Problem 1 (a)

$$P(0) = \frac{1}{2}, P(1) = \frac{1}{8}, P(2) = \frac{1}{8}, P(-1) = \frac{1}{4}$$

and so

$$E[Z] = \frac{1}{2}(0) + \frac{1}{8}(1) + \frac{1}{8}(2) + \frac{1}{4}(-1) = \frac{1}{8}$$

Problem 1 (b) If we let $E[Z] = \mu$, $Var[Z] = E[(Z-\mu)^2]$, and so

$$Var[Z] = \frac{1}{2}\left(0 - \frac{1}{8}\right)^2 + \frac{1}{8}\left(1 - \frac{1}{8}\right) + \frac{1}{8}\left(2 - \frac{1}{8}\right) + \frac{1}{4}\left(-1 - \frac{1}{8}\right) = \frac{55}{64}$$

Alternatively, you could use the formula: $Var[Z] = E[Z^2] - (E[Z])^2$:

$$\frac{1}{2}(0)^2 + \frac{1}{8}(1)^2 + \frac{1}{8}(2)^2 + \frac{1}{4}(-1)^2 \quad - \quad (1/8)^2 = \frac{55}{64}$$

Problem 1 (c) We use the formula $E[X+Y] = E[X] + E[Y]$.

$$E[X] = \frac{1}{3}(1) + \frac{1}{3}(2) + \frac{1}{3}(3) = 2$$
$$E[Y] = \frac{3}{8}(1) + \frac{3}{12}(2) + \frac{3}{8}(3) = 2$$

So, $E[X+Y] = 2 + 2 = 4$.

Problem 1 (d) Since X and Y are independent (as we will see in part (e)), we also have the formula $E[X \times Y] = E[X] \cdot E[Y]$. Thus $E[X \times Y] = 2 \cdot 2 = 4$.

Alternatively, one could do these last two problems by calculating the probabilities for $P(X+Y=2), \ldots, P(X+Y=6)$ and $P(X \times Y = 1)$,..., $P(X \times Y = 9)$, respectively, but this would take more algebra.

Problem 1 (e) The pair (X, Y) is independent since the joint probabilities $P(X = a, Y = b) = P(X = a) \cdot P(Y = b)$ for all choices of $a, b \in \{1, 2, 3\}$. There is no need to calculate however since looking at the wheel, we see that X is equally distributed, and breaking into the various subsections of the wheel (according to assumed values for Y) we still have $P(X|Y = b)$ to be equally distributed. This is enough to see that $P(X = a, Y = b) = P(X = a) \cdot P(Y = b)$.
This is the only pair that is independent since $Z = 1$ immediately forces $X = 1$ and $Y = 1$. Thus $P(X = 1|Z = 1) = 1 \neq P(X = 1)$. We get a similar equation for $P(Y = 1|Z = 1)$.

Problem 1 (f) $Z = 0$ in the SW and NE quadrants of the wheel. Using the formula $P(Y = 2|Z = 0) = \frac{P(Y=2,Z=0)}{P(Z=0)}$ or directly seeing the proportions from the wheel, we obtain the answer of 1/2.

Problem 1 (g) We must calculate $P(X = 1|Z = 0) = 5/12$, $P(X = 2|Z = 0) = 1/6$, $P(X = 3|Z = 0) = 5/12$, each of these being analogous to part (f). Thus

$$E[X|Z = 0] = \frac{5}{12}(1) + \frac{1}{6}(2) + \frac{5}{12}(3) = 2$$

Many students got the incorrect answer of 1 since they forgot to divide by 1/2 when calculating the conditional probability.

Problem 2 There are a couple ways to do this problem, one of which is to explicitly write out the event spaces A and B since the universe has only 16 elements.

$$A = \{HHHT, HHTT, HTHT, HTTT, THHH, THTH, TTHH, TTTH\}$$

$$B = \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$$

We now use Venn diagrams to see that $P(A^c \cap B) = P(B) - P(A \cap B)$ and $P(A \cap B^c) = P(A) - P(A \cap B)$.
$P(A) = 8/16 = 1/2$, $P(B) = 6/16 = 3/8$, and $P(A \cap B) = 4/16 = 1/4$ since

$$A \cap B = \{HHTT, HTHT, THTH, TTHH\}$$

So part (a) has the answer

$$\frac{6-4}{16} = 1/8 \qquad \text{and}$$

part (b) has the answer

$$\frac{8-4}{16} = 1/4$$

Alternatively, one can do this problem by writing down the elements of $A^c \cap B$ and $A \cap B^c$ and counting 2 and 4 out of sixteen possibilities, respectively.

Problem 3 If at most two heads occur, this means we are in one of the seven possibilities:

$$HHT \quad HTH \quad HTT$$

$$THH \quad THT \quad TTH \quad TTT$$

Since at least 2 heads occur in three of these, we compute the probability to be 3/7.

Problem 4 I recommend drawing/looking at the wheel for this problem (See page 12 of Paradoxes handout) Since there is one red-red stick, one black-black stick, and one black-black stick, the probability that the other end is black (given we saw a red end) is the same as the probability that the other end is red (given we saw a black end).

Thus the probability is 1/3. We will discuss how to solve more complicated questions of this type (using Bayes' Rule) this week in section.