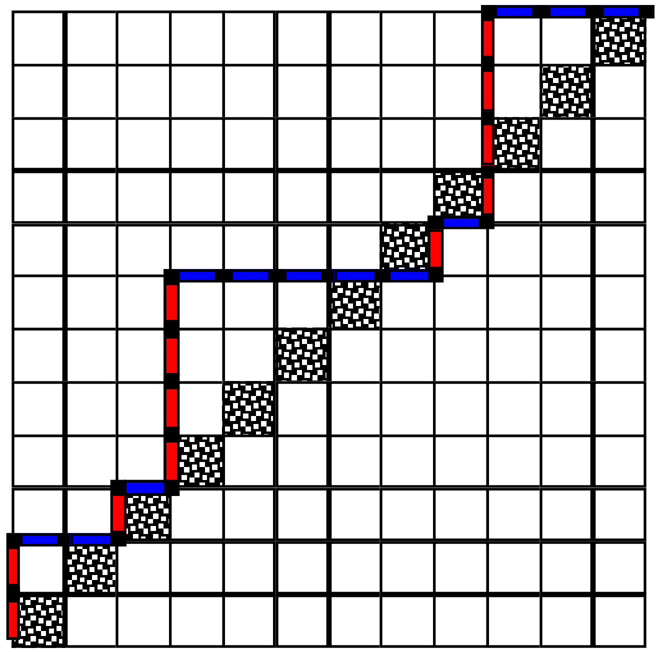
SAY RED

Introduction

This is a card game. The deck is shuffled and the cards are shown to the player one at a time. The player is to guess when the next card is red. The player says **"PASS!"** when not ready to make a guess and says **"RED!"** when he/she feels that next card has a good probability of being red. Then the player wins if the next card is "red" and loses if the next card is "black" or if the cards run out before a guess is made. Since there are an equal number of red and black cards in an ordinary deck, the player has a 50% probability of winning by saying **"RED!"** before the first card is shown. We may call this the "*Trivial*" strategy. The question is whether there is a strategy that assures a winning probability that is higher than 50%. A clever player decides to say **"PASS!"** before the first card is shown and continue to say **"PASS!"** as long as the number of "red" cards that have been shown is larger than the number of "black" cards, then say **"RED!"** the first time the number of "black" cards exceeds the number of "red" cards. This player is confident to have improved upon the Trivial strategy since, at that time, the cards that remain to be shown have one more red card than black cards and the probability of the next card to be "red" is higher than 50%. Note that under this strategy a **"RED!"** call is never made only when the number of "red" cards exceedes the number black cards at every step preceding the showing of the last card. The player is confident that this is an event of low probability. Infact, the theory assures that with a deck of $2n$ cards the probability of this event is $1/(n+1)$. In particular, with a 52 card deck this probability is less than 1.8%. Is this Player correct in assuming to have beaten the Trivial strategy? To see the answer read on.

1. Paths and Say Red.

Since the card values have no effect on the outcome of Say Red, we may assume with out loss that the cards only show the color. Moreover, to make things more colorful let us say that we have n "red" cards and n "blue" cards. It develops, that our analysis is greatly simplified by representing a shuffled deck by a lattice path in the plane. With this imagery a red card is represented by a NORTH step and a blue card by an EAST step. The figure below displays one such representation.



In this example the deck has 12 red cards and 12 blue cards. Following the path we get the sequence of colors

$$\mathbf{RRBBRBRRRRBBBBB}\cdots \tag{1}$$

Note that, if we use this deck, at each of the first 13 steps there are more red "cards" that blue "cards" shown. After the 14^{th} step 7 red cards and 7 blue cards have been shown. We should note that equality of the number of red and blue cards occurs every time an EAST step of the path goes over one of the diagonal squares (the shaded ones). Following the player's strategy, requires that we say **PASS** before each of the first 15 showings. However at the 15^{th} step, a blue card appears. At this point we have seen in total 8 blue cards and 7 red cards. The player's strategy requires that we say **RED!**. The player wins in this case, for the next card is red and the path takes a NORTH step.

Of course, the path could have proceeded as in the next figure where the initial sequence of colors is precisely as indicated as in (1), but the 16^{th} card is blue and the Player loses. In the next figure we have also depicted a 3×4 box which will necessarily contain the tail end of any path which starts with the same initial 15 steps.

We should mention that the number of our paths that are contained in an $a \times b$ box is given by the so called "*binomial coefficient*" $\binom{a+b}{a}$. This number is given by the formula

$$\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!}, \tag{2}$$

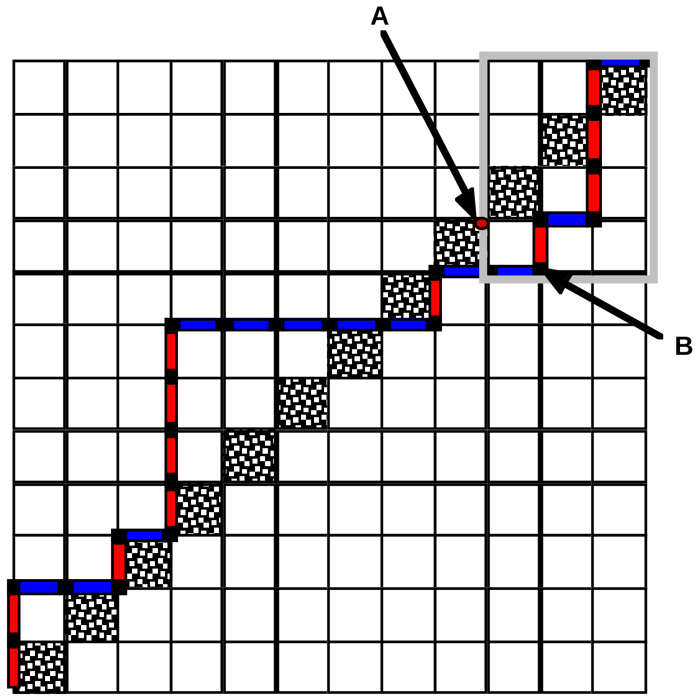
where for any integer n we set

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 \tag{3}$$

Using this formula, we obtain that the number of paths that start with the sequence in (1) is

$$\frac{(3+4)!}{3!\cdot 4!} \;=\; \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\times 4\cdot 3\cdot 2\cdot 1} \;=\; \frac{7\cdot 6\cdot 5}{3\cdot 2} \;=\; 35 \tag{4}$$

Note that the winning paths, which start with this initial sequence, must pass through the point **A** as indicated below.



Thus their tail end will be contained in a 3×3 box and their number is

$$\frac{(3+3)!}{3!\cdot 3!} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$$

Subtracting this from the result in (4), we derive that there must be a total of 15 losing paths that start with the pattern in (1). Indeed, these paths must pass through the point **B**, as indicated in the figure above, thus their tail end will be contained in a 2×4 box, and our formula gives that their number must be

$$\frac{(2+4)!}{2!\cdot 4!} = \frac{6\cdot 5}{2\cdot 1} = 15$$

which is beautifully consistent with our previous conclusion.

In summary, of the 35 paths that start with the sequence in (1) 20 are winning and 15 are losing. Thus, the probability of winning if we say **RED!** after we see the pattern in (1) is

$$\frac{20}{35} \approx .57\,.$$

which properly represents the advantage provided by the *clever* strategy. Nevertheless the probability of losing is

$$\frac{15}{35} \approx .43\,.$$

and events with such probability tend to occur roughly 43% of the time. The closeness of these two probabilities should suggest us some caution and postpone judgement about the advantages of this strategy until **all** winning and losing paths are counted.

Now it develops that there is a surprisingly simple way to solve the puzzle of "*Say Red*" once we make precise the notion of "*strategy*". To begin, let us note that the player should really say STOP! rather than RED!, since when all the red cards have been shown (†) it will be quite clear that the next card cannot be red. So let us require that the player must say *STOP*! before the last card is shown. Thus when we play with a deck with $2n$ cards, a strategy should consist of a collection $\mathcal{C}$ of "*words*" in the letters "**R**" and "**B**" (as the one given in (1)) with, at least, the following two basic properties

a) *Each word in $\mathcal{C}$ has at most $2n-1$ letters.*

b) *No initial segment of a word in $\mathcal{C}$* is in $\mathcal{C}$.

A player that uses the collection $\mathcal{C}$ as a strategy, should say *STOP*! immediately after the k^{th} card is shown if and only if the color pattern of the first k letters is a word of $\mathcal{C}$. This should make clear why property b) is included. In fact, if a word w' of $\mathcal{C}$ is an initial segment of a word w of $\mathcal{C}$ then w can be removed from $\mathcal{C}$ without loss, for w cannot be of any use, since the player will say *STOP*! as soon as he/she sees w'.

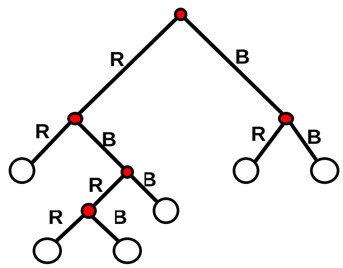
Note further that the requirement that the player must say *STOP*! before the last card is shown forces $\mathcal{C}$ to have one more additional property. Namely,

c) *Every deck has one initial color pattern of length $\leq 2n-1$ which is a word of $\mathcal{C}$.*

It turns out that there is a beautiful geometric way to produce all collections $\mathcal{C}$ with properties a), b) and c). This is best understood through an example. In the figure below we have displayed what the Computer Scientists call a "*binary tree*". The circles are the "*leaves*" of the tree and the shaded disks are called "*internal nodes*". The top internal node is called the "*root*" of the tree. From each internal note emanate two branches, one headed south-west, labelled with the letter **R** and one headed south-east labelled with the letter **B**. The maximum distance from the root to a leaf, measured by the number of branches we need to traverse to reach the leaf, is called the "*depth*" of the tree.

This given. the collection $\mathcal{C}(\mathcal{T})$ corresponding to a tree $\mathcal{T}$ is put together in a very neat way. As we follow a path that goes from the root to a leaf we simply read the labels attached to the successive branches. In this manner each leaf yields a word. This given $\mathcal{C}(\mathcal{T})$

(†) This can happen well before all the cards are shown is simply the collection of all words so obtained.



For this tree $\mathcal{T}$ the collection $\mathcal{C}(\mathcal{T})$ consists of the words.

$$\mathcal{C}(\mathcal{T}) = \{ \mathbf{RR}\,,\ \mathbf{RBRR}\,,\ \mathbf{RBRB}\,,\ \mathbf{RBB}\,,\ \mathbf{BR}\,,\ \mathbf{BB} \} \tag{5}$$

The reader should have no difficulty verifying that this collection has properties b), c) and a) when $2n-1 \geq 4$. A moment of thought should reveal that all collections satisfying a), b) and c) can be obtained in this manner.

Now for a given choice of n we could carry out the same analysis we applied to the initial word given in (1) to each of the words in (5) and obtain, for each of word $w \in \mathcal{C}(\mathcal{T})$, the number $W(w)$ of winning paths with initial pattern given by w and the number $L(w)$ of losing paths with initial pattern given by w. By summing over all words of $\mathcal{C}$ we obtain

$$W(\mathcal{T}) \;=\; \sum_{w\in\mathcal{C}(\mathcal{T})} W(w) \quad \text{and} \quad L(\mathcal{T}) \;=\; \sum_{w\in\mathcal{C}(\mathcal{T})} L(w) \tag{6}$$

as the total numbers of winning and losing paths produced by the strategy corresponding to $\mathcal{T}$. This done the probability of winning by this strategy is then given by the ratio

$$\frac{W(\mathcal{T})}{W(\mathcal{T})+L(\mathcal{T})}\,. \tag{7}$$

It turns out that we need not carry out all these calculations for there is a very simple argument that shows that

Theorem

For any deck of $2n$ cards and any tree $\mathcal{T}$ of depth less than $2n$ we have

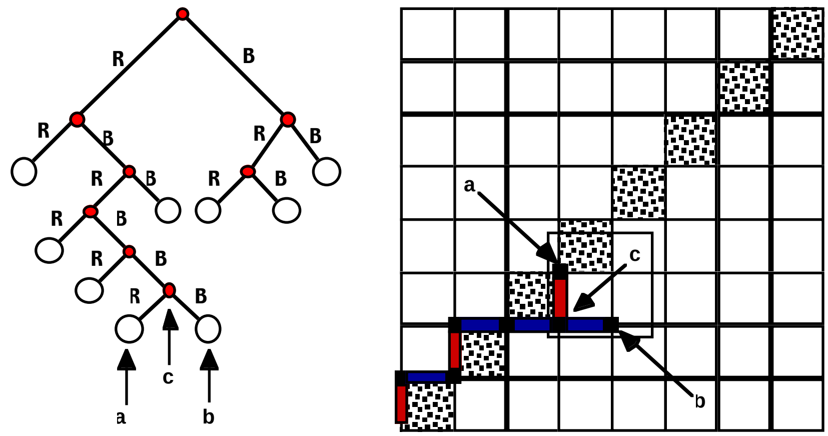
$$W(\mathcal{T}) \;=\; L(\mathcal{T})$$

Proof

Let us define the "*Goodness*" of the tree $\mathcal{T}$ and denote it by "$G(\mathcal{T})$" the difference

$$G(\mathcal{T}) \;=\; W(\mathcal{T}) - L(\mathcal{T})\,. \tag{8}$$

Of course if $G(\mathcal{T}) > 0$ then the probability of winning (i. e. the ratio in (7)) would be greater than $1/2$, this is why we use the word "*Goodness*" for this difference. We are to show that the Goodness is zero for all trees. This will be carried out by a simple process that is best explained through an example.



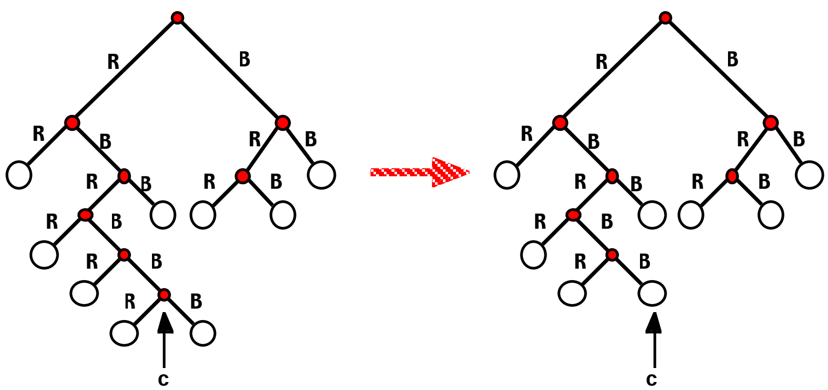
In the display above we have a tree $\mathcal{T}$ where the two lowest leaves and their parent (†) respectively labelled $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If we travel along the tree from the root to the internal node $\mathbf{c}$ and read the labels we obtain the word

$$w = \mathbf{RBRBB} \tag{9}$$

going all the way to the leaves $\mathbf{a}$ and $\mathbf{b}$ yields the two words

$$w_1 = \mathbf{RBRBBR} \quad \text{and} \quad w_2 = \mathbf{RBRBBB} \tag{10}$$

these three words are represented in the right portion of the display by the three paths ending at the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If we play by the strategy corresponding to this tree, we are to say STOP when we see one of the two color patterns in (10). Now if we remove the leaves $\mathbf{a}$ and $\mathbf{b}$ from this three (together with the branches leading to them) and make $\mathbf{c}$ into a leaf, as indicated in the display below, we obtain a tree $\mathcal{T}'$ whose corresponding strategy requires that we say STOP when we see the color pattern in (9).



We claim that this operation leaves the Goodness unchanged!

(†) The internal node immediately above them.

More precisely we will show that

$$G(\mathcal{T}) \quad = \quad G(\mathcal{T}')\,.$$

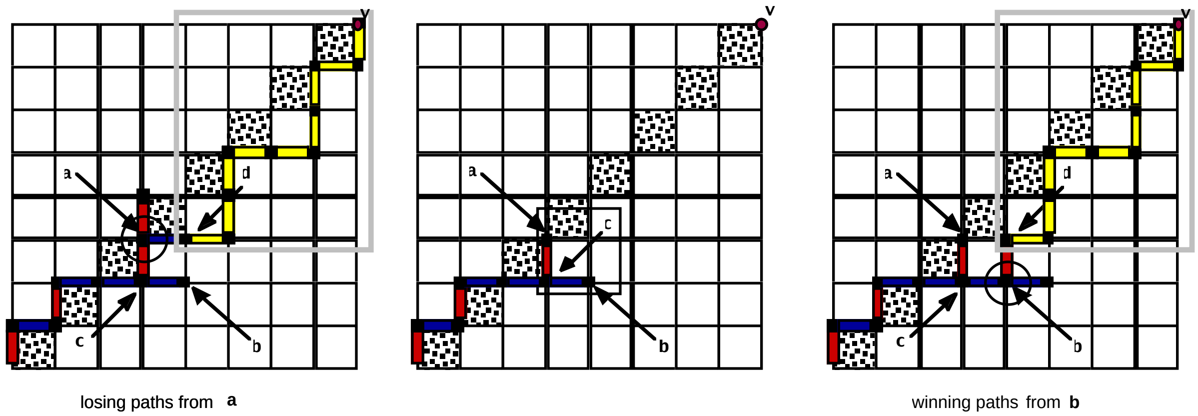
To this end note that, using our notation we have

$$\begin{aligned} &contribution\ \ to\ \ G(\mathcal{T})\ \ from\quad \mathbf{a} &=&\ W(w_1) - L(w_1)\,,\\ &contribution\ \ to\ \ G(\mathcal{T})\ \ from\quad \mathbf{b} &=&\ W(w_2) - L(w_2)\,,\\ &contribution\ \ to\ \ G(\mathcal{T}')\ \ from\quad \mathbf{c} &=&\ W(w) - L(w)\,.\end{aligned}$$

Since $\mathcal{T}$ and $\mathcal{T}'$ have so many leaves in common, we may write

$$G(\mathcal{T}) = \mathcal{R} + W(w_1) - L(w_1) + W(w_2) - L(w_2) \quad \text{and} \quad G(\mathcal{T}') = \mathcal{R} + W(w) - L(w) \tag{11}$$

where $\mathcal{R}$ denotes the contribution to the Goodness of $\mathcal{T}$ and $\mathcal{T}'$ coming from their common leaves.



However, since every losing path from **a** passes through **d** and every winning path from **b** must also pass through **d** (see figure above) we derive that

$$L(w_1) \quad = \quad W(w_2) \tag{12}$$

Using this in (11) yields that

$$G(\mathcal{T}) = \mathcal{R} + W(w_1) - L(w_2)\,. \tag{13}$$

On the other hand, the diagram in the middle of the display above makes it clear that every winning path from **c** to **v** (see figure above) passes though **a** and every losing one passes through **b**, thus

$$W(w) = W(w_1) + L(w_1) \quad \text{and} \quad L(w) = W(w_2) + L(w_2)\,.$$

Using this in (11) we derive that

$$G(\mathcal{T}') = \mathcal{R} + W(w_1) + L(w_1) - W(w_2) - L(w_2)$$

and (12) gives that

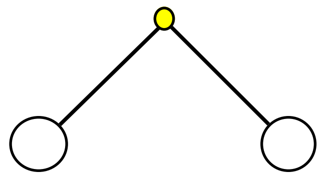
$$G(\mathcal{T}') = \mathcal{R} + W(w_1) - L(w_2)$$

Comparing with (13) proves that

$$G(\mathcal{T}) \quad = \quad G(\mathcal{T}')$$

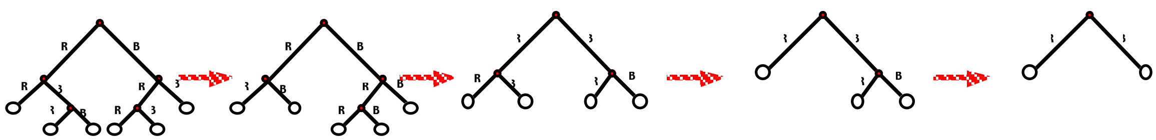
as asserted.

It should be clear from this example that the same process can be applied to any tree $\mathcal{T}$ and that any tree $\mathcal{T}'$ obtained from $\mathcal{T}$ by the removal of two contiguous lowest leaves will have the same Goodness as $\mathcal{T}$. However, by successively pealing off pairs of lowest leaves in the manner indicated we will, in any case, eventually end up with the simple tree below



whose Goodness is easily seen to be equal to zero. Thus all the trees must necessarily have Goodness zero. This completes the proof of the Theorem

The display below illustrates a particular case of the reduction process we have used in our proof.



Remark

Of course we could carry out one more step and remove these last two leaves bringing us down to the trivial tree consisting of a single leaf. Since the trivial tree corresponds to the trivial strategy, we so derive again that the Goodness of any strategy is the same as the Goodness the Trivial strategy, and therefore it must be equal to zero.

We are thus brought to the final conclusion that even the "*clever*" player strategy yields no improvement on the 50% probabilty yielded by the Trivial strategy.

2. More on the "Clever" strategy.

We have proved in the previous section that, whatever strategy we adopt, the number of winning paths is equal to the number of losing paths. This surprising conclusion should leave us in need of further explanation, not only because of its paradoxical nature, but also because pairs of collections equal in number should exhibit some kinship that explains equality. To get accross what we plan to do next we shall use a bit of contemporary imagery. Let us view each of our paths as a "*toy*". Since any rearrangement of the steps of a path (†) yields another path, we may think of them as "*transformer toys*". Could it then be that each "*winning path*" is in fact a toy that can be "*transformed*" into a "*losing path*"? Clearly, if this the case, and the resulting transformation pairs off each member of the winning family with **unique** member of the losing family, then the equality should be no longer puzzling.

Now it develops that such a "*pairing*" can be carried out for each of our strategies.

Since working in full generality might understandingly be a bit onerous at this point, we shall limit ourselves to the case of the Clever strategy. To do this we need some auxiliary facts.

Recall that under this strategy the Player loses when the path keeps above the diagonal squares. In the literature these are called "*Dyck paths*". Let us denote by $\mathcal{D}_n$ the collection of all Dyck paths in the $n\times n$ square. There is a surprisingly simple argument that shows that the cardinality of $\mathcal{D}_n$ is given by the difference

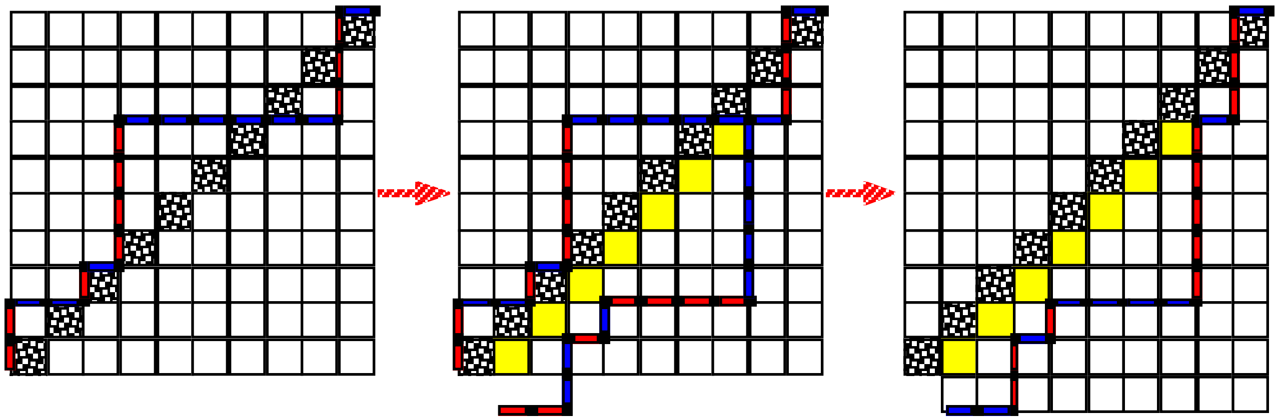
$$|\mathcal{D}_n| \;=\; \binom{2n}{n} - \binom{2n}{n-1}. \tag{14}$$

We cannot resist the temptation of schetching the proof of this beautiful formula since it leads to the "*transformer toy*" idea we need in our development.

Note first that the binomial coefficient $\binom{2n}{n}$ counts **all** paths in the $n\times n$ square, while the binomial coefficient $\binom{2n}{n-1}$ counts all the paths in an $(n-1)\times(n+1)$ rectangle. Thus to prove the identity in (14) we need only show that the paths in the $n\times n$ square that "**do not**" remain above the diagonal are counted by $\binom{2n}{n-1}$ as well!.

(†) i. e. shuffling the corresponding deck

Now this can be done by a "*transformer toy*" argument. An example should suffice to get across the idea.



We have on the left, one of the paths that does not remain above the diagonal in a 10×10 square. In the diagram in the middle, we have lightly shaded the squares immediately below the diagonal up to the point when our path goes for the first time below the diagonal. This done we have used the line of lightly shaded squares as a mirror to "*reflect* a portion of the path, and thus obtained a new path. On the diagram on the right we have simply corrected the colorings so that the NORTH steps are red and the EAST steps are blue. Note that this has produced a path in a 9×11 square... Voilà!. This is the transformation that proves (14). To complete the argument we need only show that the target path uniquely identifies the original path that created it. However this is quite simple. Any path in the 9×11 square must at some point touch the diagonal of the 10×10 square. The first point where it does is the tip of the line of lightly shaded squares that produced it.

The reader should have no difficulty concluding that this argument is valid in full generality proving (14) for any value of n.

We are now ready to tackle our Clever strategy problem. To begin we need to give a precise geometrical description of winning and losing paths in this case. From the definition of the Clever strategy we derive that

a) *A path is winning if the first time one of its* EAST *steps goes under a shaded square the next step is* NORTH.

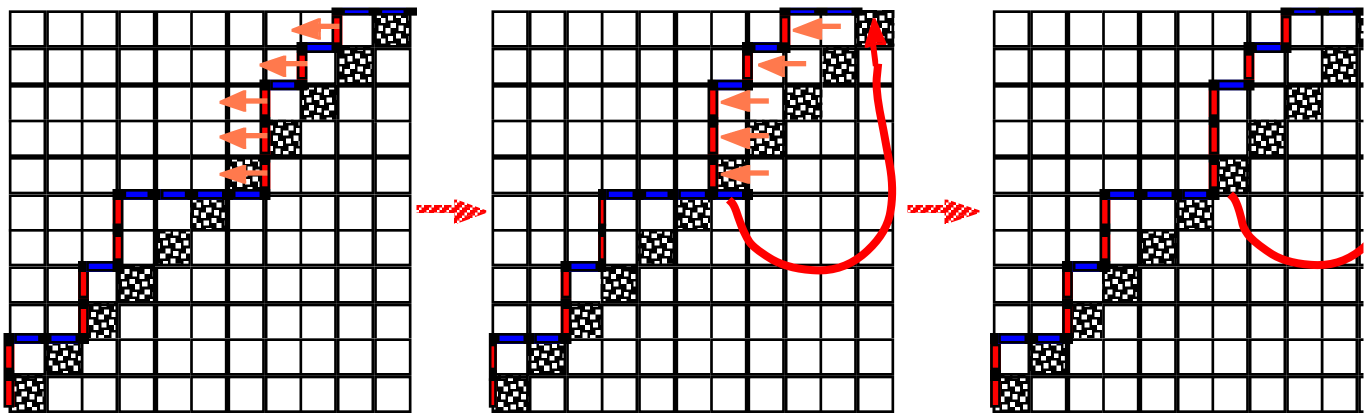
By contrast

b) *A path is losing if the path always stays above the shaded squares, or the first time one of its* EAST *steps goes under a shaded square the next step is again* EAST.

If the first alternative occurs, we call the path "*losing of type A*" and if the second alternative occurs we call the path "*losing of type B*". We can introduce an analogous distinction for winning paths as well. We shall call "*winning of type A*" a path which goes under "**only one**" diagonal square. All the other winning paths will be called "*winning of type B*".

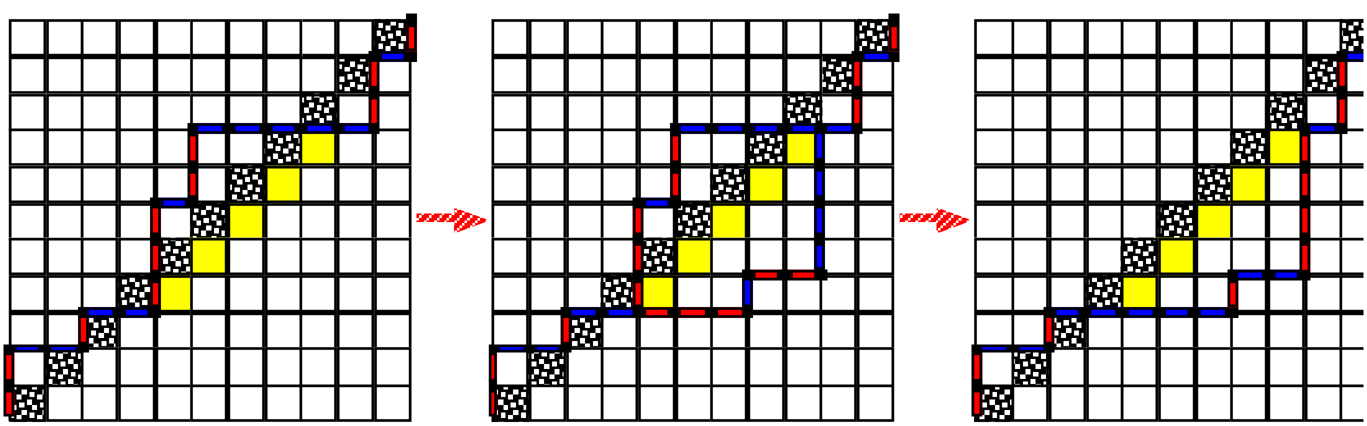
We shall show that any winning path of type A transforms into a losing path of type A and any winning path of type B transforms into a losing path of type B.

Again it will suffice to give one example in each case. To begin, in the figure below we have on the left a winning path of type A. To transform it we start by moving WEST by one square, the portion of the path that is above the EAST step that is below the diagonal, as indicated by the arrows.



This yields the diagram in the middle. Then we transfer the EAST step below the diagonal to the top side of the highest diagonal square, as indicated by the curved line. This yields the path in the right, which we can well see is a losing path of type A. Conversely, given a losing path of type A, we can easily identify the winning path of type A it came from, by starting from above and going town the path until it thouches the diagonal for the second time. This gives the portion that has to be moved back EAST to reconstruct the winning path it came from.

Let us now deal with a winning path of type B.



We see on the left a winning path of type B. We start by light-shading the line of subdiagonal squares that join the corner of the EAST-NORTH turn which makes this into a winning path to the SOUTH-EAST corner of the very next diagonal square the path goes under. This done using the line of light shaded squares as a mirror we reflect a portion of the path as indicated in the middle diagram. Finally we correct the colors of the steps of the reflected portion, so that NORTH steps are red and EAST steps are blue. This results in the path on the right, which we can very well see is losing of type B. Conversely, given a losing path of type B, we need only identify the "*mirror*" that produced it. But that is easily done, since the line of light shaded squares has to go from the mid-point of the EAST-EAST step that characterizes a losing path of type B to the SOUTH-EAST corner of the very next diagonal square the path touches.

The reader should have no difficulty seeing that the two transformations we have illustrated can be carried out in the general case as well. This completes our treatement of *SAY RED*". Hopefully we have conveyed the reader an enjoyable example of mathematical reasoning.