An Evaluation of the Dunkl-ised Vandermonde acting on the Vandermonde

The Dunkl operators

Let $\mathbf{s_{ij}}$ be the transposition that interchanges $\mathbf{x_i}$ and $\mathbf{x_j}$ then for any pair a, b we have

$$\frac{\mathbf{1}}{(\mathbf{x_i}-\mathbf{x_j})}(\mathbf{1}-\mathbf{s_{ij}})\mathbf{x_i^a x_j^b} \;=\; \begin{cases} \displaystyle\sum_{\mathbf{a\le r\le b-1}} \mathbf{x_j^r x_i^{b-1-r}} & \textbf{if } \mathbf{a \le b}\ , \\ \displaystyle\sum_{\mathbf{b\le r\le a-1}} \mathbf{x_i^r x_j^{a-1-r}} & \textbf{if } \mathbf{a > b}. \end{cases}$$

Thus the "*divided difference* " operators

$$\delta_{\mathbf{ij}} \;=\; \frac{\mathbf{1}}{(\mathbf{x_i}-\mathbf{x_j})}(\mathbf{1}-\mathbf{s_{ij}})$$

act on polynomials.

The Dunkl operators are defined by setting for $\mathbf{1 \le i \le n}$

$$\nabla_{\mathbf{i}}(\mathbf{m}) \;=\; \partial_{\mathbf{x_i}} + \mathbf{m}\sum_{\mathbf{j=1}}^{\mathbf{n}}{}^{(\mathbf{i})}\, \frac{\mathbf{1}}{\mathbf{x_i}-\mathbf{x_j}}\,(\mathbf{1}-\mathbf{s_{ij}})$$

Theorem(C. F. Dunkl)

They commute !!!

$$\nabla_{\mathbf{i}}(\mathbf{m})\nabla_{\mathbf{j}}(\mathbf{m}) \;=\; \nabla_{\mathbf{j}}(\mathbf{m})\nabla_{\mathbf{i}}(\mathbf{m})$$

The "Constant"

For a polynomial $\mathbf{P}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ we define "**the Dunkl-ised** $\mathbf{P}(\mathbf{x})$"

$$\mathbf{P}\big(\nabla(\mathbf{m})\big) \;=\; \mathbf{P}\big(\nabla_{\mathbf{1}}(\mathbf{m}), \nabla_{\mathbf{2}}(\mathbf{m}), \ldots, \nabla_{\mathbf{n}}(\mathbf{m})\big)$$

The commutativity of the Dunkl operators makes this unambiguous.

Theorem (Opdam?)

$$\prod_{\mathbf{1} \le \mathbf{i} < \mathbf{j} \le \mathbf{n}} \big(\nabla_{\mathbf{i}}(\mathbf{m}) - \nabla_{\mathbf{j}}(\mathbf{m})\big) \prod_{\mathbf{1} \le \mathbf{i} < \mathbf{j} \le \mathbf{n}} \big(\mathbf{x_i} - \mathbf{x_j}\big) \;=\; \mathbf{n}! \prod_{\mathbf{1} \le \mathbf{i} < \mathbf{j} \le \mathbf{n}} (\mathbf{jm} + \mathbf{i})$$

Easy? Hard?

OPEN PROBLEM:

Find an elementary proof

The Metha Integral

With

$$\mathbf{\Pi(x)} = \prod_{\mathbf{1 \le i < j \le n}} \left(\mathbf{x_i - x_j}\right)$$

we have

$$\frac{\mathbf{1}}{(\mathbf{2}\pi)^{\mathbf{n/2}}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathbf{\Pi(x)^{2m} e^{-|x|^2/2} d_{x_1} d_{x_2} \cdots d_{x_n}} = \prod_{\mathbf{j=1}}^{\mathbf{n}} \frac{\mathbf{(jm)!}}{\mathbf{m!}} = \mathbf{c_m}$$

Easy? Hard?

OPEN PROBLEM:

Find an elementary proof

The Selberg Integral

$$\int_0^1 \int_0^1 \cdots \int_0^1 \Big(\prod_{i=1}^{n} t_i^{x-1}(1-t_i)^{y-1} \Big) \prod_{1 \le i < j \le n} (t_i - t_j)^{2k} dt_1 dt_2 \cdots dt_n$$
$$= \prod_{j=1}^{n} \frac{\Gamma\big(x+(j-1)k)\Gamma\big(y+(j-1)k\big)}{\Gamma\big(x+y+(n+j-2)k\big)} \prod_{j=1}^{n} \frac{\Gamma(1+jk)}{\Gamma(1+k)} .$$

(More or less elementary quite ingenious proof by Selberg)
(tricky details usually skipped)

The Selberg Integral $\Longrightarrow$ The Metha Integral

(A rather messy derivation)

Enter Zeilberger

(Toward a WZ proof of the Metha integral)

For a symmetric polynomial $\mathbf{P}$

$$\sum_{i=1}^{n} \partial_{x_i}\left((\partial_{x_i}P)e^{-|x|^2/2}\Pi(x)^{2k}\right) = e^{-|x|^2/2}\Pi(x)^{2k}\left(L_m P - \sum_{i=1}^{n} x_i\partial_{x_i}P\right).$$

where

$$L_m = \nabla_1(m)^2 + \nabla_2(m)^2 + \cdots + \nabla_n(m)^2$$

Integrating gives, for $\mathbf{P}$ homogeneous of degree $\mathbf{2d}$

$$\int_{\mathbb{R}_n} L_m P \ \Pi(x)^{2m} e^{-|x|^2/2} d_{x_1}\cdots d_{x_2} = 2d\int_{\mathbb{R}_n} P \ \Pi(x)^{2m} e^{-|x|^2/2} d_{x_1}\cdots d_{x_n}$$

and by iteration

$$L_m^d P \int_{\mathbb{R}_n} \Pi(x)^{2m} e^{-|x|^2/2} d_{x_1}\cdots d_{x_2} = 2^d d! \int_{\mathbb{R}_n} P \ \Pi(x)^{2m} e^{-|x|^2/2} d_{x_1}\cdots d_{x_n}$$

setting $\mathbf{P} = \Pi(x)^2$ and $\mathbf{d} = \binom{n}{2}$

$$\frac{1}{2^d d!} L_m^d \Pi(x)^2 \ c_m = c_{m+1}$$

A DEAD END?

Zeilberger notes that the above argument shows that the Metha identity is **equivalent** to the (**simple ?**) identity

$$\mathbf{\frac{1}{2^d d!} L_m^d \Pi(x)^2 \;=\; \frac{c_{m+1}}{c_m} \;=\; n! \prod_{1\le i<j\le n}^{n} (jm+i)} \tag{$*$}$$

then offers 25$ for a simple proof of (*): I am not aware that there were any takers.

Nevertheless it turns out that (*) is a windfall for us here

For, it develops that (*) permits the reduction of the Dunk-lized Vandermonde identity to the Metha identity. In fact, we can prove that

$$\mathbf{\frac{1}{2^d d!} L_m^d \Pi(x)^2 \;=\; \Pi\big(\nabla(m)\big)\Pi(x)}$$

Some useful operator identities

For $\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$ linear operators on a vector space $\mathbf{V}$ set

$$\mathbf{D_A B} \;=\; \mathbf{AB} - \mathbf{BA}$$

then

$$\mathbf{D_A(BC)} \;=\; \mathbf{(D_A B)C} \;+\; \mathbf{B\,D_A C}$$

More generally

$$\mathbf{D_A^d B_1 B_2 \cdots B_k} \;=\; \sum_{\mathbf{p_1+p_2+\cdots+p_k=d}} \frac{\mathbf{d!}}{\mathbf{p_1!p_2!\cdots p_k!}} \mathbf{D_A^{p_1}B_1\, D_A^{p_2}B_2 \,\cdots D_A^{p_k}B_k}$$

and we also have

$$\mathbf{D_A^d B} \;=\; \sum_{\mathbf{r=0}}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{r}} (-\mathbf{1})^{\mathbf{r}} \mathbf{A^{d-r}\, B\, A^r}$$

Sheer Magics

Step 1: for $\mathbf{V} = \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}]$

$$\mathbf{D_{L_m} \underline{x}_i \;=\; 2\nabla_i(m)}$$

Step 2:

$$\mathbf{\frac{1}{a!2^a}\; D^a_{L_m} \underline{x}_i^a \;=\; \nabla_i(m)^a}$$

and since $\mathbf{L_m} = \nabla_\mathbf{1}(\mathbf{m})^\mathbf{2} + \cdots + \nabla_\mathbf{n}(\mathbf{m})^\mathbf{2}$ and $\nabla_\mathbf{i}(\mathbf{m})$ commute

$$\mathbf{D^b_{L_m} \underline{x}_i^a \;=\; 0} \qquad \forall \quad \mathbf{b > a}$$

Theorem

For all polynomials $\mathbf{P}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ homogeneous of degree $\mathbf{d}$

$$\mathbf{P\big(\nabla(m)\big) \;=\; \frac{1}{2^d d!} D^d_{L_m} \underline{P} \;=\; \sum_{r=0}^{d} \binom{d}{r} (-1)^r L_m^{d-r}\, \underline{P}\, L_m^r} \qquad (*)$$

Proof

$$\mathbf{D^d_{L_m}\, \underline{x}_1^{p_1} \cdots \underline{x}_n^{p_n} \;=\; \sum_{a_1+\cdots+a_n=d} \frac{d!}{a_1!\cdots a_n!} D^{a_1}_{L_m} \underline{x}_1^{p_1} \;\cdots D^{a_n}_{L_m} \underline{x}_n^{p_n}}$$

but (*) and $\mathbf{d} = \mathbf{p_1} + \mathbf{p_2} + \cdots + \mathbf{p_n}$ gives

$$\mathbf{\frac{1}{2^d d!} D^d_{L_m}\, \underline{x}_1^{p_1} \cdots \underline{x}_n^{p_n} \;=\; \frac{1}{2^{p_1} p_1! \cdots 2^{p_n} p_n!} D^{p_1}_{L_m} \underline{x}_1^{p_1} \;\cdots D^{p_n}_{L_m} \underline{x}_n^{p_n} \;=\; \nabla_1(m)^{p_1} \cdots \nabla_n(m)^{p_n}}$$

Punch Line

From the previous result

$$\mathbf{\Pi}\big(\nabla(\mathbf{m})\big) \;=\; \frac{\mathbf{1}}{\mathbf{2^d d!}}\mathbf{D}^{\mathbf{d}}_{\mathbf{L_m}}\mathbf{\Pi}(\underline{\mathbf{x}})$$

Thus

$$\mathbf{\Pi}\big(\nabla(\mathbf{m})\big)\mathbf{\Pi}(\mathbf{x}) \;=\; \frac{\mathbf{1}}{\mathbf{2^d d!}}\sum_{\mathbf{r=0}}^{\mathbf{d}}\binom{\mathbf{d}}{\mathbf{r}}(-\mathbf{1})^{\mathbf{r}}\mathbf{L}_{\mathbf{m}}^{\mathbf{d-r}}\;\mathbf{\Pi}(\underline{\mathbf{x}})\;\mathbf{L}_{\mathbf{m}}^{\mathbf{r}}\mathbf{\Pi}(\mathbf{x}) \qquad (*)$$

But for $\mathbf{r} > \mathbf{0}$

$$\mathbf{L}_{\mathbf{m}}^{\mathbf{r}}\mathbf{\Pi}(\mathbf{x}) \;=\; \mathbf{0}$$

and so (*) is none other than

$$\mathbf{\Pi}\big(\nabla(\mathbf{m})\big)\mathbf{\Pi}(\mathbf{x}) \;=\; \frac{\mathbf{1}}{\mathbf{2^d d!}}\mathbf{L}_{\mathbf{m}}^{\mathbf{d}}\;\mathbf{\Pi}(\underline{\mathbf{x}})\mathbf{\Pi}(\mathbf{x})$$

End of Story!