r-Qsym is free over Sym

A proof

by

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Notation

$\mathbf{X_n} = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$

$\mathbf{Q}[\mathbf{X_n}]$ denotes the algebra of polynomials in $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ with rational coefficients.

$\mathbf{V}$ a graded vector space

$\mathcal{H}_\mathbf{m}(\mathbf{V})$ denotes the subspace of the homogeneous elements of degree $\mathbf{m}$ in $\mathbf{V}$.

$$\mathbf{V} = \mathcal{H}_\mathbf{0}(\mathbf{V}) \oplus \mathcal{H}_\mathbf{1}(\mathbf{V}) \oplus \mathcal{H}_\mathbf{2}(\mathbf{V}) \oplus \cdots \oplus \mathcal{H}_\mathbf{m}(\mathbf{V}) \oplus \cdots$$

The Hilbert series of $\mathbf{V}$

$$\mathbf{F_V}(\mathbf{q}) = \sum_{\mathbf{m} \geq \mathbf{0}} \mathbf{dim}\big(\mathcal{H}_\mathbf{m}(\mathbf{V})\big) \ \mathbf{q^m}$$

Basics

Let $\mathbf{A}$ be a Finitely Generated Graded Algebra.

$$\mathbf{A} = \bigoplus_{m \geq 0} \mathcal{H}_m(\mathbf{A})$$

Let

(1) $\mathbf{n_1} =$ the order of "$\mathbf{1}$" as a pole of the Hilbert series $\mathbf{F_A(q)}$.

(2) $\mathbf{n_2} =$ the maximum number of algebraically independent elements in $\mathbf{A}$.

(3) $\mathbf{n_3} =$ the minimum number of elements $\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n} \in \mathbf{A}$ such that

$$\mathbf{dim}\ \mathbf{A}/(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n})_\mathbf{A} < \infty \qquad (*)$$

Fact: $\mathbf{n_1} = \mathbf{n_2} = \mathbf{n_3} = \mathbf{n_A} =$ the "*Krull dimension*" of $\mathbf{A}$

If (*) holds with $\mathbf{n} = \mathbf{n_A}$ then $\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}$ are called a "*System Of Parameters*". **"S.O.P."** in brief.

More Basics

Let $\mathbf{q_1, q_2, \ldots, q_n} \in \mathbf{A}$ be homogeneous of degrees $\mathbf{d_1, d_2, \ldots, d_n}$ and suppose they constitute a **S.O.P** for $\mathbf{A}$. Let $\mathbf{B} = \{\mathbf{f_1, f_2, \ldots, f_N}\}$ be a basis for the quotient

$$\mathbf{A}/(\mathbf{q_1, q_2, \ldots, q_n})_\mathbf{A} < \infty$$

Then every $\mathbf{P} \in \mathbf{A}$ has an expansion of the form

$$\mathbf{P} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{f_i}\ \mathbf{Q_i}(\mathbf{q_1, q_2, \ldots, q_n})$$

with $\mathbf{Q_1, Q_2, \ldots, Q_N} \in \mathbf{Q}[\mathbf{y_1, \ldots, y_n}]$. In other words the collection

$$\{\mathbf{f_i\, q_1^{p_1}, q_2^{p_2}, \ldots q_n^{p_p}}\}_{\mathbf{1 \le i \le N}}$$

spans $\mathbf{A}$ as a vector space. In particular it follows that

$$\mathbf{F_A(q)} << \frac{\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{q}^{\mathbf{degree(f_i)}}}{(\mathbf{1 - q^{d_1}})(\mathbf{1 - q^{d_2}}) \cdots (\mathbf{1 - q^{d_n}})}$$

The Cohen Macaulay Property

If the expansion

$$\mathbf{P} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{f_i}\ \mathbf{Q_i}(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n})$$

is **unique** for every $\mathbf{P} \in \mathbf{A}$ then the collection

$$\left\{\mathbf{f_i}\, \mathbf{q_1^{p_1}}, \mathbf{q_2^{p_2}}, \ldots \mathbf{q_n^{p_p}}\right\}_{\mathbf{1 \le i \le N}}$$

is a **basis** for $\mathbf{A}$. This holds true **if and only if** we have the the equality

$$\mathbf{F_A}(\mathbf{q}) = \frac{\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{q}^{\mathbf{degree(f_i)}}}{(\mathbf{1} - \mathbf{q^{d_1}})(\mathbf{1} - \mathbf{q^{d_2}}) \cdots (\mathbf{1} - \mathbf{q^{d_n}})}$$

Then $\mathbf{A}$ is a free module over $\mathbf{Q}[\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}]$ of rank $\mathbf{N}$ and $\mathbf{A}$ is said to be **"Cohen-Macaulay"**.

A useful criterion

Theorem A

Let $\mathbf{q_1, q_2, \ldots, q_n \in A}$ *be homogeneous of degrees* $\mathbf{d_1, d_2, \ldots, d_n}$ *and an* **S.O.P.** *for* $\mathbf{A}$. *Let*

$$\mathbf{\dim A/(q_1, q_2, \ldots, q_n)_A \ = \ M \le N}$$

with basis $\mathbf{B} = \{\mathbf{f_1, f_2, \ldots, f_M}\}$. *Then the condition*

$$\mathbf{\lim_{q \to 1}(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n}) F_A(q) \ = \ N}$$

forces $\mathbf{M} = \mathbf{N}$ *and the equality*

$$\mathbf{F_A(q) \ = \ \frac{\sum_{i=1}^{N} q^{degree(f_i)}}{(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n})}}$$

yielding that $\mathbf{A}$ *is a free module over* $\mathbf{Q[q_1, q_2, \ldots, q_n]}$ *of rank* $\mathbf{N}$ *and therefore* $\mathbf{A}$ *is a Cohen-Macaulay algebra.*

r-Quasi-Symmetric

Recall the Florent Hivert "local" $\mathbf{r}$-action of $\mathbf{S_n}$

$$\mathbf{s_i}\ \ \mathbf{x_i^a\, x_{i+1}^b} = \begin{cases} \mathbf{x_i^a\, x_{i+1}^b} & \textbf{if } \mathbf{a, b \geq r} \\ \mathbf{x_i^b\, x_{i+1}^a} & \textbf{otherwise} \end{cases} \qquad (*)$$

here $\mathbf{s_i = s_{i,i+1}}$. Since we have

$$\mathbf{s_i^2 = 1}\,, \quad \mathbf{s_i\ s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}}, \quad \mathbf{s_i\, s_j = s_j\, s_i}\ \ \forall\ \ \mathbf{|i-j| \geq 2}$$

(*) defines an action of $\mathbf{S_n}$ called the **"r-action."** Florent Hivert defines

$$\textbf{r-QSym} = \Big\{\mathbf{P(x)} : \mathbf{P(x)} \ \ \textbf{is invariant under the r-action}\Big\}$$

We have a descending chain of algebras

$$\mathbf{Q[X_n]} \supset \textbf{1-QSym}\mathbf{[X_n]} \supset \textbf{2-QSym}\mathbf{[X_n]} \supset \cdots \supset \textbf{r-QSym}\mathbf{[X_n]} \supset \cdots \supset \textbf{Sym}\mathbf{[X_n]}$$

Note that $\textbf{1-QSym}\mathbf{[X_n]}$ is the space of Gessel's **"quasi-symmetric funtions"**.

Preliminaries

For $\mathbf{p} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_k})$ and $\lambda = (\lambda_\mathbf{1}, \lambda_\mathbf{2}, \ldots, \lambda_\mathbf{h})$ with $\mathbf{k} + \mathbf{h} \leq \mathbf{n}$ set

$$\mathbf{M}_{\mathbf{p},\lambda}[\mathbf{X_n}] = \sum_{\substack{\mathbf{T} \subseteq \{\mathbf{1},\mathbf{2},\ldots,\mathbf{n}\} \\ |\mathbf{T}|=\mathbf{k}}} \mathbf{X}_\mathbf{T}^\mathbf{p}\, \mathbf{m}_\lambda[\mathbf{X_n} - \mathbf{X_T}] \tag{I.5}$$

where if $\mathbf{T} = \{\mathbf{i_1} < \mathbf{i_2} < \cdots < \mathbf{i_k}\}$ then $\mathbf{X}_\mathbf{T}^\mathbf{p} = \mathbf{x}_{\mathbf{i_1}}^{\mathbf{p_1}} \mathbf{x}_{\mathbf{i_2}}^{\mathbf{p_2}} \cdots \mathbf{x}_{\mathbf{i_k}}^{\mathbf{p_k}}$

It was show by Florent Hivert that the collection

$$\Big\{\mathbf{M}_{\mathbf{p},\lambda}[\mathbf{X_n}]\Big\}_{\lambda<\mathbf{r}\,,\,\mathbf{p}\geq\mathbf{r}}$$

is a basis for $\mathrm{rQSym}[\mathbf{X_n}]$

First Breakthroughs

(1) François and Nantel Bergeron discover that the collection of λ-free quasi-monomials

$$\left\{\mathbf{M_{p+r}}[\mathbf{X_n}]\right\}_{\mathbf{p}}$$

spans the quotient $\mathbf{rQSym}[\mathbf{X_n}]/(\mathbf{e_1},\mathbf{e_2},\ldots,\mathbf{e_n})_{\mathbf{rQSym}[\mathbf{X_n}]}$. Here

$$\mathbf{M_{p+r}}[\mathbf{X_n}] = \sum_{\mathbf{1\le i_1<i_2<\cdots<i_k\le n}} \mathbf{x_{i_1}^{p_1+r} x_{i_2}^{p_2+r} \cdots x_{i_k}^{p_k+r}}$$

(2) François and Nantel conjecture the existence of a collection $\mathbf{C_n}$ of semi-compositions such that the collection

$$\left\{\mathbf{M_{p+r}}[\mathbf{X_n}]\right\}_{\mathbf{p\in C_n}}$$

is a basis for this quotient for **all** $\mathbf{r \ge 1}$ **(!!!!!!)**

The Hilbert series of rQsym

Proposition

$$\mathbf{F}_{\mathbf{rQsym}[\mathbf{X_n}]}(\mathbf{q}) \;=\; \frac{\displaystyle\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_k(q)\; t^k}{(1-q)(1-q^2)\cdots(1-q^n)}$$

with $\mathbf{t} = \mathbf{q^r}$ and

$$\mathbf{D_n(q)} \;=\; [n]_q! \sum_{s=0}^{n} \frac{q^{\binom{s}{2}}(-1)^s}{[s]_q!} \qquad \textbf{(a q-Derangement number!!!)}$$

Proof

$$\begin{aligned}
\mathbf{F}_{\mathbf{rQsym}[\mathbf{X_n}]}(\mathbf{q}) \;&=\; \sum_{k=0}^{n} \frac{q^{r(n-k)}}{(1-q)^{n-k}} \begin{bmatrix} k+r-1 \\ r-1 \end{bmatrix}_q \\
&=\; \sum_{k=0}^{n} \frac{q^{r(n-k)}}{(1-q)^{n-k}} \;\frac{(1-q^r)(1-q^{r+1})\cdots(1-q^{r+k-1})}{(1-q)(1-q^2)\cdots(1-q^k)}. \\
&=\; \sum_{k=0}^{n} \frac{t^{n-k}}{(1-q)^{n-k}} \,\frac{(1-t)(1-tq)\cdots(1-tq^{k-1})}{(1-q)(1-q^2)\cdots(1-q^k)}
\end{aligned}$$

Use the q-binomial Theorem and rearrange terms.

The operator Tn

Let

$$\mathbf{T_n = (1, 2, \ldots, n) + (2, 3, \ldots, n) + \cdots + (n-1, n) + id}$$

Theorem(Phatarfod, Diaconis, etc)

The (right) action of $\mathbf{T_n}$ *on the group algebra of* $\mathbf{S_n}$ *is semisimple with eigenvalues* $\mathbf{(0, 1, 2, \ldots, n{-}2, n)}$ *and the multiplicity of* $\mathbf{k}$ *is the number of permuations with* $\mathbf{k}$ *fixed points. In particular the multiplicity of* $\mathbf{0}$ *is the derangement number* $\mathbf{D_n}$*. It follows from this that*

$$\sum_{\mathbf{k=0}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D_k} = \mathbf{n!}$$

A basic Fact

Set

$$\mathbf{S_{nk}} = \Big\{ \mathbf{p} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_k}) \,:\; \mathbf{0} \le \mathbf{p_i} \le \mathbf{n} - \mathbf{i}, \;\; \textbf{for} \;\; \mathbf{1} \le \mathbf{i} \le \mathbf{k} \Big\}$$

We call the elements of $\mathbf{S_{nk}}$ "*n-subtriangular semi-compositions of length* $\mathbf{k}$"

Let

$$\mathbf{I_{n,k}} = \big(\mathbf{h_{n+1-k}}[\mathbf{X_k}], \mathbf{h_{n+2-k}}[\mathbf{X_k}], \ldots, \mathbf{h_n}[\mathbf{X_k}] \big)_{\mathbf{Q}[\mathbf{X_k}]}$$

Theorem

The quotient $\mathbf{R_{n,k}} = \mathbf{Q}[\mathbf{X_k}]/\mathbf{I_{nk}}$ *affords* $\binom{\mathbf{n}}{\mathbf{k}}$ *copies of the regular representation of* $\mathbf{S_k}$, *has basis*

$$\mathbf{A_{n,k}} = \big\{ \mathbf{x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}} \big\}_{\mathbf{a} \in \mathbf{S_{nk}}}$$

and Frobenius characteristic

$$\mathbf{F\,ch\,R_{n,k}} = \begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}_{\mathbf{q}} \sum_{\lambda \vdash \mathbf{k}} \mathbf{S}_\lambda(\mathbf{x}) \sum_{\mathbf{T} \in \mathbf{ST}(\lambda)} \mathbf{q}^{\mathbf{maj}(\mathbf{T})}$$

Third BreakThrough

For $\quad \mathbf{f} = \sum_{\mathbf{p}} \mathbf{c_p X_k^p} \quad$ set $\quad \mathbf{\Gamma^{(r)}_{n,k} f} = \sum_{\mathbf{p}} \mathbf{c_p M_{p+r}[X_n]}.$

Theorem

For each $\mathbf{1 \le k \le n}$ *we have*

$$\mathbf{\Gamma^{(r)}_{n,k}\, I_{n,k} \subseteq (e_1, e_2, \ldots, e_n)_{r-Qsym[X_n]}}\,.$$

In particular $\mathbf{\Gamma^{(r)}_{n,k}}$ *is well defined as an operator from* $\mathbf{R_{n,k}}$ *into*

$$\mathbf{rQsym[X_n]/(e_1, e_2, \ldots, e_n)_{rQsym[X_n]}}\,.$$

Likewise if $\mathcal{B}_{\mathbf{n,k}} \subseteq \mathbf{Q[X_k]}$ *yields a basis for* $\mathbf{R_{n,k}}$*, for* $\mathbf{1 \le k \le n}$*, then the collection*

$$\{\mathbf{1}\} \cup \bigcup_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{\Gamma^{(r)}_{n,k}} \mathcal{B}_{\mathbf{n,k}}$$

spans $\mathbf{rQsym[X_n]}$ *as a* Λ_n *module.*

A simple consequence

Let $\mathbf{E_{nk}}$ be a basis for the eigenvectors of $\mathbf{T_k}$ on the quotient $\mathbf{R_{n,k}}$. From the previous theorem it follows that the collection

$$\{\mathbf{1}\} \cup \bigcup_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{\Gamma^{(r)}_{n,k} E_{n,k}}$$

spans the quotient

$$\mathbf{rQsym}[\mathbf{X_n}]/(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{rQsym}[\mathbf{X_n}]}$$

However since $\mathbf{R_{n,k}}$ consists of $\binom{\mathbf{n}}{\mathbf{k}}$ copies of the left regular representation of $\mathbf{S_k}$ the cardinality of this collection is

$$\mathbf{1} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{k!}$$

Too many!!!! What we have to throw away???

The crucial Happening

Since $\mathbf{R_{n,k}}$ is $\binom{\mathbf{n}}{\mathbf{k}}$ copies of the left regular representation, the subset $\mathbf{E^o_{nk}} \subset \mathbf{E_{nk}}$ of zero eigenvectors has cardinality

$$|\mathbf{E^o_{nk}}| = \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D_k}$$

and thus the subcollection

$$\{\mathbf{1}\} \cup \bigcup_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{\Gamma^{(r)}_{n,k} E^o_{n,k}}$$

has cardinality

$$\mathbf{1} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D_k} = \mathbf{n!}$$

COULD IT BE!!!???

The crucial Happening

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has cardinality

$$\mathbf{1} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D_k} = \mathbf{n!}$$

YES!!! YES!!!

IN FACT

$$\mathbf{\Gamma^{(r)}_{n,k}\left(E_{nk} - E^{o}_{nk}\right) \subset L\left[\bigcup_{s=1}^{k-1} \Gamma^{(r)}_{ns}\, E^{o}_{n,s}\right]}$$

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$$\mathbf{\Gamma^{(r)}_{nk}\big(E_{nk} - E^{o}_{nk}\big) \subset L\Big[\bigcup_{s=1}^{k-1} \Gamma^{(r)}_{ns}\, E^{o}_{n,s}\Big]}$$

Why?

IN FACT

$$\mathbf{G_{nk}}\big(\mathbf{E_{nk}} - \mathbf{E^{o}_{nk}}\big) \subset \mathbf{L}\Big[\bigcup_{\mathbf{s=1}}^{\mathbf{k-1}} \mathbf{G_{ns}}\,\mathbf{E^{o}_{n,s}}\Big]$$

Multiplication by power symmetric functions does it!

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$$\mathbf{\Gamma^{(r)}_{nk}\big(E_{nk} - E^{o}_{nk}\big) \subset L\Big[\bigcup_{s=1}^{k-1} \Gamma^{(r)}_{ns}\, E^{o}_{n,s}\Big]}$$

Proof

Note that for any semi-composition $\mathbf{b}$ of length $\mathbf{k-1}$ and any $\mathbf{1 \leq j \leq n}$ we have

$$\mathbf{p_{j+r}[X_n]M_{b+r}[X_n] \;=\; \sum_{q\in\, b\, \uplus\, j} M_{q+r}[X_n] \,+\, \sum_{s=1}^{k-1} M_{b+r+e_s(r+j)}[X_n]}$$

with $\mathbf{e_s}$ the coordinate $\mathbf{k-1}$-vector with $\mathbf{1}$ in position $\mathbf{s}$.

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Note that for any semi-composition $\mathbf{b}$ of length $k-1$ and any $1 \leq j \leq n$ we have

$$p_{j+r}[X_n]M_{b+r}[X_n] = \sum_{q\in b \,\cup\!\!\cup\, j} M_{q+r}[X_n] + \sum_{s=1}^{k-1} M_{b+r+e_s(r+j)}[X_n]$$

with e_s the coordinate $k-1$-vector with 1 in position s. This gives

$$\sum_{q\in b \,\cup\!\!\cup\, j} M_{q+r}[X_n] \equiv -\sum_{s=1}^{k-1} M_{b+r+e_s(r+j)}[X_n] \qquad \Big(\mathbf{mod}\ (e_1, e_2, \ldots, e_n)_{r-Qsym[X_n]}\Big)$$

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Thus

$$\mathbf{\sum_{q\in\, b \,\uplus\, j} M_{q+r}[X_n] \;\in\; L\Big[\Gamma^{(r)}_{n,k-1}\, E_{n,k-1}\Big] \qquad \Big(mod\ (e_1, e_2, \ldots, e_n)_{r-Qsym[X_n]}\Big)}$$

The role of $\mathbf{T_n}$

Let $\mathbf{b}[\mathbf{X_k}] \in \mathbf{E_{nk}} - \mathbf{E^o_{nk}}$ be of eigenvalue $\lambda \neq \mathbf{0}$ with

$$\mathbf{b}[\mathbf{X_k}] = \sum_{\alpha \in \mathbf{S_{nk}}} \mathbf{c}_\alpha \, \mathbf{x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}}$$

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Now setting $\mathbf{a}' = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_{k-1}})$ we have

$$\mathbf{b}[\mathbf{X_k}] = \frac{\mathbf{1}}{\lambda} \sum_{\mathbf{a} \in \mathbf{S_{nk}}} \mathbf{c_a} \, \mathbf{T_k} \, \mathbf{x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}} = \frac{\mathbf{1}}{\lambda} \sum_{\mathbf{a} \in \mathbf{S_{nk}}} \mathbf{c_a} \sum_{\mathbf{q} \in \, \mathbf{a_k} \sqcup\!\sqcup \, \mathbf{a}'} \mathbf{X_k^q}$$

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from which it follows that

$$\mathbf{\Gamma^{(r)}_{n,k}} \, \mathbf{b}[\mathbf{X_k}] = \frac{\mathbf{1}}{\lambda} \sum_{\mathbf{a} \in \mathbf{S_{nk}}} \mathbf{c_a} \sum_{\mathbf{q} \in \, \mathbf{a_k} \sqcup\!\sqcup \, \mathbf{a}'} \mathbf{M_{q+r}}[\mathbf{X_n}].$$

The role of $\mathbf{T_n}$

Let $\mathbf{b}[\mathbf{X_k}] \in \mathbf{E_{nk}} - \mathbf{E^o_{nk}}$ be of eigenvalue $\lambda \neq \mathbf{0}$ with

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$$\mathbf{\Gamma^{(r)}_{n,k}} \, \mathbf{b}[\mathbf{X_k}] = \frac{\mathbf{1}}{\lambda} \sum_{\mathbf{a} \in \mathbf{S_{nk}}} \mathbf{c_a} \sum_{\mathbf{q} \in \mathbf{a_k} \sqcup\!\sqcup \mathbf{a}'} \mathbf{M_{q+r}}[\mathbf{X_n}].$$

and the previous result gives

$$\mathbf{\Gamma^{(r)}_{n,k}} \, \mathbf{b}[\mathbf{X_k}] \in \mathbf{L}\Big[\mathbf{\Gamma^{(r)}_{n,k-1}} \, \mathbf{E_{n,k-1}}\Big] \qquad \Big(\mathbf{mod} \ (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{r-Qsym}[\mathbf{X_n}]}\Big)$$

Completion of the proof

We have established that

$$\mathbf{\Gamma^{(r)}_{n,k}}\left(\mathbf{E_{n,k}} - \mathbf{E^{o}_{n,k}}\right) \subseteq \mathbf{L}\Big[\mathbf{\Gamma^{(r)}_{n,k-1}}\,\mathbf{E_{n,k-1}}\Big] \qquad \Big(\mathbf{mod}\ (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{r-Qsym[X_n]}}\Big) \quad (*)$$

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For $\mathbf{k = 1}$

$$\mathbf{\Gamma^{(r)}_{n,1}}\mathbf{R_{n,1}} \in \mathbf{L}\big(\{\mathbf{p_{1+r}}, \mathbf{p_{2+r}}, \ldots, \mathbf{p_{n+r}}\}\big) \subseteq (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{r-Qsym[X_n]}}$$

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For $\mathbf{k = 1}$

$$\mathbf{\Gamma^{(r)}_{n,1} R_{n,1}} \in \mathbf{L}\big(\{\mathbf{p_{1+r}}, \mathbf{p_{2+r}}, \ldots, \mathbf{p_{n+r}}\}\big) \subseteq (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{r-Qsym[X_n]}}$$

So assuming by induction that (modulo the ideal $(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{r-Qsym[X_n]}}$)

$$\mathbf{\Gamma^{(r)}_{n,k-1} E_{n,k-1}} \subseteq \mathbf{L}\Big[\bigcup_{\mathbf{s=1}}^{\mathbf{k-1}} \mathbf{\Gamma^{(r)}_{n,s}}\,\mathbf{E^{o}_{n,s}}\Big]$$

Completion of the proof

We have established that

$$\mathbf{\Gamma^{(r)}_{n,k}\left(E_{n,k}-E^{o}_{n,k}\right) \subseteq L\Big[\Gamma^{(r)}_{n,k-1}\,E_{n,k-1}\Big] \qquad \Big(mod\ (e_1,e_2,\dots,e_n)_{r-Qsym[X_n]}\Big)} \quad (*)$$

For $\mathbf{k=1}$

$$\mathbf{\Gamma^{(r)}_{n,1}R_{n,1} \in L\big(\{p_{1+r},p_{2+r},\dots,p_{n+r}\}\big) \subseteq (e_1,e_2,\dots,e_n)_{r-Qsym[X_n]}}$$

So assuming by induction that (modulo the ideal $\mathbf{(e_1,e_2,\dots,e_n)_{r-Qsym[X_n]}}$)

$$\mathbf{\Gamma^{(r)}_{n,k-1}E_{n,k-1} \subseteq L\Big[\bigcup_{s=1}^{k-1}\Gamma^{(r)}_{n,s}\,E^{o}_{n,s}\Big]}$$

then (*) gives

$$\mathbf{\Gamma^{(r)}_{n,k}\big(E_{n,k}\big) \subset L\Big[\bigcup_{s=1}^{k}\Gamma^{(r)}_{n,s}\,E^{o}_{n,s}\Big]}$$

completing the induction

Wrapping thing up

We have proved that the collection

$$\{\mathbf{1}\} \cup \bigcup_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{\Gamma^{(r)}_{n,k} E^{o}_{n,k}}$$

of cardinality

$$\mathbf{1} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{D_k} \; = \; \mathbf{n}! < \infty$$

spans the quotient

$$\mathbf{rQsym[X_n]/(e_1, e_2, \ldots, e_n)_{rQsym[X_n]}}$$

Thus $\mathbf{e_1, e_2, \ldots, e_n}$ are a homogeneous S.O.P for $\mathbf{A}$ =r-Qsym and

$$\mathbf{dim}\;\; \mathbf{rQsym[X_n]/(e_1, e_2, \ldots, e_n)_{rQsym[X_n]}} \leq \mathbf{n!}$$

Recall the useful criterion

Theorem A

Let $\mathbf{q_1, q_2, \ldots, q_n} \in \mathbf{A}$ *be homogeneous of degrees* $\mathbf{d_1, d_2, \ldots, d_n}$ *and an* **S.O.P.** *for* $\mathbf{A}$. *Let*

$$\mathbf{dim}\ \mathbf{A}/(\mathbf{q_1, q_2, \ldots, q_n})_{\mathbf{A}} = \mathbf{M} \leq \mathbf{N}$$

with basis $\mathbf{B} = \{\mathbf{f_1, f_2, \ldots, f_M}\}$. *Then the condition*

$$\lim_{\mathbf{q}\to \mathbf{1}} (\mathbf{1}-\mathbf{q}^{\mathbf{d_1}})(\mathbf{1}-\mathbf{q}^{\mathbf{d_2}})\cdots(\mathbf{1}-\mathbf{q}^{\mathbf{d_n}})\mathbf{F_A}(\mathbf{q}) = \mathbf{N}$$

forces $\mathbf{M} = \mathbf{N}$ *and the equality*

$$\mathbf{F_A}(\mathbf{q}) = \frac{\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{N}} \mathbf{q}^{\mathbf{degree}(\mathbf{f_i})}}{(\mathbf{1}-\mathbf{q}^{\mathbf{d_1}})(\mathbf{1}-\mathbf{q}^{\mathbf{d_2}})\cdots(\mathbf{1}-\mathbf{q}^{\mathbf{d_n}})}$$

yielding that $\mathbf{A}$ *is a free module over* $\mathbf{Q}[\mathbf{q_1, q_2, \ldots, q_n}]$ *of rank* $\mathbf{N}$ *and therefore* $\mathbf{A}$ *is a Cohen-Macaulay algebra.*

What we have in our case

The elementary symmetric functions $\mathbf{e_1, e_2, \ldots, e_n} \in \mathbf{rQsym}$ *are homogeneous of degrees* $\mathbf{1, 2, \ldots, n}$ *and an* **S.O.P.** *for* $\mathbf{A}$. *We have shown that*

$$\mathbf{\dim\, rQsym/(q_1, q_2, \ldots, q_n)_{rQsym}} \leq \mathbf{n!}$$

with spanning set $\mathbf{B} = \{\mathbf{1}\} \cup \bigcup_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{\Gamma^{(r)}_{n,k} E^{o}_{n,k}}$ *Moreover we have*

$$\mathbf{\lim_{q\to 1}(1-q)(1-q^2)\cdots(1-q^n)F_{rQsym}(q)} = \mathbf{\lim_{q\to 1}} \sum_{\mathbf{k=0}}^{\mathbf{n}} \begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix}_{\mathbf{q}} \mathbf{D_k(q)}\ \ \mathbf{q^{rk}} = \mathbf{n!}$$

thus $\mathbf{B}$ *is a basis and we must also have*

$$\mathbf{1} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix}_{\mathbf{q}} \mathbf{D_k(q)}\ \ \mathbf{q^{rk}} = \mathbf{1} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \sum_{\mathbf{b \in E^{o}_{n,k}}} \mathbf{q^{degree(b)} q^{rk}}$$

yielding that $\mathbf{rQsym}$ *is a free module over* $\mathbf{Q[e_1, e_2, \ldots, e_n]}$ *of rank* $\mathbf{n!}$ *and therefore* $\mathbf{rQsym}$ *is a Cohen-Macaulay algebra.*

Some interesting byproducts of the proof

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}_{\mathbf{q}} \mathbf{D_k}(\mathbf{q}) \;=\; \sum_{\mathbf{b} \in \mathbf{E}^{\mathbf{o}}_{\mathbf{n},\mathbf{k}}} \mathbf{q}^{\mathbf{degree}(\mathbf{b})}$$

In particular the q-derangement polynomial

$$\mathbf{D_n}(\mathbf{q}) \;=\; [\mathbf{n}]_{\mathbf{q}}! \sum_{\mathbf{s}=\mathbf{0}}^{\mathbf{n}} \frac{\mathbf{q}^{\binom{\mathbf{s}}{\mathbf{2}}}(-\mathbf{1})^{\mathbf{s}}}{[\mathbf{s}]_{\mathbf{q}}!}$$

gives the Hilbert series of the kernel of $\mathbf{T_n}$ in the polynomial ring $\mathbf{Q}[\mathbf{X_n}]$

The Frobenius characteristic of the Kernel of $\mathbf{T_n}$ in the Group algebra of $\mathbf{S_n}$ is the symmetric polynomial

$$\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} (-\mathbf{1})^{\mathbf{i}} \mathbf{e_i} \mathbf{e_1}^{\mathbf{n}-\mathbf{i}}$$

A Remarkable Corollary

Given a standard tableau T let us set

$$\mathbf{exit(T)} = \mathbf{min}\{\mathbf{j} : j \textbf{ is in the first column of } T \textbf{ and } j+1 \textbf{ is not } \}$$

This given, Theorem 3.1 yields

Corollary 3.1 (Reiner-Wachs)

The multiplicity $\mathbf{m}_\lambda(\mathbf{k})$ *of Young's irreducible representation indexed by* λ *in the* $\mathbf{k}^{\mathbf{th}}$ *eigenspace of* $\mathbf{T_n}$ *is equal to the number of standard tableaux* $\mathbf{T}$ *of shape* λ *with* $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}$ *in the first row,* $\mathbf{k}+\mathbf{2}$ *in the first column and such that the difference* $\mathbf{exit}(\mathbf{T}) - \mathbf{k}$ *is even.*