An example of convolution

In lecture, we looked at the example of independent random variables X and Y distributed uniformly on $[0, 1]$; specifically, we asked for the probability density function $f_{X+Y}(z)$. The method we used was to start by finding the CDF of $X + Y$:

$$F_{X+Y}(z) = \iint_{x+y \leq z} f_X(x) f_y(Y)\, dx\, dy,$$

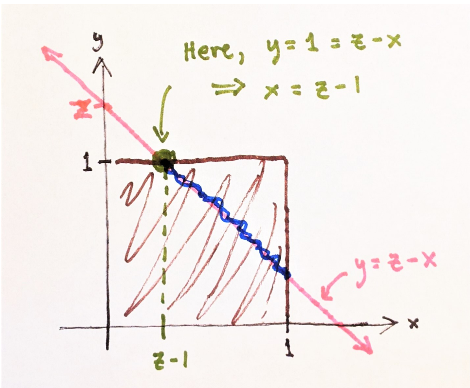
and then we differentiated the CDF to obtain the PDF. (Notice that the integrand here is $f_X(x)f_y(Y) = f_{X,Y}(x, , y)$ since X and Y are independent.)

However, for any two *independent* random variables X and Y, the convolution formula (Fact 7.1) pre-packages some parts of this calculation, and tells us the following:

$$f_{X+Y}(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx.$$

In practice, since many of the continuous random variables we work with have piecewise-defined PDFs that are often equal to 0, the main difficulty in applying the convolution formula comes down to finding the limits of integration so that *both* terms in the integrand ($f_X(x)$ and $f_Y(z - x)$) are nonzero.

The textbook goes through our example from lecture (Example 7.13) using the convolution formula rather than the approach we used in lecture. I've attached that page of the textbook at the end of this document, and I'd encourage you to work through it! But one thing I want to point out is that in the solution presented in the textbook, finding the limits of integration comes down to carefully solving a system of inequalities. While this is a useful and robust method, it is very easy to get lost and make errors. So whenever you use the convolution formula, I recommend drawing a picture to help you find the limits of integration. For example, when $X, Y \sim \text{Unif}[0, 1]$, and $z \in [1, 2]$, our picture for the integral $\int_{x=-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$ would look like this:



We know that $f_X(x)$ and $f_Y(y)$ are nonzero when $x, y \in [0, 1]$; these are all the points in the brown square. The points $(x, z - x)$ are those on the pink line. So we would like the points $(x, z - x)$ in the integrand to be precisely those that are on the pink line *and* in the brown square; i.e. the points on the blue squiggly line segment. From the picture, we see that this means x should be between $z - 1$ and 1. So for $z \in [1, 2]$ for example, we get

$$f_{X+Y}(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = \int_{x=z-1}^{1} 1 \cdot 1\, dx.$$

Example 7.13 (Convolution of uniform random variables)**.** Suppose that X and Y are independent and distributed as Unif$[0,1]$. Find the distribution of $X+Y$.

The density functions for X and Y are

$$f_X(x) = f_Y(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x < 0 \text{ or } x > 1. \end{cases}$$

$X+Y$ is always between 0 and 2, so the density function $f_{X+Y}(z)$ is zero for $z<0$ and $z>2$. It remains to consider $0 \le z \le 2$. By the convolution formula,

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx.$$

We must take care with the integration limits. The product $f_X(x)f_Y(z-x)$ is nonzero and equal to 1 if and only if $0 \le x \le 1$ and $0 \le z-x \le 1$. The second inequality is equivalent to $z-1 \le x \le z$. To simultaneously satisfy $0 \le x \le 1$ and $z-1 \le x \le z$, we take the larger of the lower bounds and the smaller of the upper bounds on x. Thus the integrand $f_X(x)f_Y(z-x)$ is 1 if and only if

$$\max(0, z-1) \le x \le \min(1, z).$$

Now for $0 \le z \le 2$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx = \int_{\max(0,z-1)}^{\min(1,z)} dx = \min(1,z) - \max(0, z-1).$$

We can simplify the result by considering the cases $0 \le z \le 1$ and $1 < z \le 2$ separately:

$$f(z) = \begin{cases} z - 0 = z, & \text{if} \quad 0 \le z \le 1, \\ 1-(z-1) = 2-z, & \text{if} \quad 1 < z \le 2, \\ 0, & \text{otherwise .} \end{cases}$$

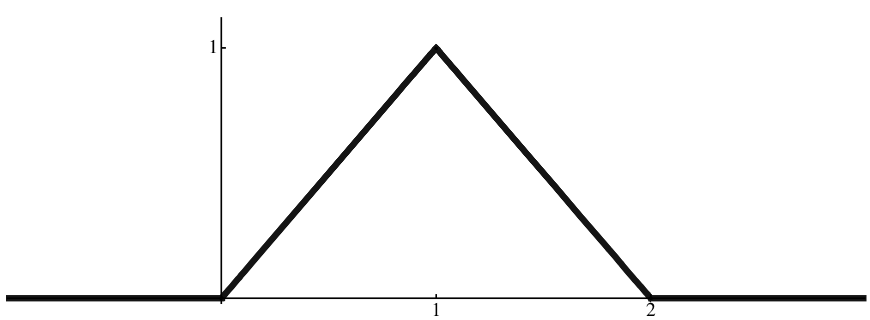


Figure 1. The p.d.f of the convolution of two Unif$[0,1]$ distributions