Some solvable cases of μ-synthesis

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The μ-synthesis problem arises in the stabilization of plants in the presence of structured uncertainty.

It generalizes the interpolation problems of Carathéodory-Fejér, Nevanlinna-Pick and Nehari.

There are numerical packages that search for solutions of μ-synthesis problems, but not yet a satisfactory analytic theory.

I will describe special cases that can be analysed.

The spectral Nevanlinna-Pick problem

Given $\begin{cases} \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{D} & \text{distinct} \\ W_1, W_2, \dots, W_n \in \mathbb{C}^{k\times k} \end{cases}$

construct an analytic function $F: \mathbb{D} \to \mathbb{C}^{k\times k}$ such that

$$F(\lambda_j) = W_j, \quad j = 1, 2, \dots, n,$$

and

$$r(F(\lambda)) \leq 1 \quad \text{for all } \lambda \in \mathbb{D}.$$

$r(A) \overset{\text{def}}{=}$ spectral radius of $A \in \mathbb{C}^{k\times k}$.

Some approaches

Bercovici, Foiaş and Tannenbaum, 1980s

Diagonalization doesn't work!

Solvable $\Leftrightarrow$ $\exists$ $\tilde{W}_j$ similar to W_j such that

$\lambda_j \mapsto \tilde{W}_j$ are solvable classical Nev Pick data.

Doyle and Stein, 1980s, "D - K iteration"

Matlab toolbox

Helton, Merino et al., 1990s

Numerical methods for the construction of

$F: \mathbb{D} \to \mathbb{C}^{k \times l}$ such that $\Gamma(F(e^{i\theta})) \leqslant 1$

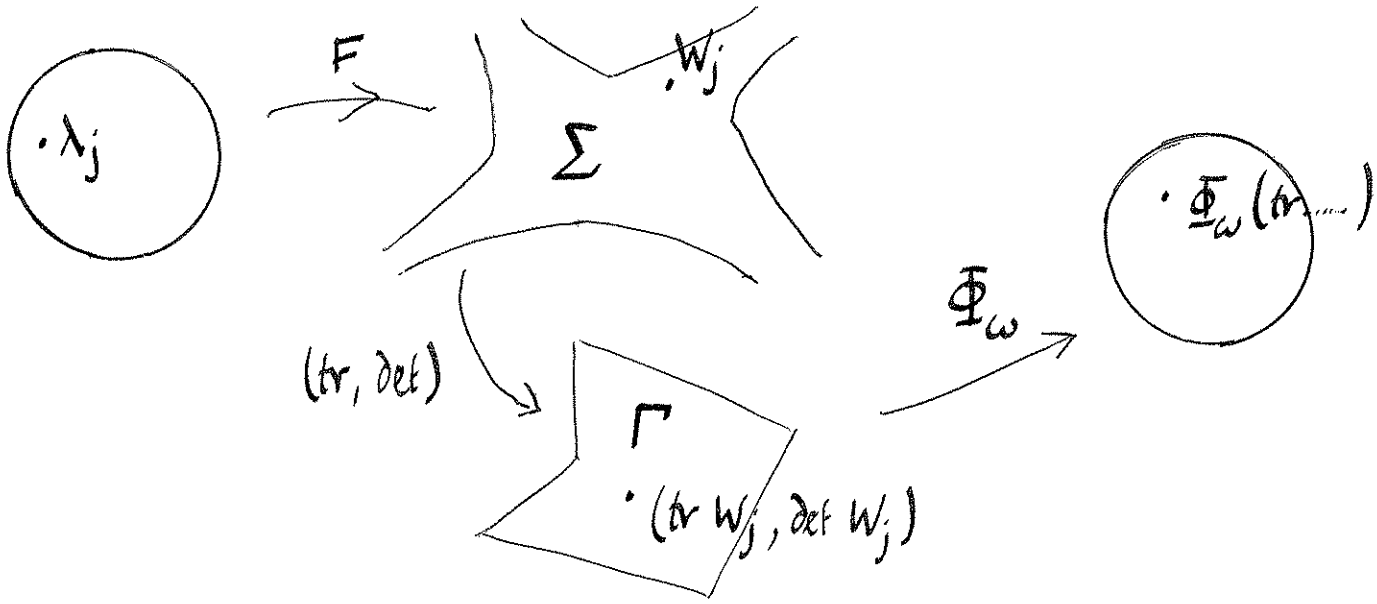
for all $\theta \in \mathbb{R}$, where Γ is a smooth function.

(Very general, but not ideal for spectral NP).

A necessary condition

Suppose $F : \mathbb{D} \to \Sigma$ is analytic & $F(\lambda_j) = W_j$.

Let $\Phi_\omega(s,p) = \dfrac{2\omega p - s}{2 - \omega s}$, $\quad \omega \in \mathbb{T}, \; s, p \in \mathbb{C}$.



$\Phi_\omega \circ (\mathrm{tr}, \det) \circ F$ maps $\mathbb{D} \to \Delta$

$$\lambda_j \mapsto \frac{2\omega \det W_j - \mathrm{tr}\, W_j}{2 - \omega\, \mathrm{tr}\, W_j}$$

By Pick's theorem, $\forall \omega \in \mathbb{T}$,

$$(\mathrm{Pick}_\omega) \qquad \left[\frac{1 - \overline{\Phi_\omega(\mathrm{tr}\, W_i, \det W_i)}\, \Phi_\omega(\mathrm{tr}\, W_j, \det W_j)}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^{n} \geqslant 0.$$

Σ and Γ

Let $\Sigma = \{A \in \mathbb{C}^{2\times 2} : r(A) \leq 1\}$,

$$\Delta = \{z \in \mathbb{C} : |z| \leq 1\}.$$

For $A \in \mathbb{C}^{2\times 2}$,

$$A \in \Sigma \Leftrightarrow \text{the zeros of } z^2 - (\operatorname{tr} A)z + \det A \text{ lie in } \Delta$$

$$\Leftrightarrow \operatorname{tr} A = z_1 + z_2, \quad \det A = z_1 z_2 \text{ for some } z_1, z_2 \in \Delta.$$

Let $\Gamma = \{(z_1+z_2, z_1 z_2) : z_1, z_2 \in \Delta\}$,

the symmetrised bidisc.

$$A \in \Sigma \Leftrightarrow (\operatorname{tr} A, \det A) \in \Gamma$$

A criterion for the 2×2 case

Agler - Y, Bercovici   TAMS 2004

When $k = 2$ the spectral Nevanlinna - Pick problem with data $\lambda_j \mapsto W_j$, $j = 1, 2, \ldots, n$, has a solution if and only if $\exists\, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C}$ such that

$$b_j c_j = \frac{1}{4} (\operatorname{tr} W_j)^2 - \det W_j \qquad \forall j$$

and

$$\left[\frac{I_2 - \tilde{W}_i^* \tilde{W}_j}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^{n} \geq 0$$

where

$$\tilde{W}_j = \begin{bmatrix} \frac{1}{2} \operatorname{tr} W_j & b_j \\ c_j & \frac{1}{2} \operatorname{tr} W_j \end{bmatrix}.$$

Spectral NP for 2 points and 2×2 matrices

Let $\lambda_1, \lambda_2 \in \mathbb{D}$ and let $W_1, W_2 \in \mathbb{C}^{2\times 2}$ be nonscalar.

There exists an analytic $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$F(\lambda_1) = W_1, \qquad F(\lambda_2) = W_2$$

and $$r(F(\lambda)) \leq 1 \qquad \forall \lambda \in \mathbb{D}$$

if and only if

$$\sup_{|\omega| = 1} \left| \frac{(s_2 p_1 - s_1 p_2)\omega^2 + 2(p_2 - p_1)\omega + s_1 - s_2}{(s_1 - \bar{s}_2 p_1)\,\omega^2 - 2(1 - p_1 \bar{p}_2)\omega + \bar{s}_2 - s_1 \bar{p}_2} \right| \leq 1$$

where $s_j = \operatorname{tr} W_j$, $p_j = \det W_j$, $j = 1, 2$.

Agler - Y, JFA 1999; Int. Eq. Op. Theory 2000.

The spectral Carathéodory-Fejér problem

Let $W_0, W_1 \in \mathbb{C}^{2\times 2}$ with W_0 nonscalar.

There exists an analytic $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$F(0) = W_0, \quad F'(0) = W_1 \quad \text{and}$$

$$r(F(\lambda)) \leq 1 \qquad \forall \lambda \in \mathbb{D}$$

if and only if

$$\sup_{|\omega|=1} \left| \frac{(s_0 p_1 - s_1 p_0)\omega^2 - 2p_1\omega + s_1}{(s_0 - \bar{s}_0 p_0)\omega^2 - 2(1-|p_0|^2)\omega + \bar{s}_0 - s_0\bar{p}_0} \right| \leq 1$$

where

$$s_0 = \operatorname{tr} W_0, \qquad p_0 = \det W_0$$

$$s_1 = \operatorname{tr} W_1, \qquad p_1 = \begin{vmatrix} w_{11}^0 & w_{12}^1 \\ w_{21}^0 & w_{22}^1 \end{vmatrix} + \begin{vmatrix} w_{11}^1 & w_{12}^0 \\ w_{21}^1 & w_{22}^0 \end{vmatrix}.$$

Huang, Marcantognini, Y 2006

The next case of μ

For $A \in \mathbb{C}^{2\times 2}$ we define

$$\frac{1}{\mu(A)} = \inf \{ \|X\| : X \in \mathbb{C}^{2\times 2}, \ X \text{ is diagonal}, \ 1 - AX \text{ is singular} \}.$$

We have $$r(A) \leq \mu(A) \leq \|A\|$$

μ is not a norm.

The μ-synthesis problem

Given $\begin{cases} \lambda_1, \dots, \lambda_n \in \mathbb{D} & \text{distinct} \\ W_1, \dots, W_n \in \mathbb{C}^{2\times 2} \end{cases}$

construct an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$

such that $F(\lambda_j) = W_j, \quad j = 1, \dots, n,$ and

$$\mu(F(\lambda)) \leq 1 \qquad \forall \lambda \in \mathbb{D}.$$

A Schwarz Lemma for μ

A. Abouhajar, M. C. White, Y, J. Geom. Anal 2007.

Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, $\zeta \in \mathbb{C}$,

$$W_1 = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} a & * \\ * & b \end{bmatrix}$$

and $p = \det W_2$. Suppose $|b| \leq |a|$.

There exists an analytic $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$F(0) = W_1$, $F(\lambda_0) = W_2$ and $\mu(F(\lambda)) \leq 1 \;\; \forall \lambda \in \mathbb{D}$

if and only if

$$\begin{cases} \dfrac{|a - \bar{b}p| + |ab - p|}{1 - |p|^2} \leq |\lambda_0| & \text{if } \zeta \neq 0 \\[2ex] |\lambda_0|^4 - \left(|a|^2 + |b|^2 + 2|ab - p|\right)|\lambda_0|^2 + |p|^2 \geq 0 & \text{if } \zeta = 0. \end{cases}$$

Observe the ill-conditioning of the problem near $\zeta = 0$.

Another Schwarz Lemma

NJY, Bull. London Math. Soc. 2008

Let $A = [A_1 \ A_2] \in \mathbb{C}^{2\times 2}$ be strictly triangular but not 0

$B = [B_1 \ B_2] = [b_{ij}] \in \mathbb{C}^{2\times 2}$ be nondiagonal.

There exists an analytic $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$F(0) = A, \qquad F'(0) = B \ \text{ and } \ \mu(F(\lambda)) \leq 1 \quad \forall \lambda \in \mathbb{D}$$

if and only if

$$\max\{|b_{11}|, |b_{22}|\} + |A_1 \wedge B_2 + A_2 \wedge B_1| \leq 1.$$