Problem 1. (16 pts.) Find the following improper integral:

\[ \int_0^\infty 4xe^{-x^2} \, dx \]

Solution

By substitution: \( t = x^2 \), \( dt = 2x \, dx \).

\[
\int 4xe^{-x^2} \, dx = \int 2e^{-t} \, dt = -2e^{-t} + C = -2e^{-x^2} + C.
\]

Hence

\[
\int_0^\infty 4xe^{-x^2} \, dx = [-2e^{-x^2}]_0^\infty = 0 - (-2\cdot0) = 2
\]

Problem 2. (16 pts.)

Does the following improper integral converge or diverge? Justify your answer.

\[ \int_1^\infty \frac{4x + 1}{x^2 + 3x + 4} \, dx \]

Solution

Observe that for all \( x \geq 1 \) we have:

\[
\frac{4x + 1}{x^2 + 3x + 4} \geq \frac{4x}{x^2 + 3x + 4} \geq \frac{4x}{x^2 + 3x^2 + 4x^2} = \frac{4x}{8x^2} = \frac{1}{2x}
\]

But

\[
\int_1^\infty \frac{1}{2x} \, dx = \frac{1}{2} \ln|\infty| = \frac{1}{2} (\infty - 0) = \infty
\]

Hence the integral \( \int_1^\infty \frac{4x + 1}{x^2 + 3x + 4} \, dx \) diverges.
Problem 3. (18 pts.) Consider the region below the graph of \( y = x^2 + 12x \) and above the \( x \)-axis over the interval \([0, 1]\). Find the volume of the solid obtained by rotating this region about the \( x \)-axis.

**Solution**

Subdivide the interval \([0, 1]\) : \( 0 = x_0 < x_1 < \ldots < x_n = 1 \). Then the volume of the slice of the solid corresponding to the interval \([x_{i-1}, x_i]\) is:

\[
V_i \approx \pi (x_i^2 + 12x_i)^2 \Delta x_i.
\]

Hence

\[
V = \lim_{n \to \infty} \sum_{i=1}^{\infty} V_i = \int_{0}^{1} \pi (x^2 + 12x)^2 \, dx = \pi \int_{0}^{1} x^4 + 24x^3 + 144x^2 \, dx = \pi \left[ \frac{1}{5} x^5 + 6x^4 + 48x^3 \right]_0^1 = \pi \left( \frac{1}{5} + 6 + 48 \right) = \frac{271}{5} \pi
\]

Problem 4. (18 pts.) Find the arc length of the graph of the function \( y = \sqrt{16 - x^2} \) from \( x = 0 \) to \( x = 4 \):

**Solution**

Use formula

\[
L = \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx
\]

In our case \( f'(x) = \frac{-x}{\sqrt{16 - x^2}} \) hence

\[
L = \int_{0}^{4} \sqrt{1 + \frac{x^2}{16 - x^2}} \, dx = 4 \int_{0}^{4} \frac{1}{\sqrt{16 - x^2}} \, dx
\]

By substitution: \( x = 4 \sin \theta, \, dx = 4 \cos \theta \, d\theta \) we get:

\[
\int \frac{1}{\sqrt{16 - x^2}} \, dx = \int \frac{4 \cos \theta \, d\theta}{\sqrt{16 - 16 \sin^2 \theta}} = \int \frac{4 \cos \theta \, d\theta}{4 \cos \theta} = \int d\theta = \theta + C = \arcsin \frac{x}{4} + C
\]

Hence

\[
L = 4 \left[ \arcsin \frac{x}{4} \right]_0^4 = 4(\arcsin 1 - \arcsin 0) = 4(\frac{\pi}{2} - 0) = 2\pi.
\]
Problem 5. (16 pts.) Find the solution of the following differential equation subject to the initial condition $y(0) = -\frac{1}{2}$:

$$\frac{dy}{dx} = (3x^2 - 2)y^2$$

Solution

Separation of variables gives: $\frac{dy}{y^2} = (3x^2 - 2)\,dx$. Integrating leads to the general solution:

$$-\frac{1}{y} = x^3 - 2x + C \quad \text{hence} \quad y = -\frac{1}{x^3 - 2x + C}$$

By initial condition $-\frac{1}{2} = -\frac{1}{C}$. Hence $C = 2$. So the answer is:

$$y = -\frac{1}{x^3 - 2x + 2}.$$ 

Problem 6. (16 pts) A bottle of juice of $40^\circ F$ was taken out of refrigerator and put in a room with $80^\circ F$ temperature. Write down and solve the corresponding Newton’s Law differential equation. If the constant $k$ is $k = 0.05$, find out what will be the juice temperature after 10 min.

Solution Newton Heating and Cooling law differential equation:

$$\frac{dH}{dt} = -k(H - H_0)$$

has solution:

$$H(t) = H_0 + Ce^{-kt}$$

In our case $H_0 = 80$, $H(0) = 40$ and $k = 0.05$. Applying this data gives: $H(0) = 40 = 80 + C$ hence $C = -40$. We obtain:

$$H(t) = 80 - 40e^{-0.05t}$$

After 10 min. the juice temperature will be:

$$H(10) = 80 - 40e^{-0.05\cdot10} = 40(2 - e^{-0.5}) \approx 55.74^\circ F$$