

**SOME NEW METHODS**  
**IN THE THEORY**  
**OF**  
**m-QUASI-INVARIANTS**

by

**J. BELL, A. M. GARSIA and N. WALLACH**

( title of the paper submitted to the Stanley Festschrift)

**BUT**

**TODAY I WILL SPEAK ON**

**ON**

**SOME OLD METHODS**  
**IN THE THEORY**  
**OF**  
***m*-QUASI-INVARIANTS**

## **$m$ -QUASI-INVARIANTS**

Denote by “ $s_{i,j}$ ” the transposition that interchanges  $x_i$  and  $x_j$ .

For any  $P(\mathbf{x}) \in \mathbf{Q}[\mathbf{X}_n]$  for some integer  $r \geq 0$  we have

$$(1 - s_{ij})P(\mathbf{x}) = (x_i - x_j)^{2r+1}P_{ij}(\mathbf{x})$$

With  $\mathbf{X}_n = \{x_1, x_2, \dots, x_n\}$  we set

$$\mathbf{QI}_m[\mathbf{X}_n] = \left\{ P(\mathbf{x}) \in \mathbf{Q}[\mathbf{X}_n] : (1 - s_{ij})P(\mathbf{x}) = (x_i - x_j)^{2m+1}P_{ij}(\mathbf{x}) \quad \forall \quad 1 \leq i < j \leq n \right\}$$

**$\mathbf{QI}_m[\mathbf{X}_n]$  is an  $S_n$ -module**

$$\mathbf{Q}[\mathbf{X}_n] = \mathbf{QI}_0[\mathbf{X}_n] \supset \mathbf{QI}_1[\mathbf{X}_n] \supset \mathbf{QI}_2[\mathbf{X}_n] \supset \cdots \supset \mathbf{QI}_m[\mathbf{X}_n] \supset \cdots \supset \mathbf{QI}_\infty[\mathbf{X}_n] = \mathbf{SYM}[\mathbf{X}_n]$$

$\mathbf{QI}_m[\mathbf{X}_n]$  is a ring

$$(1 - s_{ij})PQ = ((1 - s_{ij})P)Q + (s_{ij}P)(1 - s_{ij})Q$$

## SOME REMARKABLE RESULTS

**Theorem 1** (Etingof-Ginsburg )

$\mathbf{QI}_m[\mathbf{X}_n]$  is free over  $\mathbf{SYM}[\mathbf{X}_n]$

**Theorem 1** (Felder-Veselov )

*The quotient*

$$\mathbf{QI}_m[\mathbf{X}_n]/(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)\mathbf{QI}_m[\mathbf{X}_n]$$

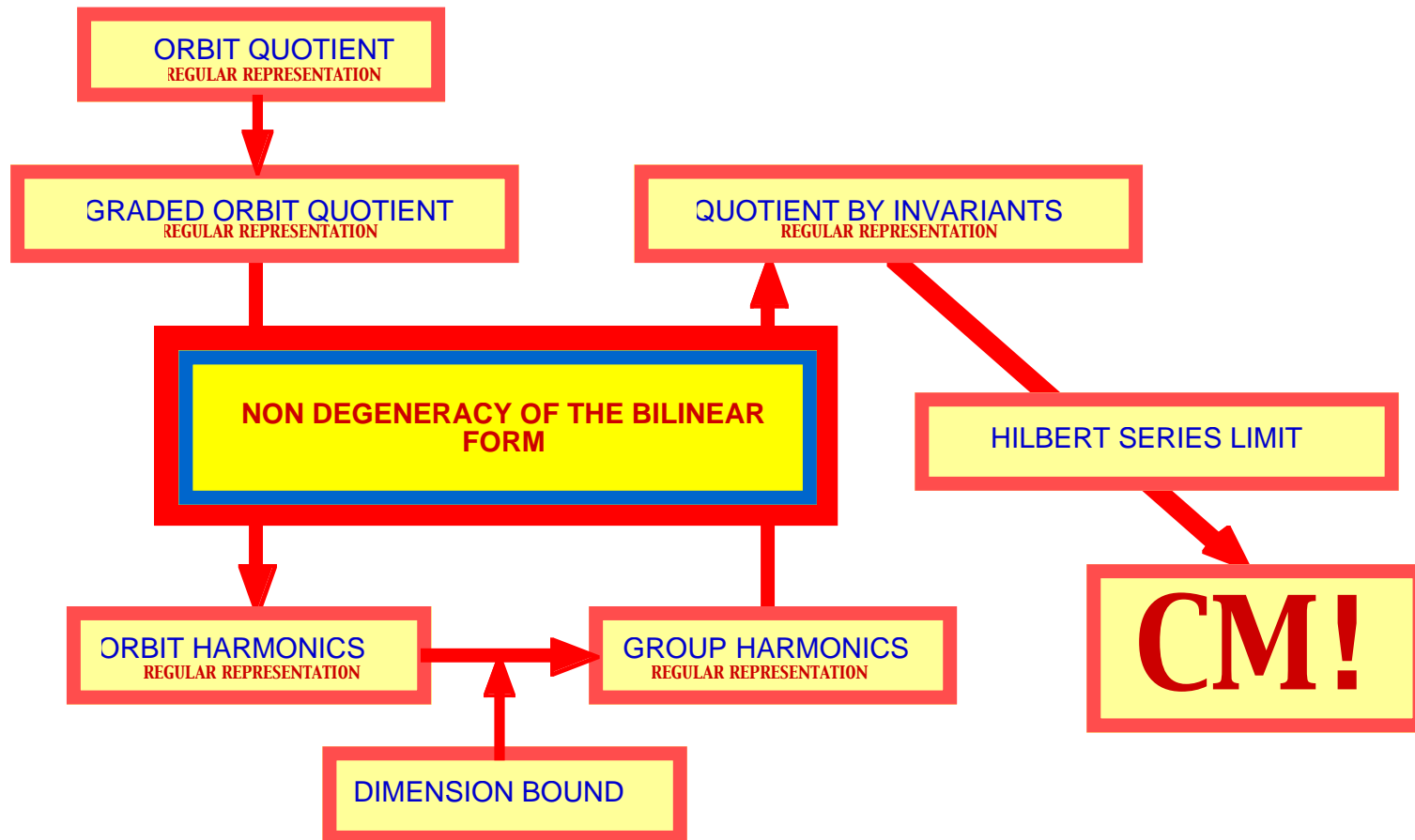
*is a graded version of the Left-Regular representation of  $\mathbf{S}_n$  with Frobenius characteristic*

$$\sum_{\lambda \vdash n} \mathbf{S}_\lambda(\mathbf{x}) \sum_{\mathbf{T} \in \mathbf{ST}(\lambda)} \mathbf{q}^{\mathbf{co}(\mathbf{T}) + \mathbf{m} \left( \binom{n}{2} - \mathbf{c}_\lambda \right)}$$

$$\mathbf{c}_\lambda = \sum_{(\mathbf{i}, \mathbf{j}) \in \lambda} (\mathbf{j} - \mathbf{i}) = \mathbf{n}_\lambda - \mathbf{n}_{\lambda'}$$

(There are  $\mathbf{m}$ -analogs of everything in sight!)

# THE PROOF OF COHEN MACAULINESS for *m*-QUASI-INVARIANTS



## THE ORBIT QUOTIENT

We select a regular point  $a = (a_1, a_2, \dots, a_n)$  and define the “*orbit of a*”

$$[a] = \{a\sigma : \sigma \in S_n\}$$

then define the “*ideal of the orbit of a*”

$$J_{[a]}(\mathbf{m}) = \left( \mathbf{P} \in \mathbf{QI}_{\mathbf{m}}[X_n] : \mathbf{P}(\mathbf{b}) = 0 \quad \forall \mathbf{b} \in [a] \right)_{\mathbf{QI}_{\mathbf{m}}[X_n]}$$

then define the “*Ring of the orbit of a*”

$$\mathbf{R}_{[a]}(\mathbf{m}) = \mathbf{QI}_{\mathbf{m}}[X_n] / J_{[a]}(\mathbf{m})$$

**Theorem**(easy)

$\mathbf{R}_{[a]}(\mathbf{m})$  has dimension  $n!$  and carries the left regular representation of  $S_n$ .

**Proof**

To get a basis of  $\mathbf{R}_{[a]}(\mathbf{m})$ , for each  $\mathbf{b} \in [a]$  we construct a polynomial

$$\phi_{\mathbf{b}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{b} \\ 0 & \text{if } \mathbf{x} \neq \mathbf{b} \end{cases} \quad \text{for } \mathbf{x} \in [a]$$

Then  $S_n$  acts on this basis precisely as it acts on itself.

## THE GRADED ORBIT QUOTIENT

For a polynomial  $\mathbf{P}$  define  $\mathbf{h}(\mathbf{P})$  to be the homogeneous component of highest degree in  $\mathbf{P}$  then set

$$\mathbf{gr} \mathbf{J}_{[\mathbf{a}]} = \left( \mathbf{h}(\mathbf{P}) : \mathbf{P} \in \mathbf{J}_{[\mathbf{a}]}(\mathbf{m}) \right)_{\mathbf{QI}_{\mathbf{m}}[\mathbf{X}_n]}$$

and define “*graded Ring of the orbit of  $a$* ”

$$\mathbf{gr} \mathbf{R}_{[\mathbf{a}]}(\mathbf{m}) = \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_n] / \mathbf{gr} \mathbf{J}_{[\mathbf{a}]}(\mathbf{m})$$

**Theorem**(easy)

$\mathbf{gr} \mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$  has dimension  $\mathbf{n}!$  and carries the left regular representation of  $\mathbf{S}_n$ .

**Proof**

A standard argument transfers properties from  $\mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$  to  $\mathbf{gr} \mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$

## THE BILINEAR FORM

**Proposition**(easy)

For any homogeneous polynomial  $q(\mathbf{x})$  of degree  $d$  we have

$$q(\partial_{\mathbf{x}}) = \frac{1}{2^d d!} \sum_{k=0}^d \binom{d}{k} \Delta^{d-k} q(\mathbf{x}) \Delta^k$$

Define

$$\mathbf{L}_m = \Delta - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j})$$

and for each  $q \in \mathbf{QI}_m[\mathbf{X}_n]$ , homogeneous of degree  $d$  set

$$\gamma_q(\mathbf{x}, \partial_{\mathbf{x}}) = \frac{1}{2^d d!} \sum_{k=0}^d \binom{d}{k} \mathbf{L}_m^{d-k} q(\mathbf{x}) \mathbf{L}_m^k$$

then extend to all of  $\mathbf{QI}_m[\mathbf{X}_n]$  by linearity.

**Theorem** (very hard)

*The map  $q \rightarrow \gamma_q$  is a ring isomorphism from  $\mathbf{QI}_m[\mathbf{X}_n]$  to the ring of operators that commute with  $\mathbf{L}_m$ . Moreover if for  $p, q \in \mathbf{QI}_m[\mathbf{X}_n]$  we set*

$$\langle p, q \rangle_m = \gamma_p q \Big|_{\mathbf{x}=0}$$

*we get a non-degenerate, symmetric, bilinear form on  $\mathbf{QI}_m[\mathbf{X}_n]$*

## ORBIT HARMONICS

Using the bilinear form define “*the  $\mathfrak{m}$ -Harmonics of  $[\mathfrak{a}]$* ” by

$$\mathbf{H}_{[\mathfrak{a}]}(\mathfrak{m}) = \{ \mathbf{P} \in \mathbf{QI}_{\mathfrak{m}}[\mathbf{X}_n] : \langle \mathfrak{q}, \mathbf{P} \rangle_{\mathfrak{m}} = 0 \quad \forall \mathfrak{q} \in \mathbf{gr} \mathbf{J}_{[\mathfrak{a}]} \}$$

### Theorem

$\mathbf{H}_{[\mathfrak{a}]}(\mathfrak{m})$  has dimension  $n!$  and carries the left regular representation of  $\mathbf{S}_n$ .

### Proof

Using the non degeneracy of the bilinear form we can transfer properties from  $\mathbf{gr} \mathbf{R}_{[\mathfrak{a}]}$  to  $\mathbf{H}_{[\mathfrak{a}]}(\mathfrak{m})$

## THE GROUP HARMONICS

Let  $\mathbf{J}_{\mathbf{S}_n}$  be the ideal

$$\mathbf{J}_{\mathbf{S}_n} = \left( \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \right)_{\mathbf{QI}_{\mathfrak{m}}[\mathbf{X}_n]}$$

then set

$$\mathbf{H}_{\mathbf{S}_n}(\mathfrak{m}) = \{ \mathbf{P} \in \mathbf{QI}_{\mathfrak{m}}[\mathbf{X}_n] : \langle \mathfrak{q}, \mathbf{P} \rangle_{\mathfrak{m}} = 0 \quad \forall \mathfrak{q} \in \mathbf{J}_{\mathbf{G}} \}$$

## THE CRUCIAL EQUALITY

**Proposition 1**

*For both  $\mathbf{H}_{[\mathbf{a}]}(\mathbf{m})$  and  $\mathbf{H}_{\mathbf{S}_n}(\mathbf{m})$  we have*

$$\mathbf{H}_{[\mathbf{a}]}(\mathbf{m}) = \{ \mathbf{P} \in \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_n] : \gamma_{\mathbf{q}}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{P} = 0 \quad \forall \mathbf{q} \in \mathbf{gr} \mathbf{J}_{[\mathbf{a}]} \}$$

$$\mathbf{H}_{\mathbf{S}_n}(\mathbf{m}) = \{ \mathbf{P} \in \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_n] : \gamma_{\mathbf{e}_i}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{P} = 0 \quad \forall i = 1, 2, \dots, n \}$$

**Proof**

Immediate application of the non-degeneracy of the form.

**Proposition 2**

$$\mathbf{H}_{[\mathbf{a}]}(\mathbf{m}) \subseteq \mathbf{H}_{\mathbf{S}_n}(\mathbf{m}) \tag{*}$$

**Proof**

Since  $\mathbf{e}_i(\mathbf{x}) - \mathbf{e}_i(\mathbf{a}) \in \mathbf{J}_{[\mathbf{a}]}$  then  $\mathbf{e}_i(\mathbf{x}) \in \mathbf{gr} \mathbf{J}_{[\mathbf{a}]}$  and (\*) follows from Proposition 1.

## THE ROLE OF THE DIMENSION BOUND

**Theorem**

$$\dim \mathbf{H}_{\mathbf{S}_n}(\mathbf{m}) \leq n! \implies \mathbf{H}_{[\mathbf{a}]}(\mathbf{m}) = \mathbf{H}_{\mathbf{S}_n}(\mathbf{m})$$

*and thus  $\mathbf{H}_{\mathbf{S}_n}(\mathbf{m})$  carries the left regular representation*

## Two simple observations

Recall

$$q(\partial_x) = \frac{1}{2^d d!} \sum_{k=0}^d \binom{d}{k} \Delta^{d-k} q(x) \Delta^k \quad \gamma_q(x, \partial_x) = \frac{1}{2^d d!} \sum_{k=0}^d \binom{d}{k} L_m^{d-k} q(x) L_m^k$$

$$L_m = \Delta - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j})$$

It follows that

$$\gamma_q(x, \partial_x) = q(\partial_x) + \sum_{|\mathbf{p}| < \text{degree}(q)} c_{\mathbf{p}}(x) \partial_x^{\mathbf{p}}$$

with  $c_{\mathbf{p}}(x)$  having only the factors  $x_i - x_j$  in the denominators.

**NOTE:**

- (1)  $c_{\mathbf{p}}(x)$  is OK as long as  $x$  is not in one of the reflecting hyperplanes
- (2) the difference  $\gamma_q(x, \partial_x) - q(\partial_x)$  is of lower order than  $q(\partial_x)$

## THE DIMENSION BOUND

**Theorem** (Feigin-Veselov)

$$\dim \mathbf{H}_{\mathbf{S}_n}(\mathbf{m}) \leq n!$$

**Proof**

Classical ( $\mathbf{m} = \mathbf{0}$ ) Harmonic theory gives that for every monomial  $\mathbf{x}^{\mathbf{p}}$  we have

$$\mathbf{x}^{\mathbf{p}} = \sum_{\mathbf{0} \leq \boldsymbol{\epsilon}_i \leq \mathbf{i}-\mathbf{1}} \mathbf{x}^{\boldsymbol{\epsilon}} \mathbf{A}_{\boldsymbol{\epsilon}}(\mathbf{x}) \quad (\text{with } \mathbf{A}_{\boldsymbol{\epsilon}} \in \mathbf{R}^{\mathbf{S}_n})$$

and for any  $\mathbf{q} \in \mathbf{H}_{\mathbf{S}_n}(\mathbf{m})$  and any  $\mathbf{x}^{\mathbf{p}}$  we can write

$$\partial_{\mathbf{x}}^{\mathbf{p}} \mathbf{q}(\mathbf{x}) = \sum_{\mathbf{0} \leq \boldsymbol{\epsilon}_i \leq \mathbf{i}-\mathbf{1}} \partial_{\mathbf{x}}^{\boldsymbol{\epsilon}} \gamma_{\mathbf{A}_{\boldsymbol{\epsilon}}}(\mathbf{x}, \partial_{\mathbf{x}}) \mathbf{q}(\mathbf{x}) + \sum_{\mathbf{0} \leq \boldsymbol{\epsilon}_i \leq \mathbf{i}-\mathbf{1}} \partial_{\mathbf{x}}^{\boldsymbol{\epsilon}} \left( \mathbf{A}_{\boldsymbol{\epsilon}}(\partial_{\mathbf{x}}) - \gamma_{\mathbf{A}}(\mathbf{x}, \partial_{\mathbf{x}}) \right) \mathbf{q}(\mathbf{x}) \quad (*)$$

Since

$$\mathbf{A}_{\boldsymbol{\epsilon}}(\mathbf{x}) = \mathbf{A}_{\boldsymbol{\epsilon}}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$$

it follows that for any  $\mathbf{q} \in \mathbf{H}_{\mathbf{S}_n}(\mathbf{m})$  we get

$$\gamma_{\mathbf{A}_{\boldsymbol{\epsilon}}}(\mathbf{x}, \partial_{\mathbf{x}}) \mathbf{q}(\mathbf{x}) = \mathbf{A}_{\boldsymbol{\epsilon}}(\mathbf{0}) \mathbf{q}(\mathbf{x})$$

so (\*) gives for a regular point  $\mathbf{x}_o$

$$\partial_{\mathbf{x}}^{\mathbf{p}} \mathbf{q}(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_o} = \sum_{\mathbf{0} \leq \boldsymbol{\epsilon}_i \leq \mathbf{i}-\mathbf{1}} \mathbf{A}_{\boldsymbol{\epsilon}}(\mathbf{0}) \partial_{\mathbf{x}}^{\boldsymbol{\epsilon}} \mathbf{q}(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_o} + \dots \text{(lower order derivatives of } \mathbf{q}(\mathbf{x}) \text{ at } \mathbf{x}_o)$$

and by induction we get that  $\mathbf{q}(\mathbf{x})$  is determined by  $n!$  derivatives at  $\mathbf{x}_o$ .

## THE HILBERT SERIES LIMIT

**Theorem** (known more or less)

*Let  $\mathbf{A}$  be a finitely generated graded algebra and let*

$$\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathbf{A}$$

*be homogeneous of degrees*

$$\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_k$$

*and suppose that*

$$\dim \mathbf{A}/(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k)_{\mathbf{A}} \leq \mathbf{N}$$

*with  $\mathbf{k}$  minimal. Then*

$$\lim_{q \rightarrow -1} (1 - q^{\mathbf{d}_1})(1 - q^{\mathbf{d}_2}) \cdots (1 - q^{\mathbf{d}_k}) \mathbf{F}_{\mathbf{A}} = \mathbf{N}$$

*Implies that  $\mathbf{A}$  is COHEN-MACAULAY*

## THE $m$ -QUASI-INVARIANT SANDWICH

### Theorem

*The Hilbert series of the Ring  $\mathbf{QI}_m[\mathbf{X}_n]$  of  $m$ -Quasi-Invariants satisfies the coefficient-wise inequalities*

$$\frac{q^{2m\binom{n}{2}}}{(1-q)^n} \ll \mathbf{F}_{\mathbf{QI}_m[\mathbf{X}_n]}(q) \ll \frac{1}{(1-q)^n}$$

*Thus*

$$\lim_{q \rightarrow -1} (1-q)(1-q^2) \cdots (1-q^n) \mathbf{F}_{\mathbf{QI}_m[\mathbf{X}_n]}(q) = n!$$

### Proof

Every polynomial  $\mathbf{P}(\mathbf{x}) \in \mathbb{Q}[\mathbf{X}_n]$  multiplied by

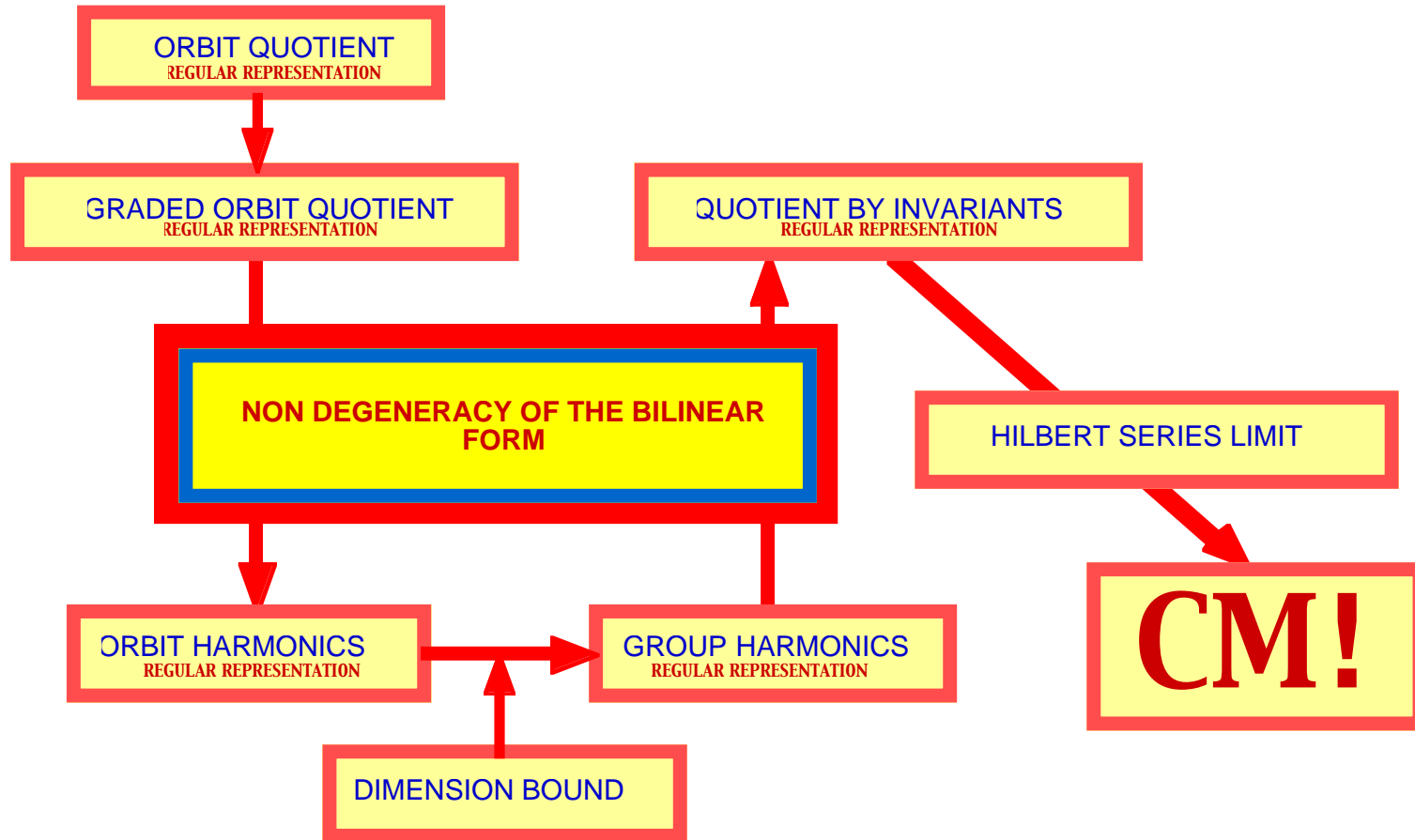
$$\Pi(\mathbf{x})^{2m} = \prod_{1 \leq r < s \leq n} (x_r - x_s)^{2m}$$

gives an  $m$ -Quasi-Invariant. In fact for all  $1 \leq i < j \leq n$  we have

$$(1 - s_{ij})\mathbf{P}(\mathbf{x})\Pi(\mathbf{x})^{2m} = \Pi(\mathbf{x})^{2m}(1 - s_{ij})\mathbf{P}(\mathbf{x})$$

**QED!**

# THE PROOF OF COHEN MACAULINESS for *m*-QUASI-INVARIANTS



## REFERENCES

- [1] Yu. Berest, P. Etingof and V. Ginzburg, *Cherednik algebras and differential operators on  $m$ -quasi-invariants*, math. QA/011005.
- [2] E. R. Berlekamp, *Algebraic Coding Theory*, Aegean Park Press (1984) pp. 21-30.
- [3] O. A. Chalykh and A. P. Veselov, *Commutative Rings of Partial Differential Operators and Lie Algebras*, Comm. in Math. Phys. **126** 597-611 (1990).
- [4] P. Etingof and V. Ginzburg, *On  $m$ -quasi-invariants of a Coxeter group*, arXiv:math.QA/0106175 v1 Jun 2001.
- [5] P. Etingof and E. Strickland, *Lectures on quasi-invariants of Coxeter groups and the Cherednik algebra*, arXiv:math.QA/0204104 v1 9 Apr 2002.
- [6] G. Felder and A. P. Veselov, *Action of Coxeter Groups on  $m$ -harmonic Polynomials and KZ equations* arXiv:math.QA/0108012 v2 3 Oct 2001.
- [7] M. Feigin and A. P. Veselov, *Quasi-invariants of Coxeter groups and  $m$ -harmonic polynomials*, arXiv:math-ph/0105014 v1 11 May 2001.
- [8] A. P. Veselov, K.L. Styrkas and O. A. Chalykh, *Algebraic integrability for the Schrödinger equation and finite reflection groups*, (translated from Teo. i Math. Fizika, Vol 94 No 2 pp 253-275)