New Methods

in

The Theory of m-Quasi-Invariants

(joint work with J. Bell and N. Wallach)

The divided difference operator

Set $\mathbf{X_n} = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$. For $\mathbf{P} \in \mathbb{Q}[\mathbf{X_n}]$ we will write $\mathbf{P}(\mathbf{x})$ for $\mathbf{P}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$.

Let $\mathbf{s_{ij}}$ denote the transposition which interchanges $\mathbf{x_i}$ with $\mathbf{x_j}$.

Proposition

The divided difference operator

$$\delta_{\mathbf{ij}} \quad = \quad \frac{\mathbf{1}}{\mathbf{x_i} - \mathbf{x_j}}(\mathbf{1} - \mathbf{s_{ij}}) \tag{1}$$

sends polynomials into polynomials and satisfies the "Leibnitz" identity

$$\delta_{\mathbf{ij}} \big(\mathbf{P}(\mathbf{x})\mathbf{Q}(\mathbf{x})\big) \quad = \quad \big(\delta_{\mathbf{ij}}\, \mathbf{P}(\mathbf{x})\big)\, \mathbf{Q}(\mathbf{x}) \quad + \quad \big(\mathbf{s_{ij}}\mathbf{P}(\mathbf{x})\big)\, \delta_{\mathbf{ij}}\, \mathbf{Q}(\mathbf{x}) \tag{2}$$

Proof

We trivially have

$$(\mathbf{1} - \mathbf{s_{ij}})\mathbf{PQ} \quad = \quad \mathbf{PQ} - (\mathbf{s_{ij}}\mathbf{P})(\mathbf{s_{ij}}\mathbf{Q}) \quad = \quad \big(\mathbf{P} - \mathbf{s_{ij}}\mathbf{P}\big)\mathbf{Q} \quad + \quad (\mathbf{s_{ij}}\mathbf{P})\big(\mathbf{Q} - \mathbf{s_{ij}}\mathbf{Q}\big)$$

and (2) follows upon dividing both sides by $\mathbf{x_i} - \mathbf{x_j}$

More identities

Setting

$$\mathbf{u = x_i + x_j}\ , \qquad \mathbf{v = x_i - x_j}$$

we can write

$$\mathbf{x_i} = \frac{\mathbf{u+v}}{\mathbf{2}}\ , \qquad \mathbf{x_j} = \frac{\mathbf{u-v}}{\mathbf{2}}$$

Thus any polynomial P has an expansion of the form

$$\mathbf{P} \;=\; \mathbf{a_{oo}} \;+\; \sum_{\mathbf{r+s\geq 1}} \mathbf{a_{rs}\, u^r\, v^s} \qquad \text{(with } \mathbf{a_{ij}} \text{ polynomials in the remaining variables)}$$

Since $\mathbf{s_{ij}\, u = u}$ *and* $\mathbf{s_{ij}\, v = -v}$ *it follows that*

$$\mathbf{(1 - s_{ij})P} \;=\; \sum_{\mathbf{s\geq 1}} \mathbf{a_{rs}\, u^r\, (1-(-1)^s) v^s} \;=\; \mathbf{2} \sum_{\mathbf{s\ odd}} \mathbf{a_{rs}\, u^r\, v^s}$$

Thus the highest power of $\mathbf{x_i - x_j}$ *that divides* $\mathbf{(1 - s_{ij})P}$ *is necessarily odd, in particular*

$$\frac{\mathbf{1}}{\mathbf{x_i - x_j}} \mathbf{(1 - s_{ij})P}$$

is a polynomial

The m-Quasi-invariants

$\mathbf{P}$ *is* $\mathbf{m}$*-quasi-invariant if, for all* $1 \le i < j \le n$, $(\mathbf{1} - \mathbf{s_{ij}})\mathbf{P}$ *is divisible by* $(\mathbf{x_i} - \mathbf{x_j})^{\mathbf{2m+1}}$

Some basic facts:

(1) *The space* $\mathbf{QI_m}[\mathbf{X_n}]$ *of* $\mathbf{m}$*-quasi-invariants in* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ *is an algebra*

(2) $\mathbb{Q}[\mathbf{X_n}] = \mathbf{QI_0}[\mathbf{X_n}] \supset \mathbf{QI_1}[\mathbf{X_n}] \supset \ldots \supset \mathbf{QI_m}[\mathbf{X_n}] \supset \cdots \supset \mathbf{QI_\infty}[\mathbf{X_n}] = \mathbf{SYM_n}$ (Symmetric polynomials)

(3) *Each* $\mathbf{QI_m}[\mathbf{X_n}]$ *is an* $\mathbf{S_n}$ *module*

Proof

(1): $$(\mathbf{1} - \mathbf{s_{ij}})(\mathbf{PQ}) = \big(\mathbf{P} - \mathbf{s_{ij}P}\big)\mathbf{Q} + (\mathbf{s_{ij}P})\big(\mathbf{Q} - \mathbf{s_{ij}Q}\big)$$

(2): $(\mathbf{x_i} - \mathbf{x_j})^{\mathbf{2m+1}}$ *"divides"* $\mathbf{P} - \mathbf{s_{ij}P}$ *for all* $\mathbf{m}$ *if and only if* $\mathbf{P} - \mathbf{s_{ij}P} = \mathbf{0}$

(3): *If* $\sigma \in \mathbf{S_n}$ *then*

$$\begin{aligned}(\mathbf{1} - \mathbf{s_{ij}})\sigma^{-\mathbf{1}}\mathbf{P} &= \sigma^{-\mathbf{1}}\sigma(\mathbf{1} - \mathbf{s_{ij}})\sigma^{-\mathbf{1}}\mathbf{P} = \sigma^{-\mathbf{1}}(\mathbf{1} - \mathbf{s}_{\sigma_\mathbf{i}\sigma_\mathbf{j}})\mathbf{P} \\ &= \sigma^{-\mathbf{1}}(\mathbf{x}_{\sigma_\mathbf{i}} - \mathbf{x}_{\sigma_\mathbf{j}})^{\mathbf{2m+1}}\mathbf{Q} = (\mathbf{x_i} - \mathbf{x_j})^{\mathbf{2m+1}}\sigma^{-\mathbf{1}}\mathbf{Q}\end{aligned}$$

Further properties of m-Quasi-invariants

(1) $$\mathbf{QI_m[X_n]} = \mathcal{H}_0[\mathbf{QI_m}] \oplus \mathcal{H}_1[\mathbf{QI_m}] \oplus \mathcal{H}_2[\mathbf{QI_m}] \oplus \cdots \oplus \mathcal{H}_r[\mathbf{QI_m}] \oplus \cdots$$

where $\mathcal{H}_r[\mathbf{QI}_m]$ *denotes the subspace of homogeneous* **m**-*Quasi-invariants of degree* **r**.

(2) *Setting* $\mathbf{\Pi(x)} = \prod_{1\le i<j\le n}(\mathbf{x_i} - \mathbf{x_j})$ *we have*

$$\mathbf{\Pi(x)^{2m+1}} \in \mathbf{QI_m[X_n]}$$

(3) *More generally for every polynomial* $\mathbf{P} \in \mathbb{Q}[\mathbf{X_n}]$ *we have*

$$\mathbf{\Pi(x)^{2m}P(x)} \in \mathbf{QI_m[X_n]}$$

Proof

(1): *If* $\mathbf{P} = \sum_{r=0}^{k} \mathbf{P_r}$ *and* $\mathbf{Q} = \sum_{r=0}^{h} \mathbf{Q_r}$ *with* $\mathbf{P_r}, \mathbf{Q_r}$ *homogeneous of degree* **i** *then*

$$(\mathbf{1} - \mathbf{s_{ij}})\mathbf{P} = (\mathbf{x_i} - \mathbf{x_j})^{\mathbf{2m+1}}\mathbf{Q} \quad \Longrightarrow \quad \sum_{i=0}^{k}(\mathbf{1} - \mathbf{s_{ij}})\mathbf{P_r} = \sum_{r=0}^{h}(\mathbf{x_i} - \mathbf{x_j})^{\mathbf{2m+1}}\mathbf{Q_r}$$

(This also shows that the first $2m$ components of P must be symmetric)

(3): $(\mathbf{1} - \mathbf{s_{ij}})\mathbf{\Pi(x)^{2m}P(x)} = \mathbf{\Pi(x)^{2m}}(\mathbf{1} - \mathbf{s_{ij}})\mathbf{P(x)} = \mathbf{\Pi(x)^{2m}}(\mathbf{x_i} - \mathbf{x_j})\mathbf{Q(x)}$

Some deeper facts

Theorem (Etingof-Ginsburg, etc)

$\mathbf{QI_m[X_n]}$ *is free over* $\mathbf{SYM[X_n]}$ *and the quotient*

$$\mathbf{QI_m[X_n]/(e_1, e_2, \ldots, e_n)_{QI_m}}$$

is a graded version of the Regular representation of $\mathbf{S_n}$

Theorem (Felder-Veselov using a KZ connection) (KZ=Knizhnik-Zamolodchikov)

The Frobenius characteristic of $\mathbf{QI}_m[X)n]$ *as a graded* S_n*-module is*

$$\frac{\sum_{\lambda \vdash \mathbf{n}} \mathbf{S}_\lambda(\mathbf{x}) \sum_{\mathbf{T} \in \mathbf{ST}(\lambda)} \mathbf{q}^{\mathbf{co(T)+m}\left(\binom{\mathbf{n}}{\mathbf{2}} - \mathbf{c}_\lambda\right)}}{\mathbf{(1-q)(1-q^2)\cdots(1-q^n)}}$$

$$\mathbf{c}_\lambda = \sum_{(\mathbf{i},\mathbf{j}) \in \lambda} (\mathbf{j} - \mathbf{i})$$

(There are m*-analogs of everything in sight!)*

Classical Harmonics of S_n

$$H_n = \left\{ q(\partial_x)\Pi(x) : q \in R \right\} \qquad \text{with} \quad \Pi(x) = \prod_{1\le i<j\le n} (x_i - x_j)$$

m- Harmonics of S_n

Note if $q(x)$ *is homogeneous of degree* d

$$q(\partial_x) = \frac{1}{2^d d!}\sum_{r=0}^{d} \binom{d}{2} (-1)^r \Delta^{d-r} q(x) \Delta^r \qquad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$$

For $q \in QI_m[X_n]$ *set*

$$L_q(m) = \frac{1}{2^d d!}\sum_{r=0}^{d} \binom{d}{2} (-1)^r \Delta(m)^{d-r} q(x) \Delta(m)^r$$

with

$$\Delta(m) = \Delta - 2m \sum_{1\le i<j\le n} \frac{1}{x_i - x_j}\left(\partial_{x_i} - \partial_{x_j}\right)$$

Then

$$H_n(m) = \left\{ L_q(m)\Pi(x)^{2m+1} : q \in QI_m[X_n] \right\}$$

A bit on Finitely generated graded algebras

(1) $\mathbf{A} \; = \; \mathcal{H}_{\mathbf{0}}(\mathbf{A}) \oplus \mathcal{H}_{\mathbf{1}}(\mathbf{A}) \oplus \mathcal{H}_{\mathbf{2}}(\mathbf{A}) \oplus \mathcal{H}_{\mathbf{3}}(\mathbf{A}) \oplus \cdots$ *with*

$$\mathcal{H}_r(\mathbf{A}) \times \mathcal{H}_s(\mathbf{A}) \subseteq \mathcal{H}_{r+s}(\mathbf{A})$$

(2) *For some* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_m} \in \mathbf{A}$ *the collection* $\left\{\mathbf{a_1^{p_1} a_2^{p_2} \cdots a_m^{p_m}}\right\}$ *spans* $\mathbf{A}$.

The Hilbert Series

$$\mathbf{F_A(t)} \; = \; \sum_{\mathbf{r} \geq \mathbf{0}} \mathbf{t^r} \;\; \dim \mathcal{H}_{\mathbf{r}}(\mathbf{A})$$

Theorem

$\mathbf{F_A(t)}$ *is a rational function and if* $\mathcal{B}$ *is a homogeneous basis for* $\mathbf{A}$ *then*

$$\mathbf{F_A(t)} \; = \; \sum_{\mathbf{b} \in \mathcal{B}} \mathbf{t^{degree(b)}}$$

If $\mathbf{a_i}$ *is homogeneous of degree* $\mathbf{d_i}$ *then*

$$(\mathbf{2}): \quad \Longrightarrow \quad \mathbf{F_A(t)} << \frac{\mathbf{1}}{(\mathbf{1}-\mathbf{t^{d_1}})(\mathbf{1}-\mathbf{t^{d_2}})\cdots(\mathbf{1}-\mathbf{t^{d_m}})}$$

and thus $F_{\mathbf{A}}(q)$ *has a pole of order at most* $\mathbf{m}$ *at* $\mathbf{t} = \mathbf{1}$

The Krull dimension

Theorem

The following three integers are equal (with $\mathbf{b_i} \in \mathbf{A}$ *homogeneous)*

$\mathbf{n_1}$: *The order of the pole of* $F_{\mathbf{A}}(t)$ *at* $\mathbf{t} = \mathbf{1}$

$\mathbf{n_2}$: *The maximum* $\mathbf{m}$ *such that* $\left\{\mathbf{b_1^{p_1} b_2^{p_2} \cdots b_m^{p_m}}\right\}_{\mathbf{p_i \geq 0}}$ *is an independent set*

$\mathbf{n_3}$: *The minimum* $\mathbf{m}$ *such that* $\mathbf{d} = \dim\ \mathbf{A}/(\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m})_{\mathbf{A}} \ < \ \infty$ $(*)$

Set $\mathbf{m_A} = \mathbf{n_1} = \mathbf{n_2} = \mathbf{n_3}$

Note that if $\{\mathbf{Q_1}, \mathbf{Q_2}, \ldots, \mathbf{Q_d}\} \subseteq \mathbf{A}$ *is any basis for the quotient in (*) then (*) implies that the collection*

$$\left\{\mathbf{Q_i}\ \mathbf{b_1^{p_1} b_2^{p_2} \cdots b_m^{p_m}}\right\}_{\mathbf{1 \leq i \leq d\,;\, p_i \geq 0}} \qquad (**)$$

spans $\mathbf{A}$*. When* $\mathbf{m} = \mathbf{m_A}$ *we say that* $\{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}\}$ *is an HSOP*

HSOP="Homogeneous System Of Parameters"

Note that if (**) is a basis for $\mathbf{A}$ *then* $\mathbf{A}$ *is called "Cohen-Macaulay" and of course then* $\mathbf{A}$ *is a free module over* $\mathbb{Q}[\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}]$ *of rank* $\mathbf{d}$ *moreover if* $\mathbf{d_i} = \mathbf{degree}(\mathbf{b_i})$ *then*

$$\mathbf{F_A(q)} \ = \ \frac{\sum_{\mathbf{i=1}}^{\mathbf{m}} \mathbf{t^{degree(Q_i)}}}{(\mathbf{1} - \mathbf{t^{d_1}})(\mathbf{1} - \mathbf{t^{d_2}}) \cdots (\mathbf{1} - \mathbf{t^{d_n}})}$$

The Classic Example

$$\mathbf{R_n} = \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}]$$

with

$$\mathbf{F_{R_n}(t)} = \frac{\mathbf{1}}{(\mathbf{1}-\mathbf{t})^{\mathbf{n}}} = \frac{[\mathbf{n}]_{\mathbf{t}}!}{(\mathbf{1}-\mathbf{t})(\mathbf{1}-\mathbf{t^2})\cdots(\mathbf{1}-\mathbf{q^n})} \qquad [\mathbf{r}]_{\mathbf{t}} = \mathbf{1}+\mathbf{t}+\cdots+\mathbf{t^{r-1}} \qquad (*)$$

Theorem

The collection $\mathcal{ART}_{\mathbf{n}} = \left\{\mathbf{x^r} = \mathbf{x_2^{r_2} x_3^{r_3} \cdots x_n^{r_n}}\right\}$ *with* $\mathbf{0} \le \mathbf{r_i} \le \mathbf{i}-\mathbf{1}$ *spans the quotient*

$$\mathbf{R}_n/(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{R}}$$

with

$$\mathbf{e_k} = \sum_{\mathbf{1} \le \mathbf{i_1} < \mathbf{i_2}, \ldots < \mathbf{i_k} \le \mathbf{n}} \mathbf{x_{i_1} x_{i_2} \cdots x_{i_k}}$$

Then it follows from (*) that the collection

$$\left\{\mathbf{x^r}\, \mathbf{e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}}\right\}_{\mathbf{x^r} \in \mathcal{ART}_{\mathbf{n}}}$$

is a basis for $\mathbf{R}$.
Thus $\mathbf{R}$ *is a Cohen-Macaulay algebra, free over* $\mathbb{Q}[\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}] = \mathbf{SYM_n}$ *of rank* $\mathbf{n}!$

A few words about GORDAN ("Gröbner") bases

Theorem 1 *(Gordan: Liouville Journal 1900)*
Any collection of monomials in $\mathbf{x_1, x_2, \ldots, x_n}$ *has a finite number of minimal elements under the division partial order.*

For $\mathbf{P} \in \mathbf{R}$ denote by $\mathbf{h}(\mathbf{R})$ the homogeneous component of highest degre in $\mathbf{P}$ and let $\mathbf{lm}(\mathbf{P})$ be **dlex** largest monomial in $\mathbf{P}$ (leading monomial)

Theorem 2 *(Gordan: Liouville Journal 1900)*
If $\mathbf{J}$ *is any ideal in* $\mathbf{R}$ *and* $\mathbf{m_1, m_2, \ldots, m_k}$ *are the minimal elements of*

$$\mathcal{M}_{\mathbf{J}} = \{\mathbf{lm}(\mathbf{P}) \ : \ \mathbf{P} \in \mathbf{J}\,\}$$

and

$$\mathbf{m_1} = \mathbf{h}(\mathbf{f_1})\,,\ \mathbf{m_2} = \mathbf{h}(\mathbf{f_2})\,, \ldots,\ \mathbf{m_k} = \mathbf{h}(\mathbf{f_k})$$

then every $\mathbf{P} \in \mathbf{J}$ *may be expressed in the form*

$$\mathbf{P} = \mathbf{A_1 f_1} + \mathbf{A_2 f_2} + \cdots + \mathbf{A_k f_k} \qquad \big(\textbf{with } \mathbf{A_i} \in \mathbf{R} \textbf{ and } \textbf{degree}(\mathbf{A_i f_i}) \leq \textbf{degree}(\mathbf{P})\big)$$

Moreover the complement of $\mathcal{M}_{\mathbf{J}}$ *yields a monomial basis for the quotient* $\mathbf{R}/\mathbf{J}$

An Application

Theorem

For $\mathbf{J} = (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{R}}$ *we have*

$$\mathbf{min}\, \mathcal{M}_{\mathbf{J}} \;=\; \{\mathbf{x_1}, \mathbf{x_2^2}, \ldots, \mathbf{x_n^n}\}$$

Proof

The "dlex" GORDAN basis of the ideal $(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{R}}$ is simply given by the complete homogeneous symmetric functions

$$\mathbf{h_k}(\mathbf{x_k}, \mathbf{x_{k+1}}, \ldots, \mathbf{x_n}) \;=\; \mathbf{x_k^k} \;+\; \sum_{\mathbf{r=0}}^{\mathbf{k-1}} \mathbf{x_k^r}\, \mathbf{h_{k-r}}(\mathbf{x_{k+1}}, \ldots, \mathbf{x_n}) \qquad (\text{for } \mathbf{1} \leq \mathbf{k} \leq \mathbf{n})$$

It thus follows from the Gordan Theorem that the collection

$$\mathcal{ART}_{\mathbf{n}} \;=\; \left\{\mathbf{x_2^{r_2} x_3^{r_3} \cdots x_n^{r_n}}\right\} \qquad (\text{with } \mathbf{0} \leq \mathbf{r_i} \leq \mathbf{i} - \mathbf{1})$$

is a basis for the quotient

$$\mathbf{R}/(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})_{\mathbf{R}}$$

More on m-Quasi-Invariants

It follows from the classic example that every polynomial $\mathbf{P} \in \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}]$ *has an expansion of the form*

$$\mathbf{P}(\mathbf{x}) \;=\; \sum_{\mathbf{b} \in \mathcal{ART}_\mathbf{n}} \mathbf{b}(\mathbf{x})\, \mathbf{A_b}(\mathbf{x})$$

where the coefficients $\mathbf{A_b}$ *are symmetric polynomials uniquely determined by* P.

Example: *For* $\mathbf{n} = \mathbf{3}$ *any* $\mathbf{P} \in \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]$ *may be uniquely expressed in the form*

$$\mathbf{P} \;=\; \mathbf{A_{000}} \;+\; \mathbf{A_{010}}\,\mathbf{x_2} \;+\; \mathbf{A_{001}}\mathbf{x_3} \;+\; \mathbf{A_{002}}\,\mathbf{x_3^2} \;+\; \mathbf{A_{011}}\,\mathbf{x_2}\mathbf{x_3} \;+\; \mathbf{A_{012}}\,\mathbf{x_2}\mathbf{x_3^2}$$

Thus we may represent each $\mathbf{P} \in \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]$ *by a sixtuple of symmetric polynomials*

$$\mathbf{P} \iff \left(\mathbf{A_{000}}\,,\; \mathbf{A_{010}}\,,\; \mathbf{A_{001}}\,,\; \mathbf{A_{002}}\,,\; \mathbf{A_{011}}\,,\; \mathbf{A_{012}}\right)$$

Theorem

The algebra $\mathbf{QI}_m[\mathbf{X_n}]$ *of* **m**-*Quasi-Invariants is finitely generated*

Proof

It easily follows from the Hilbert basis theorem (or constructively from the Gordan Theorem) that a module of $\mathbf{N}^{\mathbf{tuples}}$ over a polynomial ring is finitely generated. (use induction on $\mathbf{N}$ and apply the result to the ring $\mathbb{Q}[\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}]$.

A not so well known CM criterion

Theorem

Let $\mathbf{A}$ *be a finitely generated graded algebra and let* $\mathbf{b_1, b_2, \ldots, b_k}$ *be an HSOP of* $\mathbf{A}$ *and let* $\mathbf{d_i} = \mathbf{degree}(\mathbf{b_i})$. *Suppose that*

$$\mathbf{a)} \qquad \lim_{\mathbf{q} \to \ -\mathbf{1}} (\mathbf{1} - \mathbf{q}^{\mathbf{d_1}})(\mathbf{1} - \mathbf{q}^{\mathbf{d_2}}) \cdots (\mathbf{1} - \mathbf{q}^{\mathbf{d_k}}) \mathbf{F_A}(\mathbf{q}) \ = \ \mathbf{N}$$

and

$$\mathbf{b)} \qquad \mathbf{dim A}/(\mathbf{b_1, b_2, \ldots, b_k})_{\mathbf{A}} \ = \ \mathbf{N}$$

then $\mathbf{A}$ *is free over* $\mathbb{Q}[\mathbf{b_1, b_2, \ldots, b_k}]$ *of rank* $\mathbf{N}$ *and thus a Cohen-Macaulay algebra In particular if* $\{\mathbf{Q_1, Q_2, \ldots, Q_N}\} \subseteq \mathbf{A}$ *is a homogeneous basis for* $\mathbf{A}/(\mathbf{b_1, b_2, \ldots, b_k})_{\mathbf{A}}$ *then*

$$\mathbf{F_A}(\mathbf{q}) \ = \ \frac{\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{q}^{\mathbf{degree}(\mathbf{Q_i}}}{(\mathbf{1} - \mathbf{q}^{\mathbf{d_1}})(\mathbf{1} - \mathbf{q}^{\mathbf{d_2}}) \cdots (\mathbf{1} - \mathbf{q}^{\mathbf{d_k}})}$$

Back to $\mathbf{m}$-Quasi-Invariants

Corollary

The algebra $\mathbf{QI_m}$ *of* $\mathbf{m}$*-Quasi-Invariants in* $\mathbf{x_1, x_2, \ldots, x_n}$ *is a free module over* $\mathbb{Q}[\mathbf{e_1, e_2, \ldots e_n}]$ *of rank* $\mathbf{n}!$

Proof

It can be shown that

$$\mathbf{dim QI_m/(e_1, e_2, \ldots e_n)_{QI_m}} \;=\; \mathbf{n}!$$

On the other hand the inclusions

$$\mathbf{\Pi_n(x)^{2m+1}}\mathbb{Q}[\mathbf{x_1, x_2, \ldots, x_n}] \;\subseteq\; \mathbf{QI_m} \;\subseteq\; \mathbb{Q}[\mathbf{x_1, x_2, \ldots, x_n}]$$

imply the component wise Hilbert series inequalities

$$\frac{\mathbf{q}^{(\mathbf{2m+1})\binom{\mathbf{n}}{\mathbf{2}}}}{(\mathbf{1-q})^{\mathbf{n}}} \;<<\; \mathbf{F_{QI_m}(q)} \;<<\; \frac{\mathbf{1}}{(\mathbf{1-q})^{\mathbf{n}}}$$

Thus

$$\mathbf{lim_{q\to\ -1}(1-q)(1-q^2)\cdots(1-q^n)F_A(q)} \;=\; \mathbf{n}!$$

On $\mathbf{m}$-Quasi-Invariants of $\mathbf{S_3}$

Theorem

The polynomial

$$\mathbf{P_1} = \mathbf{A}\,(\mathbf{x_2} + \mathbf{x_3}) + \mathbf{B}\,\mathbf{x_2 x_3}$$

with A, B symmetric, is m-quasi-invariant if and if only we have

$$\textbf{a)}\quad \mathbf{A} = -\delta_{\mathbf{12}}\mathbf{x_1}(\mathbf{x_1} - \mathbf{x_3})^{\mathbf{2m}}\,\mathbf{M}(\mathbf{x}) \qquad\qquad \mathbf{B} = \delta_{\mathbf{12}}(\mathbf{x_1} - \mathbf{x_3})^{\mathbf{2m}}\,\mathbf{M}(\mathbf{x})$$

where $\mathbf{M}$ *is any polynomial that satisfies the two conditions* (∗)

$$\textbf{a)}\quad \mathbf{s_{13}}\,\mathbf{M}(\mathbf{x}) \;=\; \mathbf{M}(\mathbf{x})\,, \qquad \textbf{b)}\quad \delta_{\mathbf{23}}\delta_{\mathbf{12}}(\mathbf{x_1} - \mathbf{x_3})^{\mathbf{2m}}\mathbf{M}(\mathbf{x}) \;=\; \mathbf{0}\,,$$

Problem

Construct these mysterious polynomials $\mathbf{M}(\mathbf{x})$

The reduction to a Gordan basis problem

$$\begin{aligned} \mathbf{a}) \quad & \delta_{13}\delta_{12}(\mathbf{x_2} - \mathbf{x_3})^{\mathbf{2m}}\,\mathbf{M}(\mathbf{x}) \;=\; \mathbf{0}\,, \\ \mathbf{b}) \quad & \mathbf{s_{23}}\,\mathbf{M}(\mathbf{x}) \;=\; \mathbf{M}(\mathbf{x})\,. \end{aligned} \qquad (*)$$

Now b) implies that

$$\mathbf{M}(\mathbf{x}) \;=\; \mathbf{a} + \mathbf{b}(\mathbf{x_2} + \mathbf{x_3}) \;+\; \mathbf{c}\,\mathbf{x_2}\mathbf{x_3} \qquad (\textbf{with } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{SYM_3})$$

and we also have

$$(\mathbf{x_2} - \mathbf{x_3})^{\mathbf{2m}} \;=\; \mathbf{A_m} + \mathbf{B_m}(\mathbf{x_2} + \mathbf{x_3}) \;+\; \mathbf{C_m}\,\mathbf{x_2}\mathbf{x_3} \qquad (\textbf{with } \mathbf{A_m}, \mathbf{B_m}, \mathbf{C_m} \in \mathbf{SYM_3})$$

Now it turns out that a) simply requires that

$$\mathbf{c}\,\overline{\mathbf{A}}_{\mathbf{m}} \;+\; \mathbf{b}\,\overline{\mathbf{B}}_{\mathbf{m}} \;+\; \mathbf{a}\,\overline{\mathbf{C}}_{\mathbf{m}} \;=\; \mathbf{0}$$

with

$$\overline{\mathbf{A}}_{\mathbf{m}} \;=\; \mathbf{A_m} + \mathbf{B_m}\mathbf{e_1} + \mathbf{C_m}\mathbf{e_2}\,, \qquad \overline{\mathbf{B}}_{\mathbf{m}} \;=\; \mathbf{B_m} + \mathbf{C_m}\mathbf{e_1}\,, \qquad \overline{\mathbf{C}}_{\mathbf{m}} \;=\; \mathbf{C_m}$$

Theorem 3.3

The space of triplets of symmetric functions

$$\mathcal{M}_m(e) \;=\; \Big\{(a,b,c) \,:\, c\,\overline{A}_m \;+\; b\,\overline{B}_m \;+\; a\,\overline{C}_m \;=\; 0\Big\}$$

is a free $\mathbb{Q}[e_1,e_2,e_3]$-module of rank 2. With Hilbert series

$$F_{\mathcal{M}_m(e)}(t) \;=\; \frac{t^m + t^{m+1}}{(1-t)(1-t^2)(1-t^3)} \qquad (*)$$

“Proof”

(*) Follows from the two identities

$$t^{2m-2}F_{\mathcal{M}_m(e)}(t) \;=\; F_{(A_m,B_m,C_M)}(t) \;+\; \frac{t^{2m-2}+t^{2m-1}+t^{2m}}{(1-t)(1-t^2)(1-t^3)}$$

$$F_{(A_m,B_m,C_M)}(t) \;=\; \frac{(1+t^m+t^{m-1})(1-t^m)(1-t^{m-1})}{(1-t)(1-t^2)(1-t^3)}$$