} \]


The sum of three squares representation is derived from

\[
\begin{pmatrix}
x^2 \\
y^2 \\
z^2 \\
xy \\
xz \\
yz
\end{pmatrix}^T
\begin{pmatrix}
25 & -55/2 & -55/2 & 0 & 0 & 21 \\
-55/2 & 25 & 25 & 0 & 0 & 0 \\
-55/2 & 25 & 25 & 0 & 0 & 0 \\
0 & 0 & 0 & 21 & -21 & 0 \\
0 & 0 & 0 & -21 & 21 & 0 \\
21 & 0 & 0 & 0 & 0 & -84
\end{pmatrix}
\begin{pmatrix}
x^2 \\
y^2 \\
z^2 \\
xy \\
xz \\
yz
\end{pmatrix}
\]
Twenty-Eight Bitangents

Theorem (Plücker 1834)

*Every smooth quartic curve has precisely 28 bitangent lines.*

Figure: The Edge quartic and some of its 28 bitangents
Computing the Bitangents Symbolically?

Let $K$ denote the splitting field of the 28 bitangents, that is, the smallest field extension of $\mathbb{Q}$ over which they are defined.

The Galois group $\text{Gal}(K, \mathbb{Q})$ is much smaller than the symmetric group $S_{28}$. If the coefficients $c_{ijk}$ of $f(x, y, z)$ are general enough, it is the Weyl group of $E_7$ modulo its center,

$$\text{Gal}(K, \mathbb{Q}) \cong \text{W}(E_7)/\{\pm 1\} \cong \text{Sp}_6(\mathbb{Z}/2\mathbb{Z}).$$

This group has order $8! \cdot 36 = 1451520$, and it is not solvable.

Some 19th century mathematicians who worked on quartic curves (and abelian functions of genus 3): Aronhold, Cayley, Frobenius, Hesse, Klein, Riemann, Schottky, Steiner, Sturm, Zeuthen, …
The Real Picture

Theorem (Zeuthen 1873; Klein 1876)

There are six possible topological types for a smooth quartic curve \( \mathcal{V}_R(f) \) in the real projective plane. Each of the types corresponds to precisely one connected component in the complement of the discriminant \( \Delta \) in the 14-dimensional projective space of quartics.

<table>
<thead>
<tr>
<th>The real curve</th>
<th>Cayley octad</th>
<th>real bitangents</th>
<th>real Gram matrices</th>
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</thead>
<tbody>
<tr>
<td>4 ovals</td>
<td>8 real points</td>
<td>28</td>
<td>63</td>
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<tr>
<td>3 ovals</td>
<td>6 real points</td>
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<tr>
<td>2 non-nested ovals</td>
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<td>1 oval</td>
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<td>2 nested ovals</td>
<td>0 real points</td>
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<tr>
<td>empty curve</td>
<td>0 real points</td>
<td>4</td>
<td>15</td>
</tr>
</tbody>
</table>

Table: The six types of smooth quartics in the real projective plane.
Convex Algebraic Geometry

**Theorem (Hilbert 1888)**

A ternary quartic is non-nonnegative if and only if it can be written as a sum of squares of quadrics. Here, three squares always suffice.

**Theorem (Coble 1929; Powers-Reznick-Scheiderer-Sottile 2004)**

Every smooth quartic has 63 representations as sums of three squares over \( \mathbb{C} \). Precisely eight of these are real sums of squares.

**Theorem (Helton-Vinnikov 2007)**

Every real quartic can be written as \( f(x, y, z) = \det(xA+yB+zC) \) where \( A, B \) and \( C \) are real symmetric \( 4 \times 4 \)-matrices. This net of quadrics contains a matrix \( x_0A+y_0B+z_0C \) that is positive definite if and only if the real curve \( V_{\mathbb{R}}(f) \) consists of two nested ovals.

**Point:** Our algorithms compute these representations (in sage).
Input: A Helton-Vinnikov Curve

The following quartic defines two nested ovals

\[ f(x, y, z) = 2x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2. \]

\textbf{Helton-Vinnikov:} The interior convex region is a \textit{spectrahedron}. 
Output: A Linear Matrix Inequality

\[
\begin{pmatrix}
ux + y & 0 & az & bz \\
0 & ux - y & cz & dz \\
az & cz & x + y & 0 \\
bz & dz & 0 & x - y
\end{pmatrix} \succeq 0
\]

The scalars in this matrix are

\[
\begin{align*}
u &= \sqrt{2} = 1.414213562373095048\ldots \\
a &= -0.57464203209296160548032752478263\ldots \\
b &= 1.03492595196395554058118944258225\ldots \\
c &= 0.69970597091301262923557093892256\ldots \\
d &= 0.48004865038024320108560278354988\ldots
\end{align*}
\]

Their maximal ideal in \( \mathbb{Q}[a, b, c, d, u] \) expresses them in radicals:

\[
\langle u^2 - 2, 256d^8 - 384d^6u+256d^6-384d^4u+672d^4-336d^2u+448d^2-84u+121, 23c+7584d^7u+10688d^7-5872d^5u-8384d^5+1806d^3u+2452d^3-181du-307d, 23b+5760d^7u+8192d^7-4688d^5u-6512d^5+1452d^3u+2200d^3-212du-232d, 23a - 1440d^7u-2048d^7+1632d^5u+2272d^5-570d^3u-872d^3+99du+81d \rangle.
\]
An Algorithm for Quartic Curves

**Theorem:** Let $f \in \mathbb{Q}[x, y, z]$ be a quartic whose curve $\mathcal{V}_\mathbb{C}(f)$ is smooth. Suppose $f(x, 0, 0) = x^4$ and $f(x, y, 0)$ is squarefree. Then we can compute a determinantal representation

$$f(x, y, z) = \det(xI + yD + zR) \quad (1)$$

where $I$ is the identity matrix, $D$ is a diagonal matrix, $R$ is a symmetric matrix, and the entries of $D$ and $R$ are expressed in radicals over the splitting field $K$. Here, the entries of $D$ and $R$ can be real numbers if and only if $\mathcal{V}_\mathbb{R}(f)$ consists of two nested ovals.

**Algorithm:** We write a Helton-Vinnikov curve as a spectrahedron in radicals over the splitting field $K$ of its 28 bitangents.

**Details:** The identity (1) specifies a system of 14 polynomial equations in the 14 unknown entries of $D$ and $R$. This system has $6912 = 36 \cdot 24 \cdot 8$ complex solutions. We compute these.
Sums of Squares

A Gram matrix for $f$ is a symmetric $6 \times 6$ matrix $G$ over $\mathbb{C}$ such that

$$f = v^T \cdot G \cdot v$$

where $v = (x^2, y^2, z^2, xy, xz, yz)^T$.

If $G = H^T H$, where $H$ is an $r \times 6$-matrix and $r = \text{rank}(G)$, then the factorization $f = (Hv)^T (Hv)$ writes $f$ as the sum of $r$ squares.

No Gram matrix with $r \leq 2$ exists when $f$ is smooth, and there are infinitely many for $r \geq 4$. We compute all Gram matrices for $r = 3$.

Theorem (and Algorithm)

Let $f \in \mathbb{Q}[x, y, z]$ be a smooth quartic and $K$ the splitting field for its 28 bitangents. Then $f$ has precisely 63 Gram matrices $G$ of rank 3. We compute them all using rational arithmetic over $K$. 
Example: An Empty Curve

Let \( f = \det(M) \) where \( M \) is the matrix

\[
\begin{pmatrix}
52x + 12y - 60z & -26x - 6y + 30z & 48z & 48y \\
-26x - 6y + 30z & 26x + 6y - 30z & -6x + 6y - 30z & -45x - 27y - 21z \\
48z & -6x + 6y - 30z & -96x & 48x \\
48y & -45x - 27y - 21z & 48x & -48x
\end{pmatrix}
\]

Here \( V_C(f) \) is smooth and \( V_R(f) \) is empty. The corresponding Cayley octad \( O \) consists of four pairs of complex conjugates:

\[
\begin{pmatrix}
i & -i & 0 & 0 & -6 + 4i & -6 - 4i & 3 + 2i & 3 - 2i \\
1 + i & 1 - i & 0 & 0 & -4 + 4i & -4 - 4i & 7 - i & 7 + i \\
0 & 0 & i & -i & -3 + 2i & -3 - 2i & -86/39 - 4/13i & -86/39 + 4/13i \\
0 & 0 & 1 + i & 1 - i & 1 - i & 1 + i & 4/39 - 20/39i & 4/39 + 20/39i
\end{pmatrix}
\]

The bitangent matrix \( O^T M O \) is defined over \( K = \mathbb{Q}(i) \), and hence so are all 63 rank-3 Gram matrices. Precisely 15 of these are real:

\[
f = 288 \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}^T \begin{pmatrix} 45500 & 3102 & -9861 & 5718 & -9246 & 4956 \\ 3102 & 288 & -747 & 882 & -18 & -144 \\ -9861 & -747 & 3528 & -864 & -1170 & -504 \\ 5718 & 882 & -864 & 4440 & 1104 & -2412 \\ -9246 & -18 & -1170 & 1104 & 11814 & -5058 \\ 4956 & -144 & -504 & -2412 & -5058 & 3582 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}
\]
The Gram Spectrahedron

of a quartic $f$ is the set of its positive semidefinite Gram matrices.

This spectrahedron is the intersection of the cone of positive semidefinite $6 \times 6$-matrices with a 6-dimensional affine subspace:

$$
\text{Gram}(f) = \{ \lambda \in \mathbb{R}^6 : \begin{bmatrix}
    c_{400} & \lambda_1 & \lambda_2 & \frac{1}{2}c_{310} & \frac{1}{2}c_{301} & \lambda_4 \\
    \lambda_1 & c_{040} & \lambda_3 & \frac{1}{2}c_{130} & \lambda_5 & \frac{1}{2}c_{031} \\
    \lambda_2 & \lambda_3 & c_{004} & \lambda_6 & \frac{1}{2}c_{103} & \frac{1}{2}c_{202} \\
    \frac{1}{2}c_{310} & \frac{1}{2}c_{130} & \lambda_6 & c_{220} - 2\lambda_1 & \frac{1}{2}c_{211} - \lambda_4 & \frac{1}{2}c_{121} - \lambda_5 \\
    \frac{1}{2}c_{301} & \lambda_5 & \frac{1}{2}c_{103} & \frac{1}{2}c_{211} - \lambda_4 & c_{202} - 2\lambda_2 & \frac{1}{2}c_{112} - \lambda_6 \\
    \lambda_4 & \frac{1}{2}c_{031} & \frac{1}{2}c_{013} & \frac{1}{2}c_{121} - \lambda_5 & \frac{1}{2}c_{112} - \lambda_6 & c_{022} - 2\lambda_3
\end{bmatrix} \succeq 0 \} 
$$

Hilbert: $\text{Gram}(f)$ is non-empty if and only if $f$ is non-negative.

The *Steiner graph* of the Gram spectrahedron is the graph on the eight vertices of rank 3 whose edges represent edges of $\text{Gram}(f)$.

**Theorem**

*The Steiner graph of the Gram spectrahedron of a general positive quartic $f$ is the disjoint union $K_4 \sqcup K_4$ of two complete graphs. The relative interiors of these edges consist of rank-5 matrices.*
The Bigger Picture

- Plane quartics are canonical curves of genus 3
- The 28 bitangents are the odd theta characteristics
- The 36 Cayley octads are the even theta characteristics
- The 63 Steiner complexes and rank-3 Gram matrices correspond to the 2-torsion points on the Jacobian
- 3-phase solutions of the Kadomtsev-Petviashvili equation
- Period matrices to theta functions to plane quartics (and back)

Classical
- Today’s talk on plane quartics
- Abelian varieties moduli of curves

Tropical
- Tropical quartics
- Tropical bitangents
- The tropical Torelli map

Concrete

Abstract

How to manipulate genus 3 curves over a field such as $K = \overline{\mathbb{Q}(\epsilon)}$?
Trichotomy for Nets of Quadrics in $\mathbb{P}^3$

Proposition (Calabi 1964; “S-Lemma”)
Let $\mathcal{P}$ be a pencil of homogeneous quadrics in $n$ unknowns. Then precisely one of the following two cases holds:
(a) The quadrics in $\mathcal{P}$ have a common point in $\mathbb{P}^{n-1}(\mathbb{R})$.
(b) The pencil $\mathcal{P}$ contains a positive definite quadric.

Theorem
Let $\mathcal{N}$ be a net of homogeneous quadrics in four unknowns with $\Delta(\mathcal{N}) \neq 0$. Then precisely one of the following three cases holds:
(a) The quadrics in $\mathcal{N}$ have a common point in $\mathbb{P}^3(\mathbb{R})$.
(b) The net $\mathcal{N}$ contains a positive definite quadric.
(c) The $4 \times 4$-determinant restricted to $\mathcal{N}$ is a sum of squares.

Proof.
The net $\mathcal{N}$ defines a Cayley octad $O$ and ternary quartic $f$. Either $O$ has a real point, or $\mathcal{V}_R(f)$ is Helton-Vinnikov, or $\mathcal{V}_R(f) = \emptyset$. \qed