Problem 1. (16 pts.) Velocity $v(t)$ of a car, measured in ft/sec, increases according to the table below. To approximate the total distance traveled find:

(a) An upper estimate  
(b) A lower estimate

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

How frequently must the velocity be recorded in order to estimate the total distance traveled with precision up to 20 ft.

Solution

upper estimate = right estimate = $(15 + 15 + 20 + 30) \cdot 2 = 80 \cdot 2 = 160$

lower estimate = left estimate = $(5 + 15 + 15 + 20) \cdot 2 = 55 \cdot 2 = 110$

If we record the velocity $n$ times during the 8 seconds to have the estimation of the total distance traveled with precision up to 20 ft then:

$$|v(8) - v(0)| \frac{8 - 0}{n} \leq 20$$

Hence $\frac{200}{n} \leq 20$ which is equivalent to $10 \leq n$.

The velocity must be recorded at least 10 times to have the estimate of the total distance traveled with precision up to 20 ft.

Problem 2. (14 pts.)

Find the following derivative:

$$\frac{d}{dx} \int_1^{x^2 + 2} \ln(t^4 + 3) \, dt$$

Solution

Put $F(x) := \int_1^{x^2} \ln(t^4 + 3) \, dt$. Hence by FTC

$$\frac{d}{dx} \int_1^{x^2 + 2} \ln(t^4 + 3) \, dt = (F(x^2 + 2))' = F'(x^2 + 2) \cdot 2x = 2x \ln((x^2 + 2)^4 + 3).$$
Problem 3. (14 pts.) Find the area between the $x$-axis and the graph of $y = \sin x$ over the interval $[0, \pi]$.

Solution

The area $= \int_0^\pi \sin x dx = [-\cos x]_0^\pi = -\cos(\pi) + \cos 0 = 1 + 1 = 2$

Problem 4. (16 pts.) A projectile is shot upward from the ground with initial velocity 64 ft/sec. What is the maximal height the projectile will reach. Recall that: $v(t) = v_0 - gt$ and $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$, where $g = 32$ ft/sec$^2$.

Solution

Observe that: $v_0 = 64$ and $s_0 = 0$. The projectile will reach the maximal height when $v(t) = 64 - 32t = 0$. This gives $t = 2$.

So the maximum height is:

$$s(2) = 64 \cdot 2 - 16 \cdot 2^2 = 128 - 64 = 64 \text{ ft.}$$
Problem 5. (24 pts.) Find the following integrals:

(a) \( \int \sin x \, e^{3+\cos x} \, dx \)

(b) \( \int e^x \sin x \, dx \)

Solution

(a) Integration by substitution. Take \( u = 3 + \cos x \) then \( du = -\sin x \). Hence:

\[
\int \sin x \, e^{3+\cos x} \, dx = \int e^{u} \, du = -e^u + C = -e^{3+\cos x} + C
\]

(b) Integration by parts twice.

\[
\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx
\]

\[
\int e^x \sin x \, dx = \frac{1}{2} (-e^x \cos x + e^x \sin x) + C
\]

Problem 6. (16 pts) Find the following integral:

\[
\int \frac{8}{(x - 5)(x + 3)} \, dx
\]

Solution

\[
\frac{8}{(x - 5)(x + 3)} = \frac{A}{x - 5} + \frac{B}{x + 3} = \frac{(A + B)x + 3A - 5B}{(x - 5)(x + 3)}.
\]

From this \( A + B = 0 \) and \( 3A - 5B = 8 \). Hence \( B = -A \) and \( 3A + 5A = 8A = 8 \). So \( A = 1 \) and \( B = -1 \). We have:

\[
\int \frac{8}{(x - 5)(x + 3)} \, dx = \int \frac{1}{x - 5} \, dx - \int \frac{1}{x + 3} \, dx = \ln |x - 5| - \ln |x + 3| = \ln \left| \frac{x - 5}{x + 3} \right| + C.
\]