Finding a probability density function from a CDF

In lecture, we defined uniform random variables; in particular, if X is a uniform random variable on the interval $[1,3]$, it has probability density function (PDF)

$$f_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x \in [1,3] \\ 0 & \text{if } x > 3 \end{cases}.$$

In words, this just says that X is equally likely to take any value in the interval $[1,3]$.

Now, let $Y = X^2$ (i.e., we pick a uniformly random number between 1 and 3, and compute its square). In this note, I'm going to work through how we would find the PDF of Y, which we'll call $f_Y(y)$, and how we can use that to find $E(Y)$. Here are a few ingredients from lecture that we'll use:

First, let's remember how/why this came up. One of the examples in lecture was to find the variance of a uniform random variable (this is Example 3.50 in the book). Since $\text{Var}(X) = E(X^2) - [E(X)]^2$, we needed to find $E(Y) = E(X^2)$. The easiest approach to this computation is to start this way:

$$E(Y) = E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx$$

This comes from applying our formula for the expectation of a function of a random variable (Fact 3.33); here the function is $Y = g(X) = X^2$. Then, for $X \sim \text{Unif}[1,3]$ specifically, the calculation finishes off as follows:

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx \\ &= \cancelto{0}{\int_{-\infty}^{1} x^2 \cdot 0\,dx} + \int_{1}^{3} x^2 \cdot \frac{1}{2}\,dx + \cancelto{0}{\int_{3}^{\infty} x^2 \cdot 0\,dx} \\ &= \int_{1}^{3} x^2 \cdot \frac{1}{2}\,dx \\ &= \tfrac{13}{3}. \end{aligned}$$

However, it is also certainly possible to compute $E(Y)$ directly from the definition of expectation. How do we do this? We would need to evaluate the integral:

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy.$$

In order to finish this computation, we would then need to find $f_Y(y)$. In practice, when we're trying to find the PDF of a random variable, it's almost always easiest to start by finding the CDF, then differentiating. This works because

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt,$$

which by the Fundamental Theorem of Calculus Part 1, says that

$$\frac{d}{dy}F_Y(y) = f_Y(y).$$

(this is also written in the book as Fact 3.13)

So let's start by finding the CDF of Y, which we'll write as $F_Y(y)$. By definition,

$$F_Y(y) = P(Y \le y) = P(X^2 \le y).$$

If $y < 0$, this is just 0, since X^2 can never be negative. If $y \ge 0$, this is the same as $P(X \le \sqrt{y})$. And notice, this is just the CDF of X! (Not a factorial, just excitement...) In lecture, we showed that the CDF of X is:

$$F_X(s) = \begin{cases} 0 & \text{if } s < 1 \\ \frac{1}{2}(s-1) & \text{if } s \in [1,3] \\ 1 & \text{if } s > 3 \end{cases}.$$

(This example is also in the textbook as Example 3.12 if you'd like to take a second/slower look at it). So putting it all together,

$$\begin{aligned} F_X(s) = P(X^2 < y) &= \begin{cases} 0 & \text{if } y < 0 \text{ or } \sqrt{y} < 1 \\ \frac{1}{2}(\sqrt{y}-1) & \text{if } \sqrt{y} \in [1,3] \\ 1 & \text{if } \sqrt{y} > 3 \end{cases} \\ &= \begin{cases} 0 & \text{if } y < 1 \\ \frac{1}{2}(\sqrt{y}-1) & \text{if } y \in [1,9] \\ 1 & \text{if } y > 9 \end{cases}. \end{aligned}$$

Then, to find $f_Y(y)$, we just take a derivative:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ \frac{1}{4}y^{-1/2} & \text{if } y \in [1,9] \\ 0 & \text{if } y > 9 \end{cases}$$

And to finish off, we can check that indeed

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy \\ &= \int_1^9 y \cdot \tfrac{1}{4}y^{-1/2}\, dy \\ &= \tfrac{13}{3}, \end{aligned}$$

just like we found earlier using the other (simpler) method.

The main takeaways here are:

- If we already know the PDF of a random variable X, it's much easier to find the expectation of a function $Y = g(X)$ using the formula above (where we only need f_X) than by using the definition of expectation directly (which requires us to find f_Y).
- If we know stuff about X and want to find f_Y, for a function $Y = g(X)$, we will almost always start by finding the CDF of Y, since this is a concrete probability we can get our hands on, and then differentiating F_Y to get f_Y.