Chapter 3.2–3.4
Spanning Tree Algorithms

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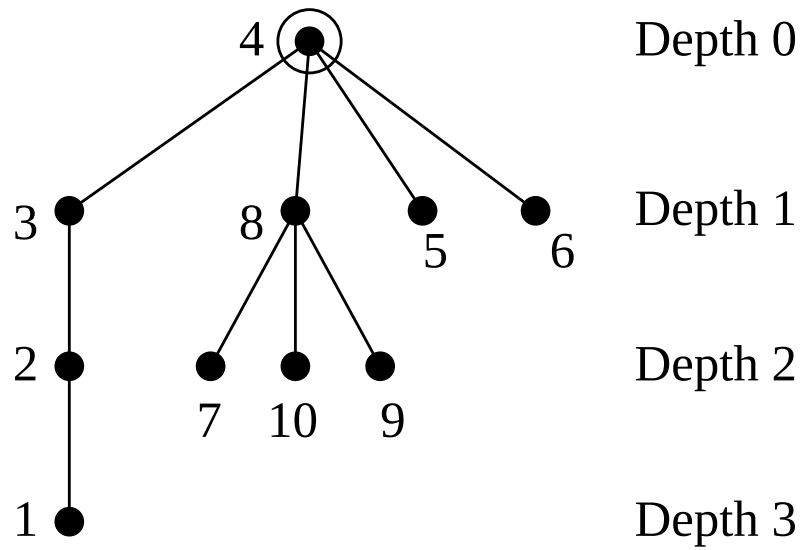
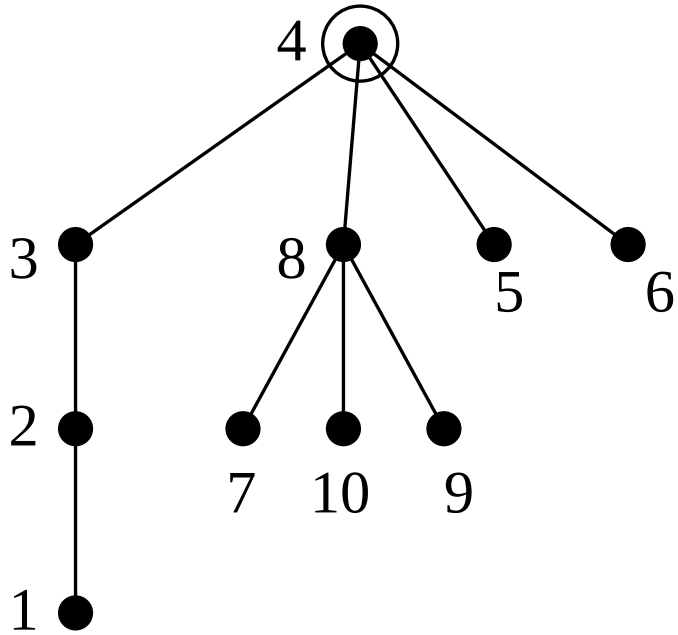
Math 154
Winter 2020

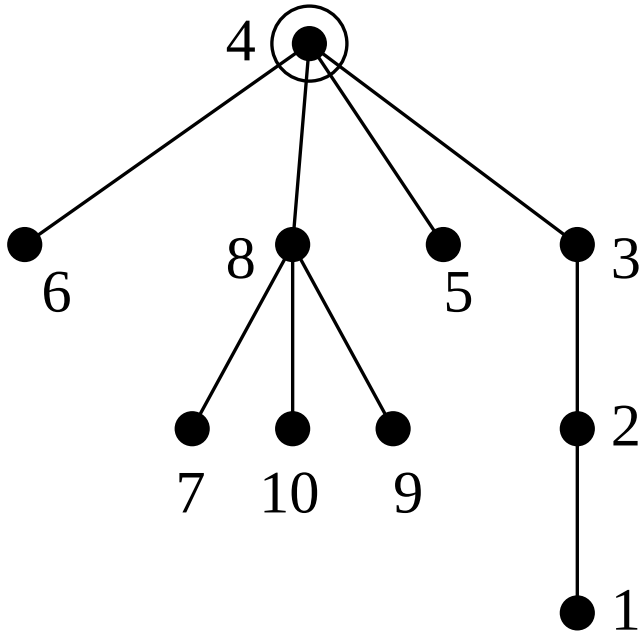
Depth



- The *depth* of v is the length of the path from the root to v.
- The *height* of the tree is the maximum depth.

Ordered trees

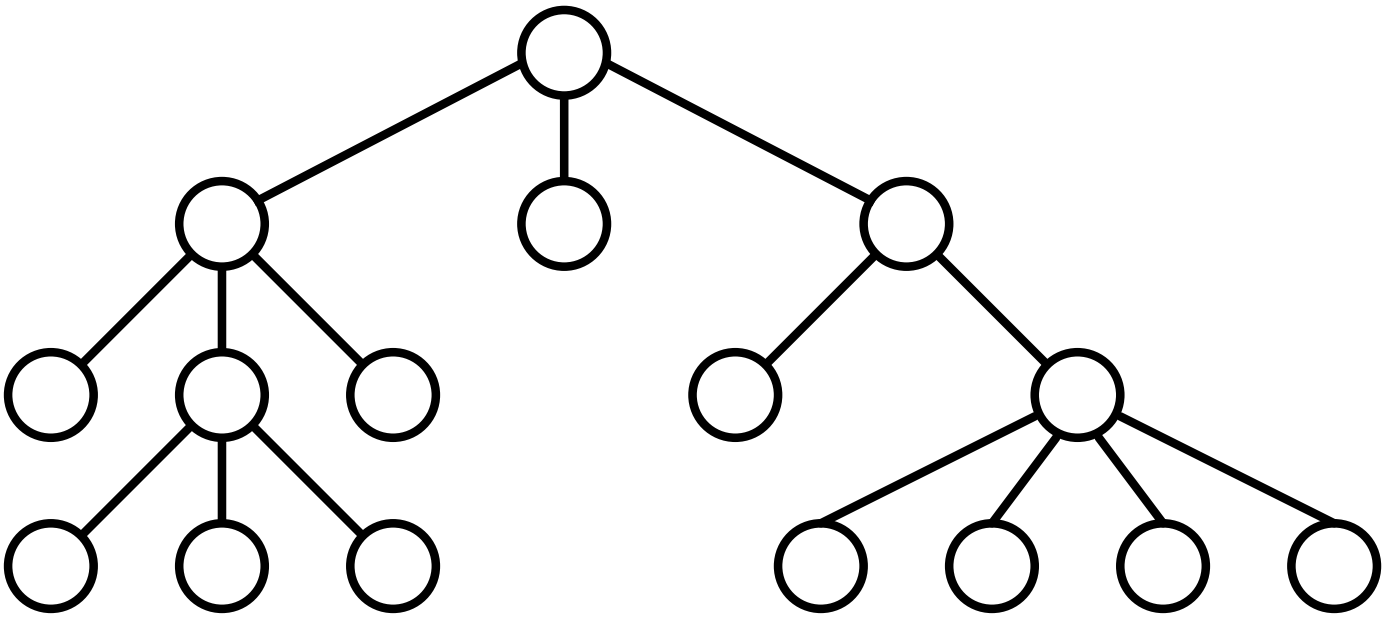




- An *ordered tree* puts the children of each node into a specific order (pictorially represented as left-to-right).
- The diagrams shown above are the same as *unordered trees*, but are different as *ordered trees*.
- We'll be looking at graphs and trees drawn in other ways; there will be other orderings besides left-to-right.

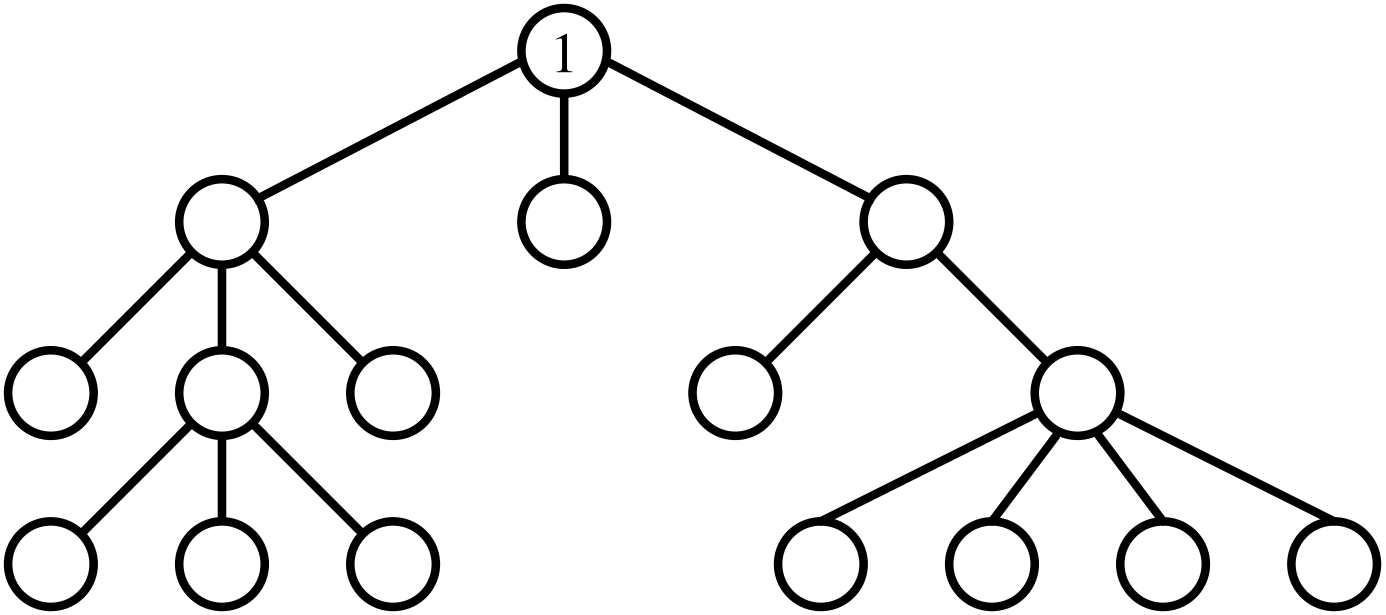
Depth first search of a tree

Depth first search of a tree



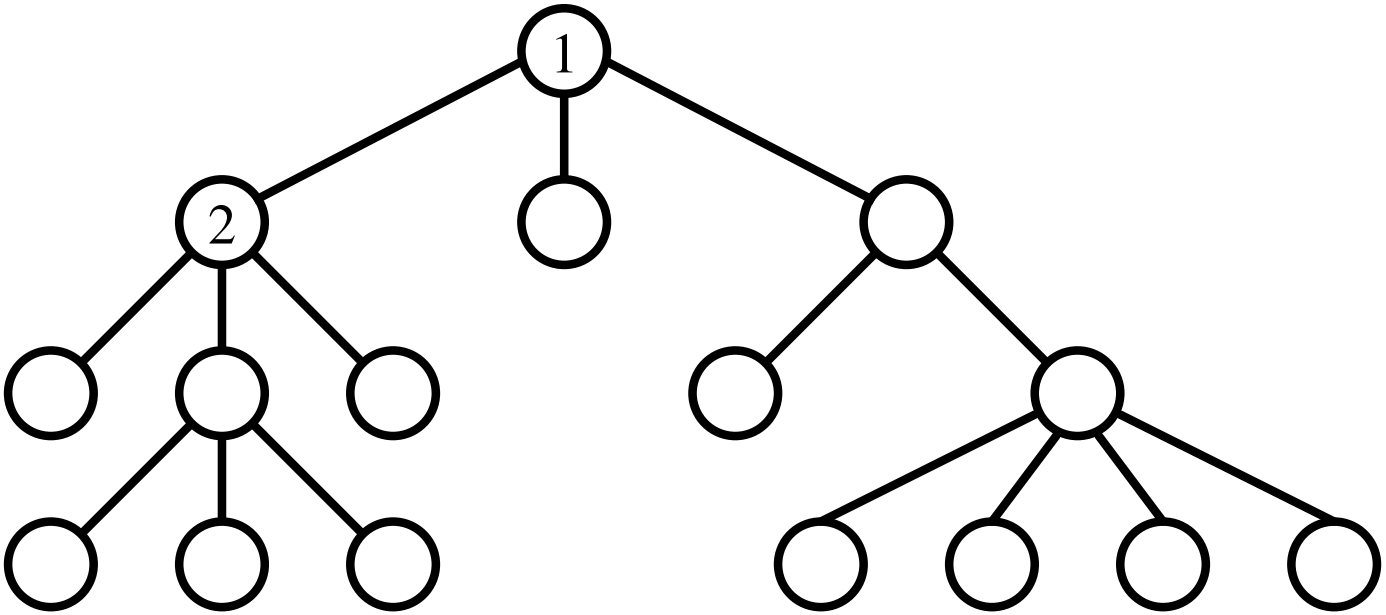
- We will number the vertices by a *depth first search* (DFS), also called a *depth first traversal*.

Depth first search of a tree



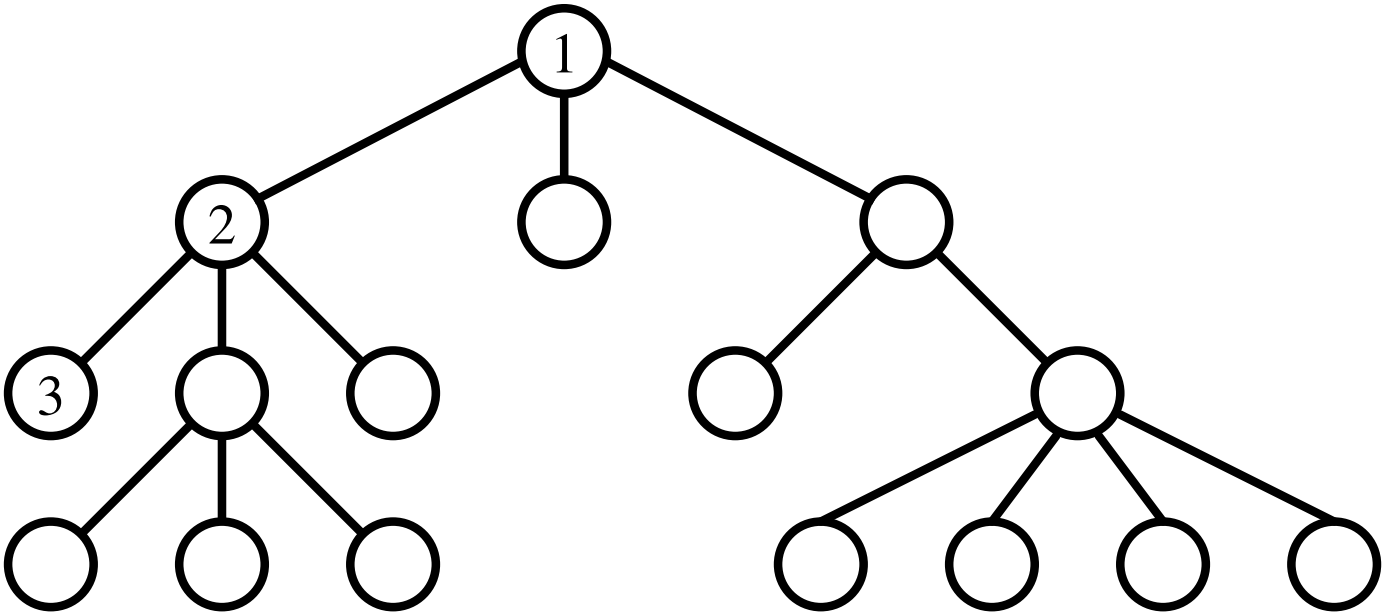
- Start at the root r (the top vertex in this diagram), and assign it the number 1.

Depth first search of a tree



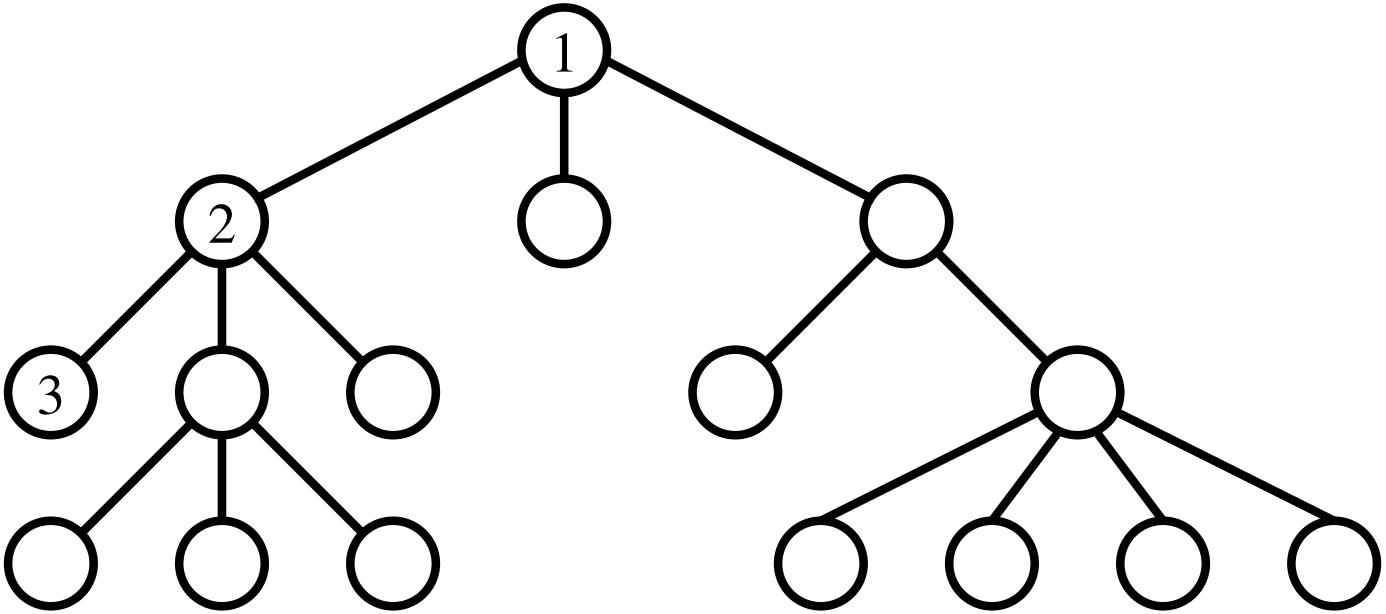
- Visit 1's first child (in left-to-right order), and assign it the number 2.

Depth first search of a tree



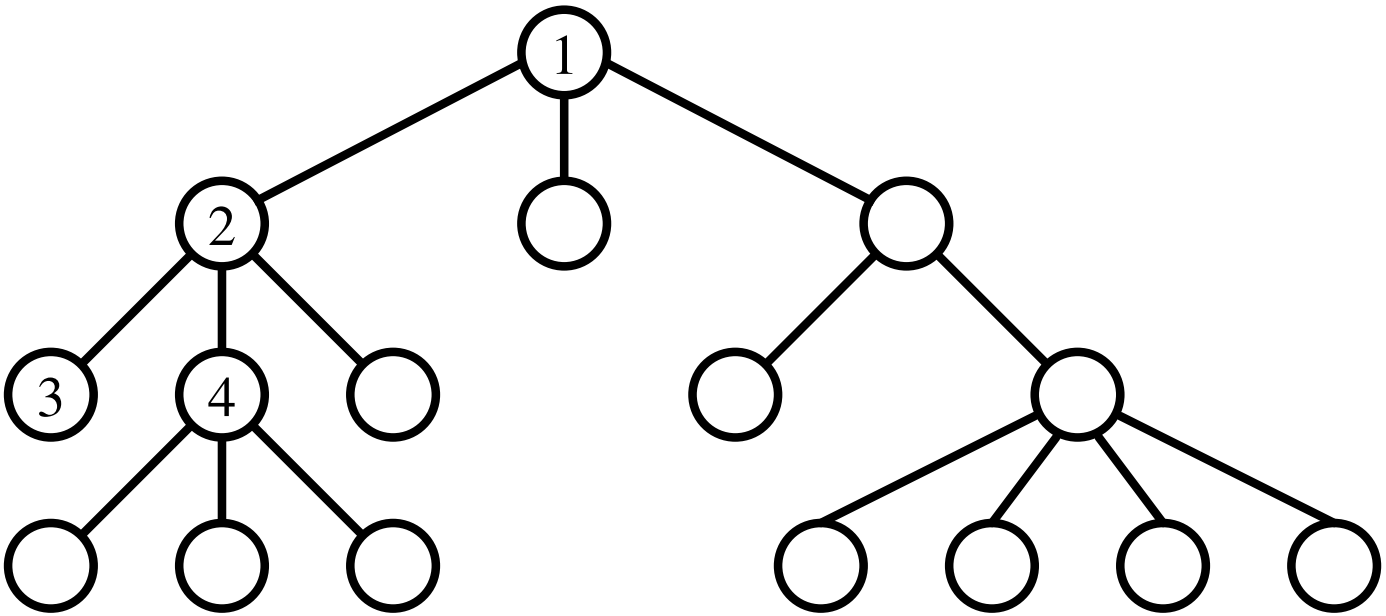
- Visit 2's first child (in left-to-right order), and assign it the number 3.

Depth first search of a tree



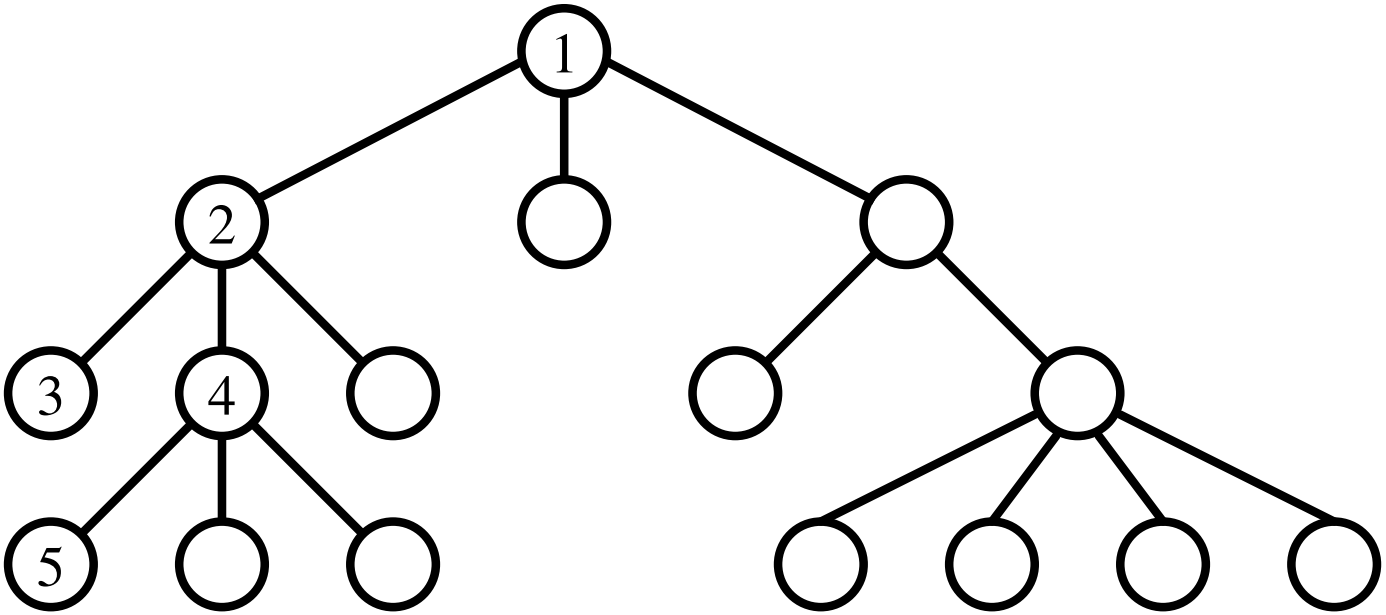
- 3 is a leaf, with no children!
- We explored as far as possible, but now have to backtrack to explore more.
- Back up how we came until a vertex (2) with at least one unnumbered child.

Depth first search of a tree



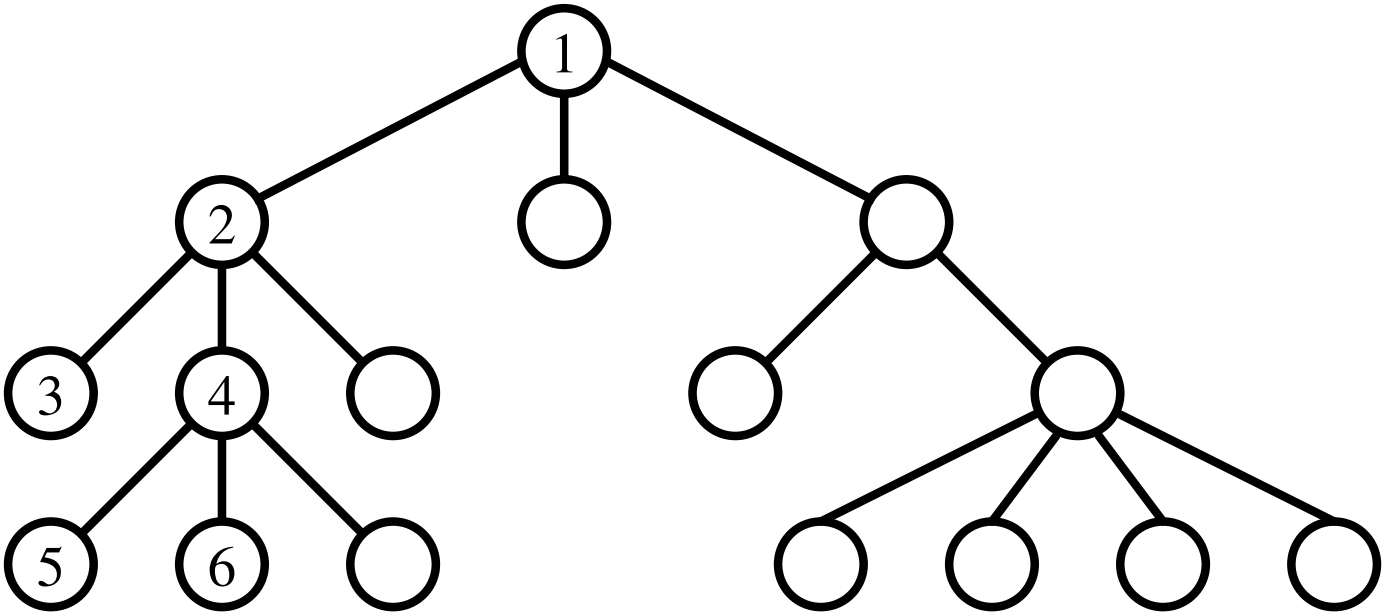
- The next child of 2 is numbered 4. Continue going down this branch, choosing the leftmost option available.

Depth first search of a tree



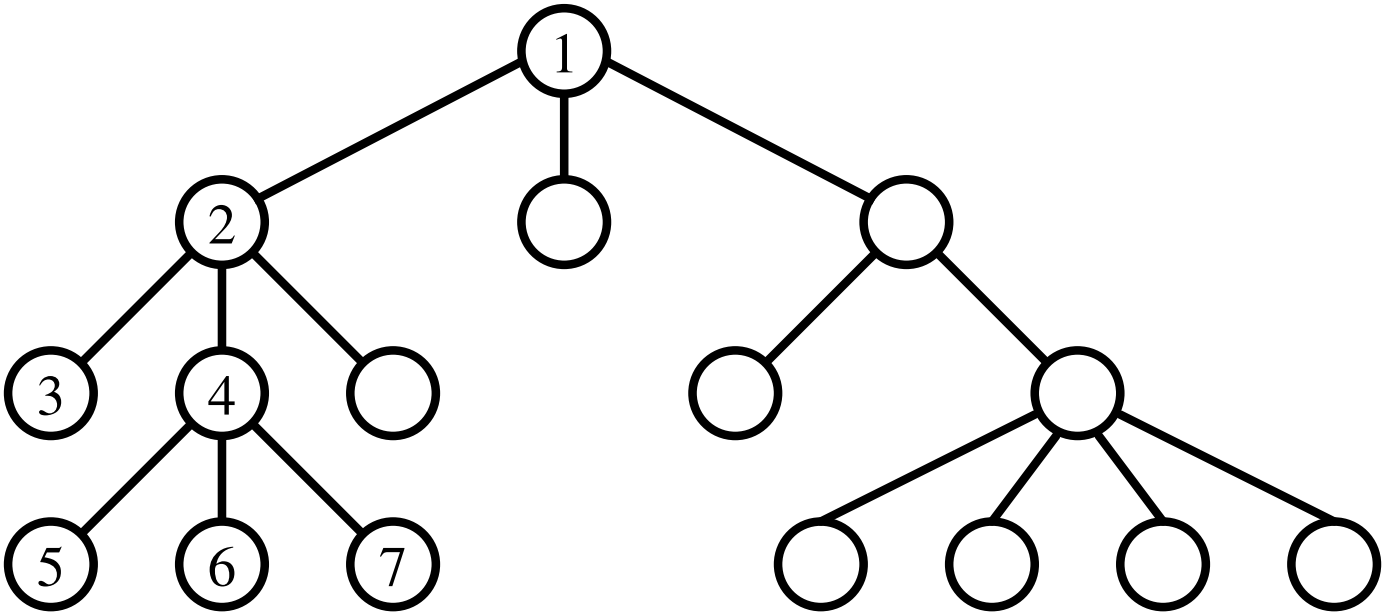
- Visit 4's first child and assign it the number 5.

Depth first search of a tree



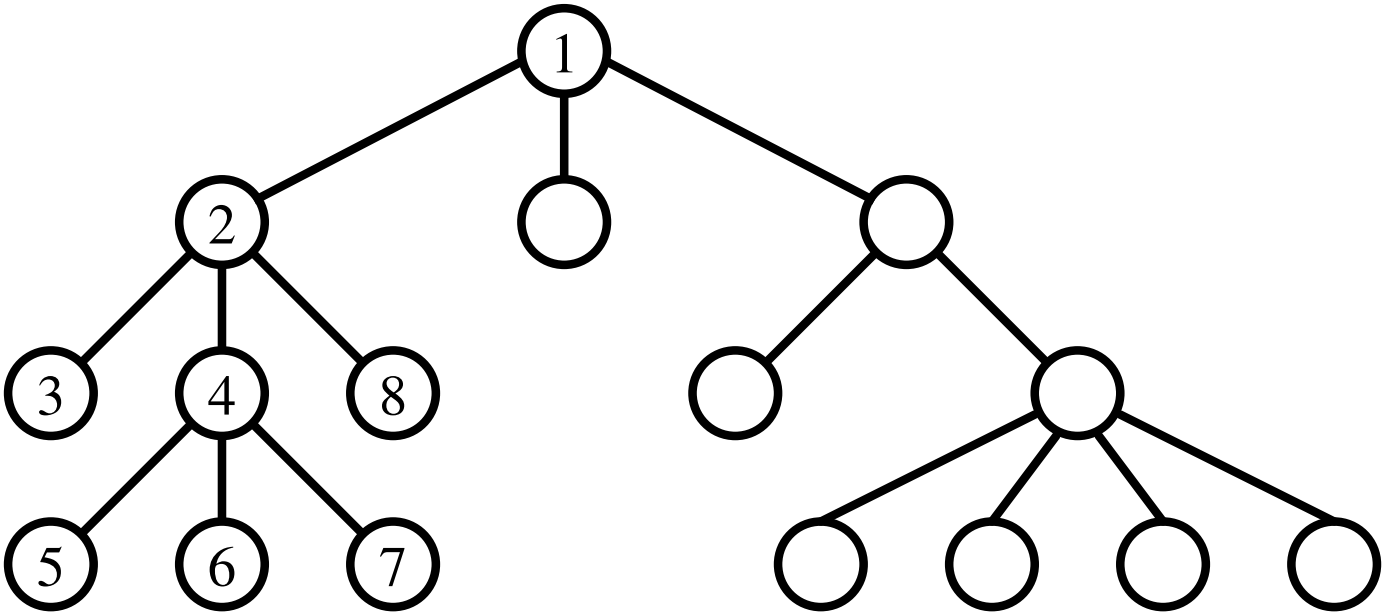
- 5 has no children.
- Back up to the first vertex (4) that has a child not yet numbered.
- Assign the first remaining child the number 6.

Depth first search of a tree



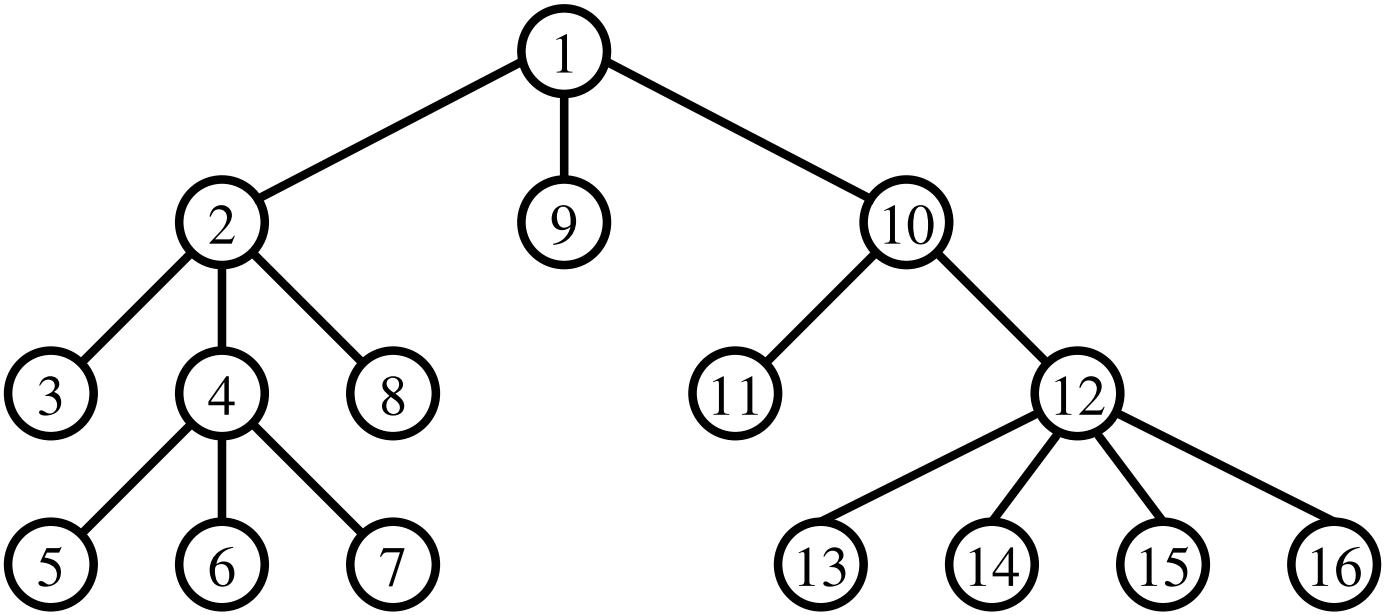
- 6 has no children.
- Back up to the first vertex (4) that has a child not yet numbered.
- Assign the first remaining child the number 7.

Depth first search of a tree



- 7 has no children.
- Back up to the first vertex (2) that has a child not yet numbered.
- Assign the first remaining child the number 8.

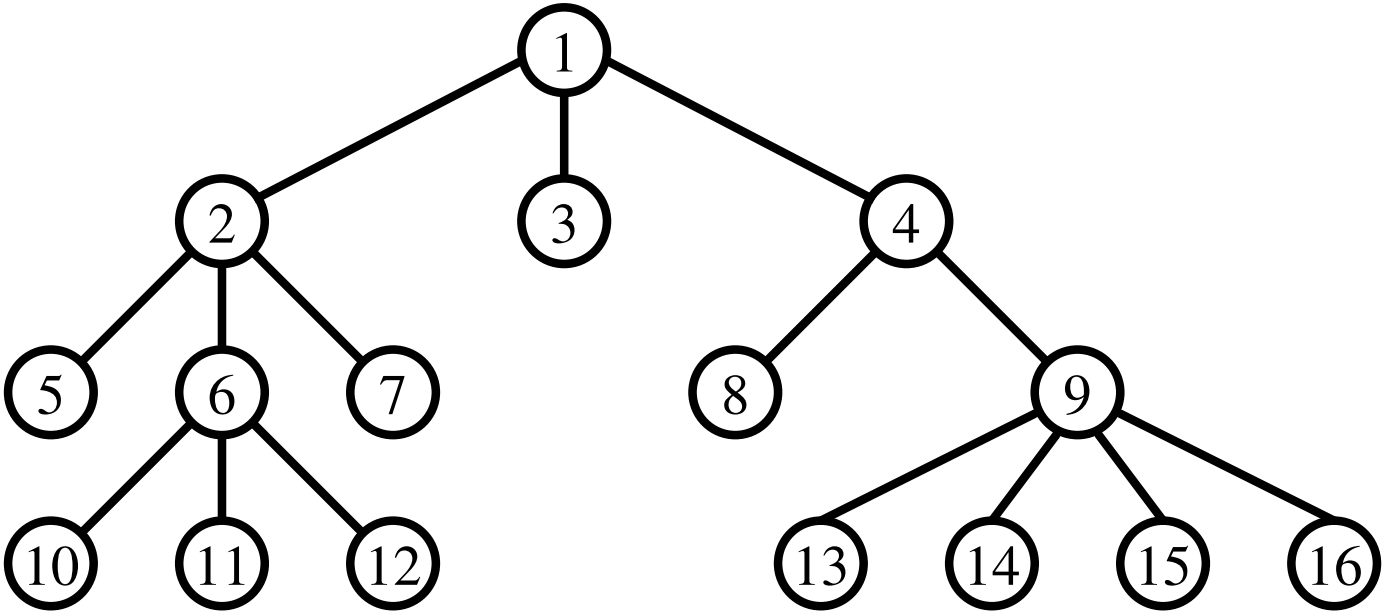
Depth first search of a tree



- Continue in this fashion until all vertices are numbered.

Breadth first search of a tree

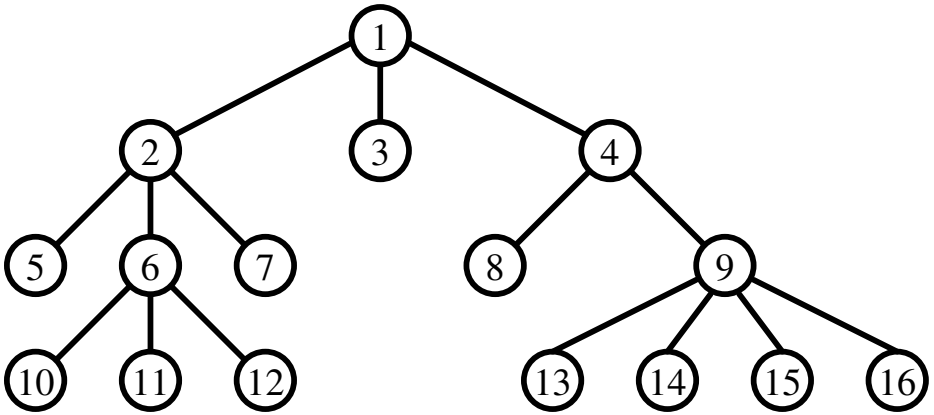
Breadth first search of a tree



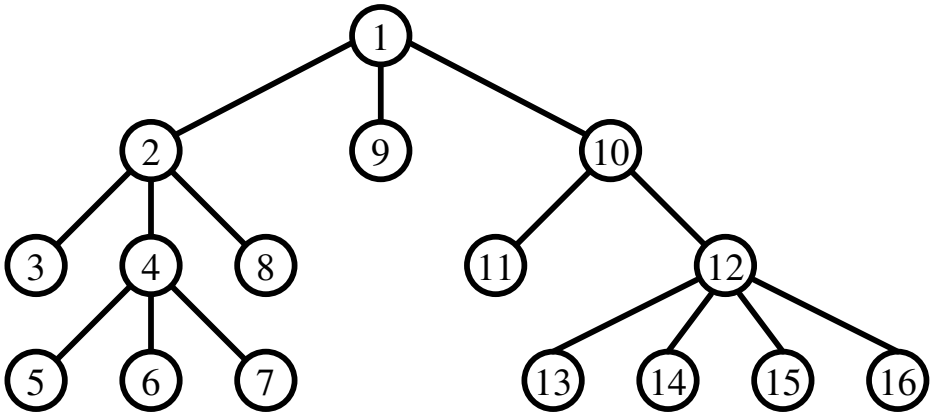
- While depth first search explores as deep as possible, *Breadth first search* (BFS) works one layer (depth) at a time.
- **Depth 0:** Number the root 1.
- **Depth 1:** Consecutively number all neighbors of 1.
- **Depth 2:** The depth 2 elements are all the neighbors of the depth 1 elements not yet accounted for. Number them consecutively.
- Continue in this way for depth 3, 4, etc.

Breadth first search of a tree

Depth first search

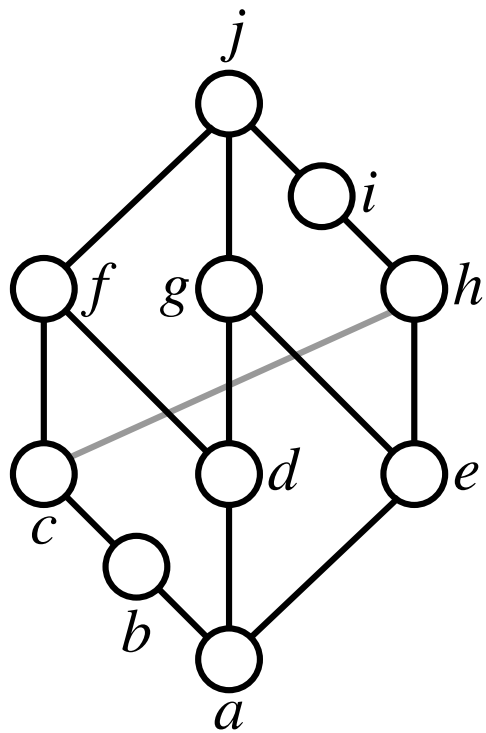


Breadth first search



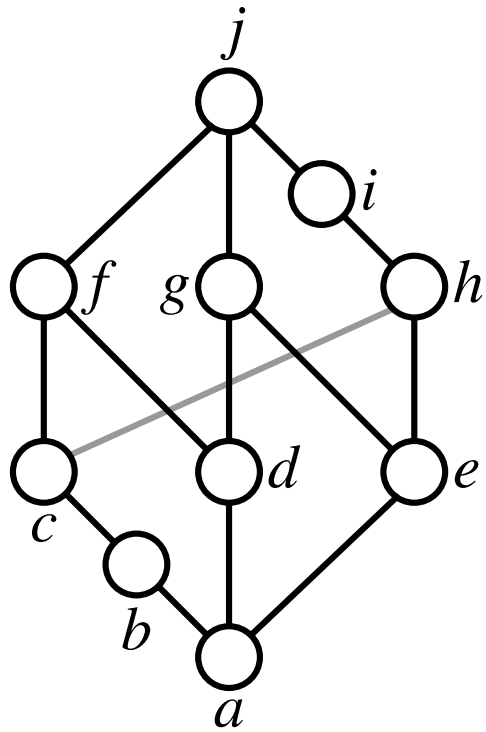
Depth first search (DFS) in a connected graph

Depth first search (DFS) in a connected graph



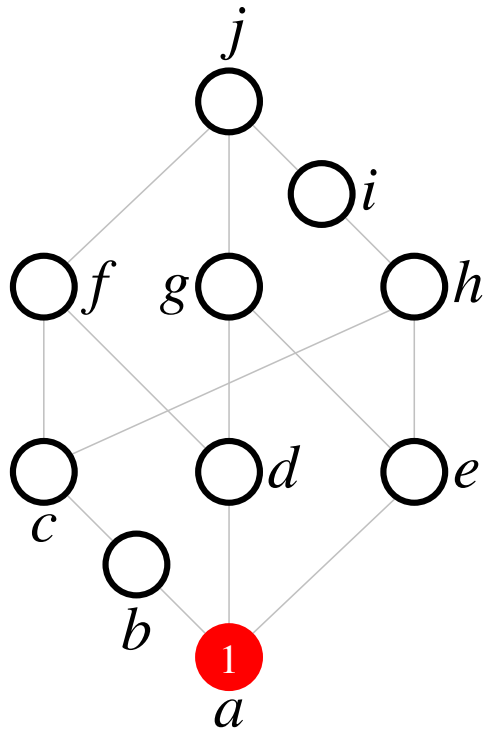
- For trees, we showed the root at the top, and children below in left-to-right order. That doesn't apply for arbitrary graph drawings.

Depth first search (DFS) in a connected graph



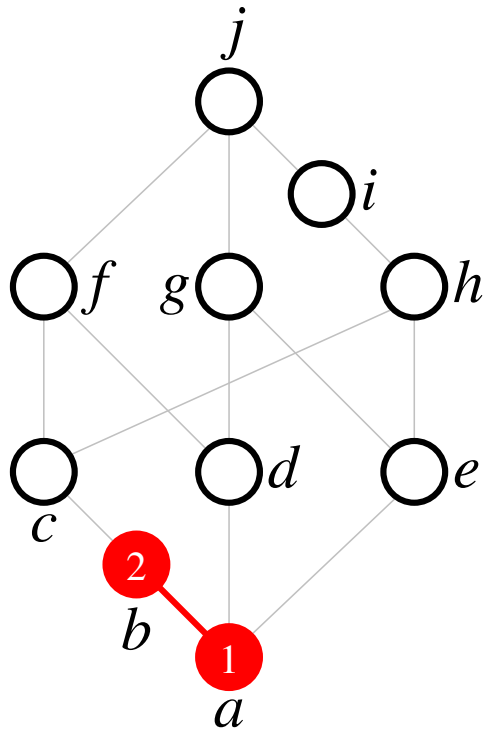
- **Set up a counter:** time $= 0$
  - We'll keep adding 1 to it as we number the vertices.
- **Pick starting vertex:** a
  - The starting vertex can be *any* vertex; a is just an example. You may get a different tree, depending on where you start.
  - This takes the place of the root in a rooted tree.
  - In some applications, it's called the *source*.

Depth first search (DFS) in a connected graph



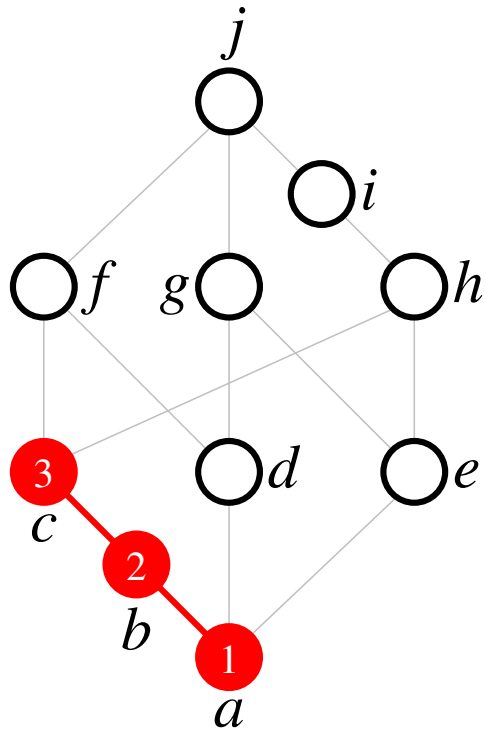
- **Current vertex:** $u = a$
- **Color u red:** a is *discovered*
- **Time stamp u:** $T(a) = \text{time} = 1$
- **Neighbors of u:** b, d, e (we'll use alphabetical order)
- **1st undiscovered neighbor:** $v = b$
- **Draw red edge $\{u, v\}$:** $\{a, b\}$
- **Continue exploring:** DFS(b)

Depth first search (DFS) in a connected graph



- **Current vertex:** $u = b$
- **Color u red:** b is *discovered*
- **Time stamp u:** $T(b) = \text{time} = 2$
- **Neighbors of u:** a, c
- **1st undiscovered neighbor:** $v = c$
- **Draw red edge $\{u, v\}$:** $\{b, c\}$
- **Continue exploring:** DFS(c)

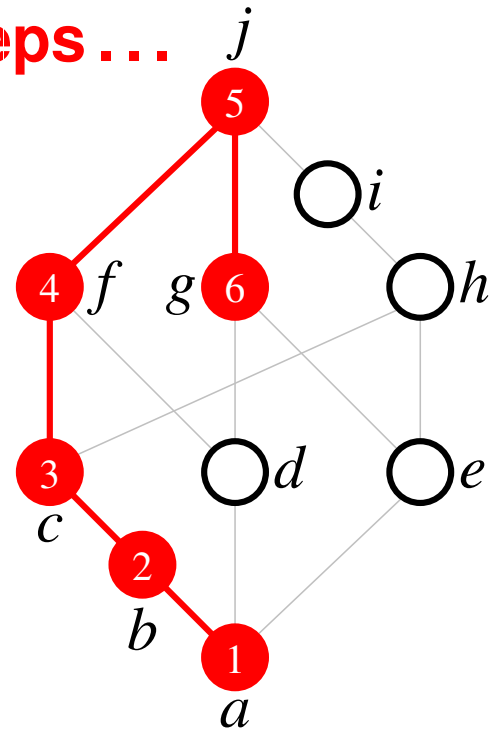
Depth first search (DFS) in a connected graph



- **Current vertex:** $u = c$
- **Color u red:** c is *discovered*
- **Time stamp u:** $T(c) = \text{time} = 3$
- **Neighbors of u:** b, f, h
- **1st undiscovered neighbor:** $v = f$
- **Draw red edge $\{u, v\}$:** $\{c, f\}$
- **Continue exploring:** DFS(f)

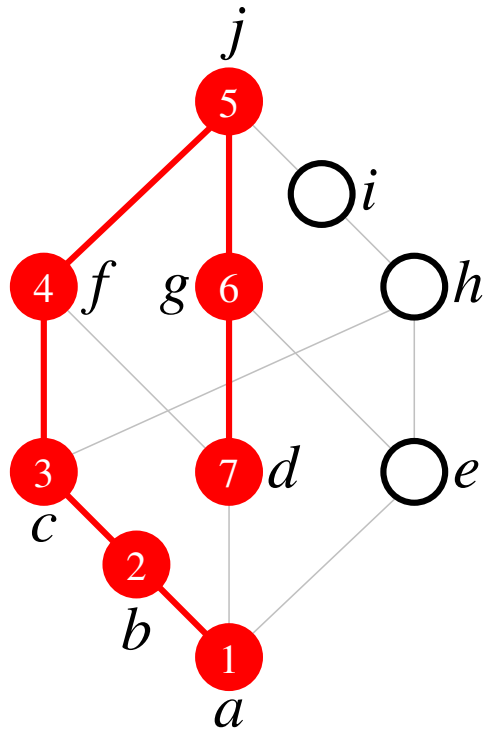
Depth first search (DFS) in a connected graph

Skipping ahead a few steps . . .



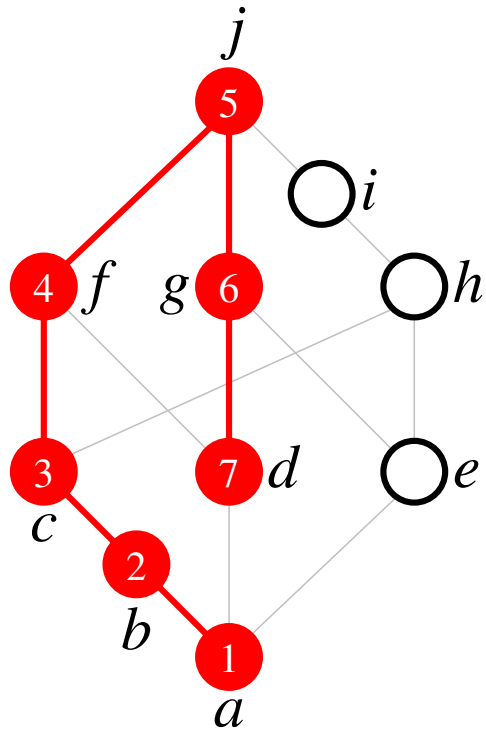
- **Current vertex:** $u = g$
- **Color u red:** g is *discovered*
- **Time stamp u:** $T(g) = \text{time} = 6$
- **Neighbors of u:** d, e, j
- **1st undiscovered neighbor:** $v = d$
- **Draw red edge $\{u, v\}$:** $\{g, d\}$
- **Continue exploring:** DFS(d)

Depth first search (DFS) in a connected graph



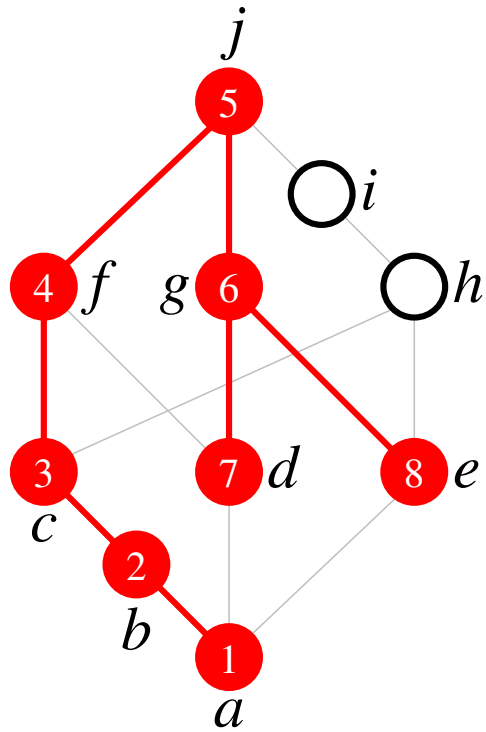
- **Current vertex:** $u = d$
- **Color u red:** d is *discovered*
- **Time stamp u:** $T(d) = \text{time} = 7$
- **Neighbors of u:** a, f, g
- But all neighbors of u have already been discovered!
- *Backtrack* to find a vertex (g) with an undiscovered neighbor.

Depth first search (DFS) in a connected graph



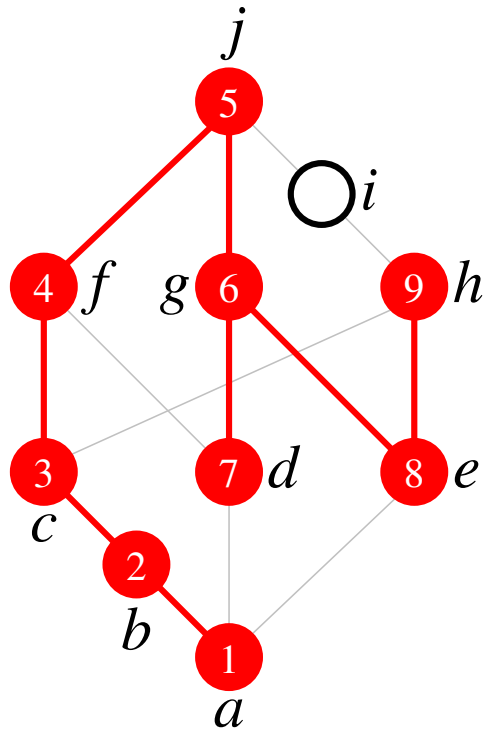
- **Current vertex:** $u = g$
- **Neighbors of u:** d, e, j
- **1st undiscovered neighbor:** $v = e$
- **Draw red edge $\{u, v\}$:** $\{g, e\}$
- **Continue exploring:** DFS(e)

Depth first search (DFS) in a connected graph



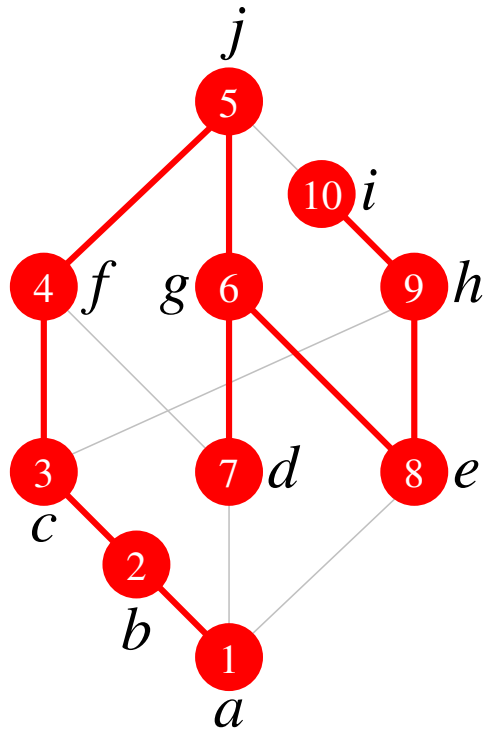
- **Current vertex:** $u = e$
- **Color u red:** e is *discovered*
- **Time stamp u:** $T(e) = \text{time} = 8$
- **Neighbors of u:** a, g, h
- **1st undiscovered neighbor:** $v = h$
- **Draw red edge $\{u, v\}$:** $\{e, h\}$
- **Continue exploring:** DFS(h)

Depth first search (DFS) in a connected graph



- **Current vertex:** $u = h$
- **Color u red:** h is *discovered*
- **Time stamp u:** $T(h) = \text{time} = 9$
- **Neighbors of u:** c, e, i
- **1st undiscovered neighbor:** $v = i$
- **Draw red edge $\{u, v\}$:** $\{h, i\}$
- **Continue exploring:** DFS(i)

Depth first search (DFS) in a connected graph



- **Current vertex:** $u = i$
- **Color u red:** i is *discovered*
- **Time stamp u:** $T(i) = \text{time} = 10$
- **Neighbors of u:** h, j
- All neighbors are already discovered.
- Backtracking doesn't give any new branch, so we're done.

Depth first search pseudocode, using recursion

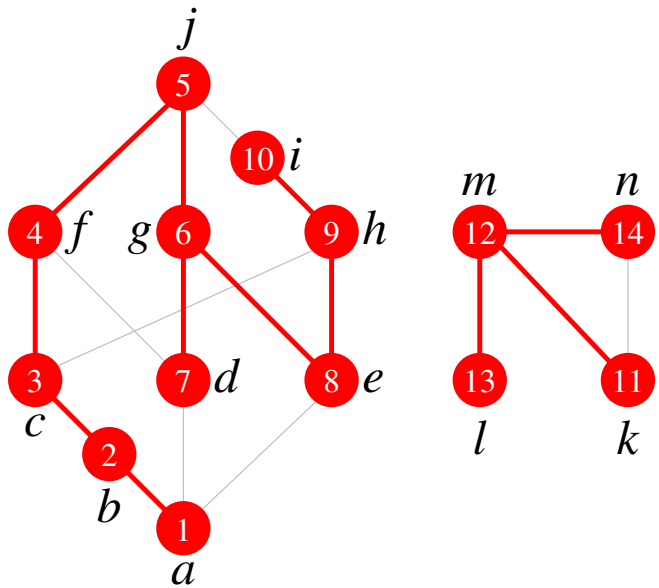
Initialize: discovered$[\cdots] \leftarrow$ false
parent$[\cdots] \leftarrow$ null
$T[\cdots] \leftarrow \infty$ ▷ Time stamp
time $\leftarrow 0$

Start with: DFS(root)

procedure DFS(u)

```
discovered[u] ← true          ▷ We did this by coloring u red
time ← time + 1               ▷ Visit u by doing something with it,
T[u] ← time                   ▷ such as recording a time stamp.
for all v ∈ N(u) do
    if (not discovered[v]) then
        parent[v] ← u         ▷ Add tree edge
        DFS(v)                ▷ Recursively explore further
```

Extending traversals to disconnected graphs



- DFS only finds vertices reachable from the source.
- To find a *spanning forest* for a disconnected graph, loop over all vertices. If a vertex isn't marked as discovered, do DFS starting there to get a spanning tree for that component. This also lets you count the connected components.
- This also applies to breadth first search (BFS), coming up next.
- DFS and BFS also work for a directed graph (explore $v \in N^+(u)$). Similar issues arise if it's not strongly connected.

Breadth first search (BFS) in a connected graph

Queues

- Consider customers waiting in a line, called a *queue*.

  Queue: B, C, E, A, D, H

- The first customer is B. We *dequeue* B and process their order.

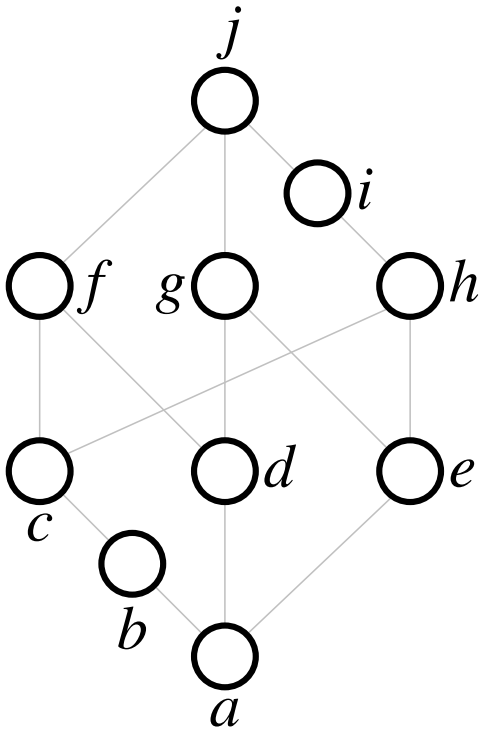
  Queue: C, E, A, D, H

- While processing B, customer J comes along and is added to the end of the queue (called *enqueuing*); no cuts allowed!

  Queue: C, E, A, D, H, J

- In Computer Science, a *queue* is a data structure that works in the same way.
  - New items are added to the end of the queue (*enqueued*).
  - Items at the front of the line are *dequeued* and processed.
  - We don't have the complete sequence when we start. The queue grows and shrinks over time as items are enqueued and dequeued.

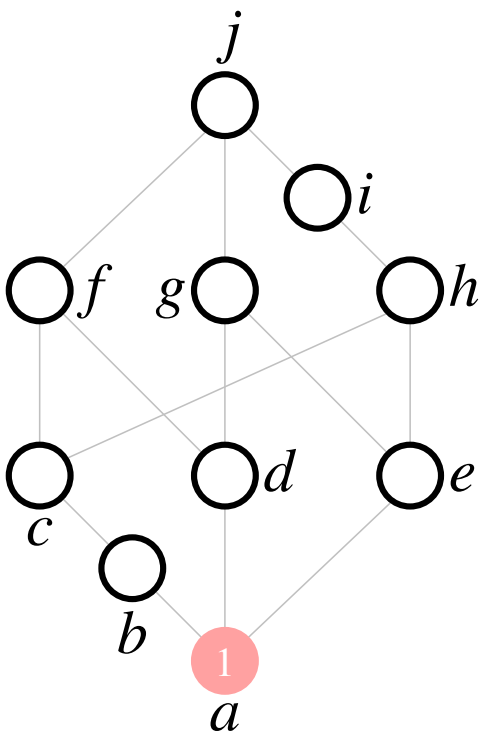
Breadth first search (BFS) in a connected graph



$\boldsymbol{Q}$: a
$\boldsymbol{u}$:

- Pick a starting vertex. We'll use a.
- Add a to the end of the queue, Q, and mark a as *discovered*.
  - The undiscovered nodes are white.
  - The discovered nodes are pink. By hand, just make a small mark like a dot.

Breadth first search (BFS) in a connected graph

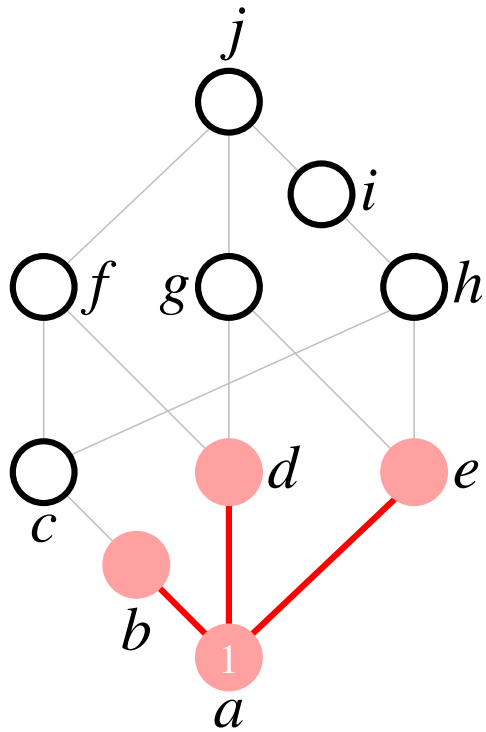


$\boldsymbol{Q}$: ~~a~~

$\boldsymbol{u}$: a

- **Dequeue a vertex:** $u = a$.
- **Time stamp $\boldsymbol{u}$:** $T(a) = \text{time} = 1$

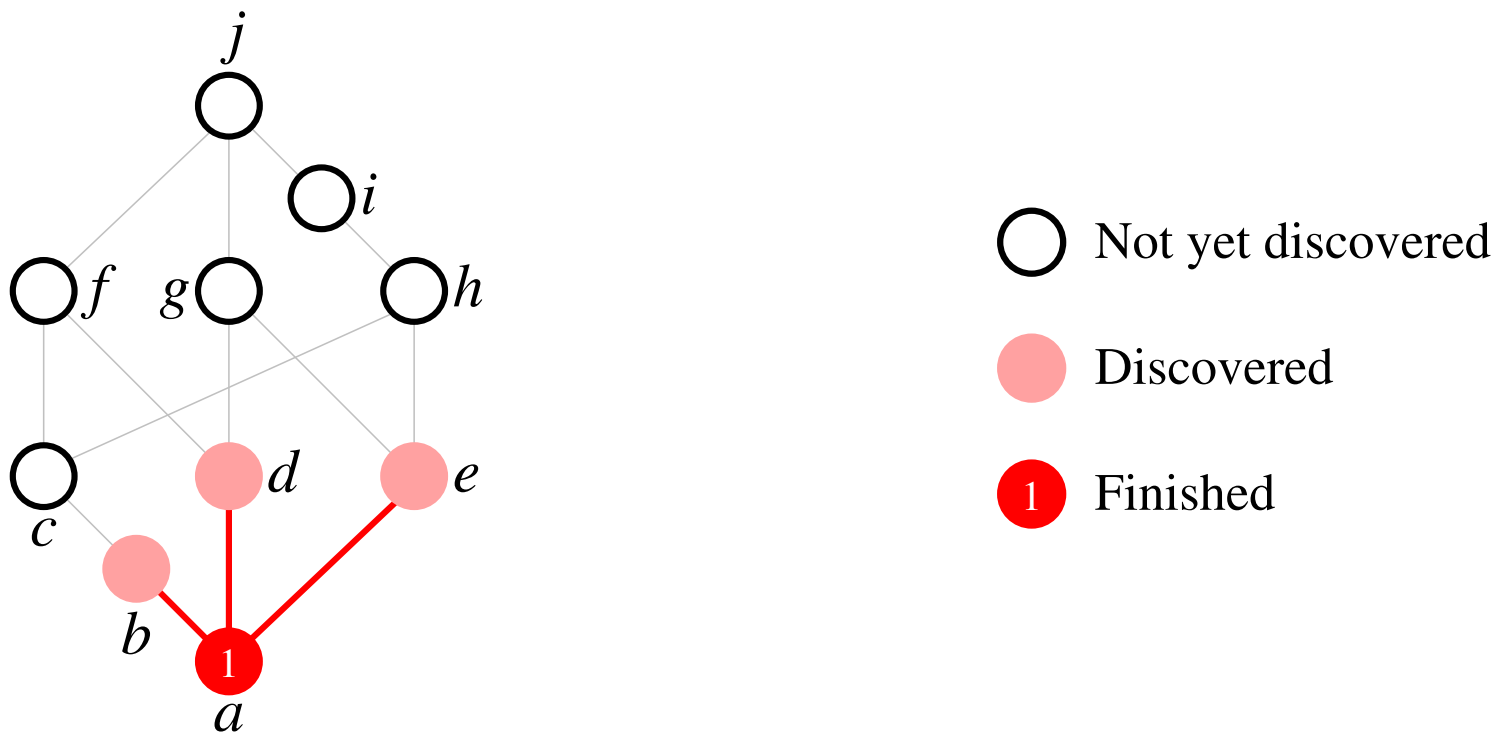
Breadth first search (BFS) in a connected graph



$\boldsymbol{Q}$: $\cancel{a}, b, d, e$
$\boldsymbol{u}$: a

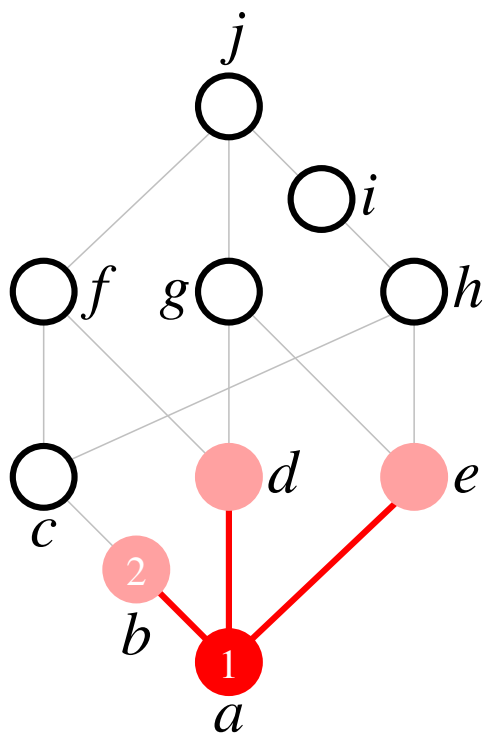
- Add any undiscovered neighbors of u to the end of the queue: b, d, e. This is called *enqueueing*.
- Mark those neighbors as discovered (color them pink).
- Add an edge from u to each of those neighbors to the tree (color the edges red).

Breadth first search (BFS) in a connected graph



- Mark a as *finished* (we'll color it red).
- **By hand:** Don't literally color vertices. Just make a mark (like a dot) to show discovery, and write the number to show it's finished.
- **In software:**
  - There's a variable `discovered` or `visited` for each vertex.
  - Various algorithms using DFS or BFS use 0, 2, or 3 of these states.

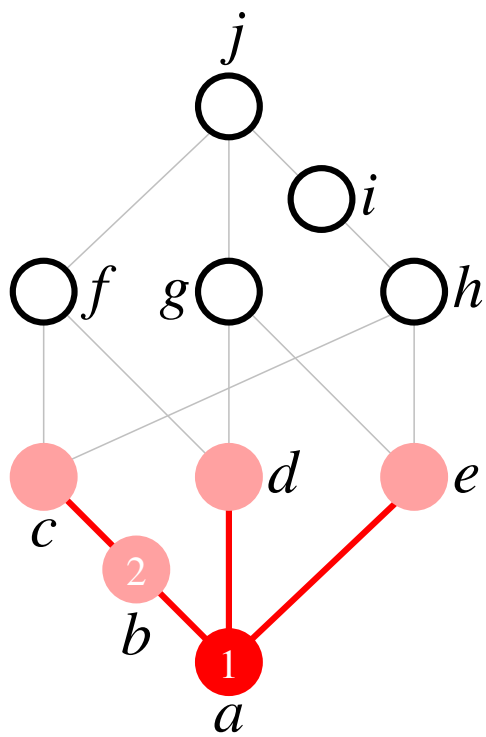
Breadth first search (BFS) in a connected graph



$\boldsymbol{Q}$: ~~a~~, ~~b~~, d, e
$\boldsymbol{u}$: b

- **Dequeue:** $u = b$
- **Time stamp u:** $T(b) = \text{time} = 2$
- **Undiscovered neighbors:** c

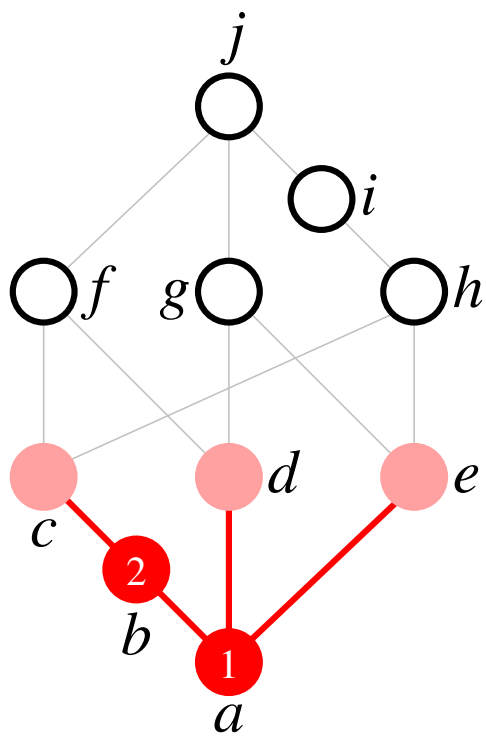
Breadth first search (BFS) in a connected graph



$\boldsymbol{Q}$: ~~a~~, ~~b~~, d, e, c
$\boldsymbol{u}$: b

- **Dequeue:** $u = b$
- **Time stamp $\boldsymbol{u}$:** $T(b) = \text{time} = 2$
- **Undiscovered neighbors:** c
- **Append them to queue, mark them discovered (pink), and add tree edges to them (red)**

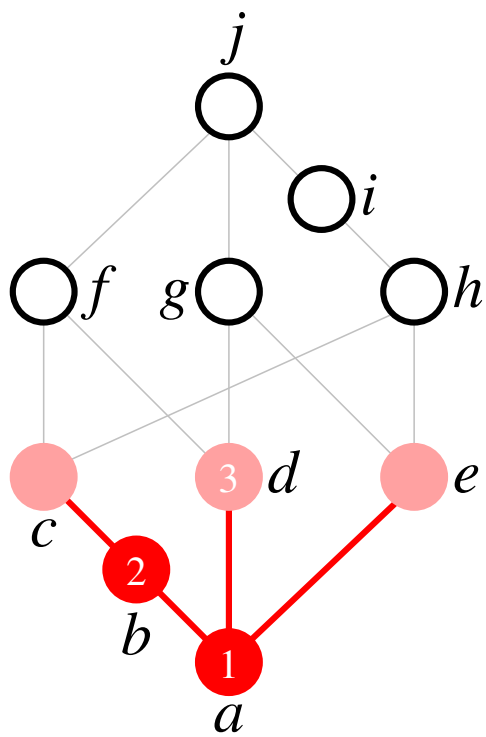
Breadth first search (BFS) in a connected graph



$\boldsymbol{Q}$: ~~a~~, ~~b~~, d, e, c
$\boldsymbol{u}$: b

- **Dequeue:** $u = b$
- **Time stamp u:** $T(b) = \text{time} = 2$
- **Undiscovered neighbors:** c
- **Append them to queue, mark them discovered (pink), and add tree edges to them (red)**
- **Mark u finished (red)**

Breadth first search (BFS) in a connected graph

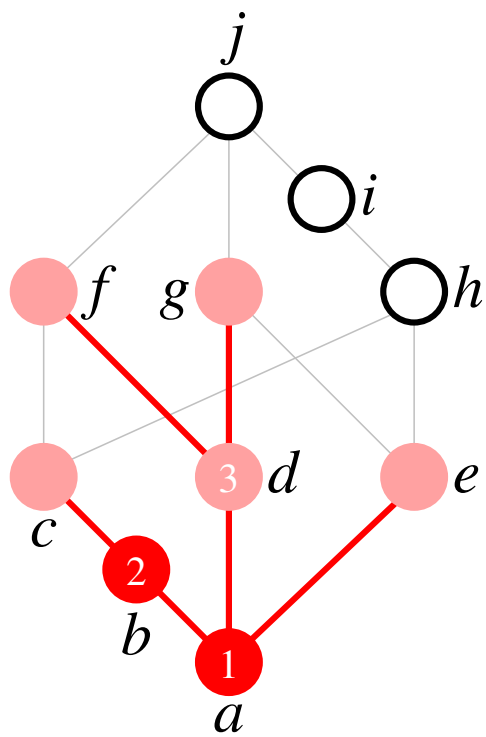


$\boldsymbol{Q}$: ~~a~~, ~~b~~, ~~d~~, e, c

$\boldsymbol{u}$: d

- **Dequeue:** $u = d$
- **Time stamp u:** $T(d) = \text{time} = 3$
- **Undiscovered neighbors:** f, g

Breadth first search (BFS) in a connected graph



$\boldsymbol{Q}$: ~~a~~, ~~b~~, ~~d~~, e, c, f, g
$\boldsymbol{u}$: d

- **Dequeue:** $u = d$
- **Time stamp $\boldsymbol{u}$:** $T(d) = \text{time} = 3$
- **Undiscovered neighbors:** f, g
- **Append them to queue, mark them discovered (pink), and add tree edges to them (red)**

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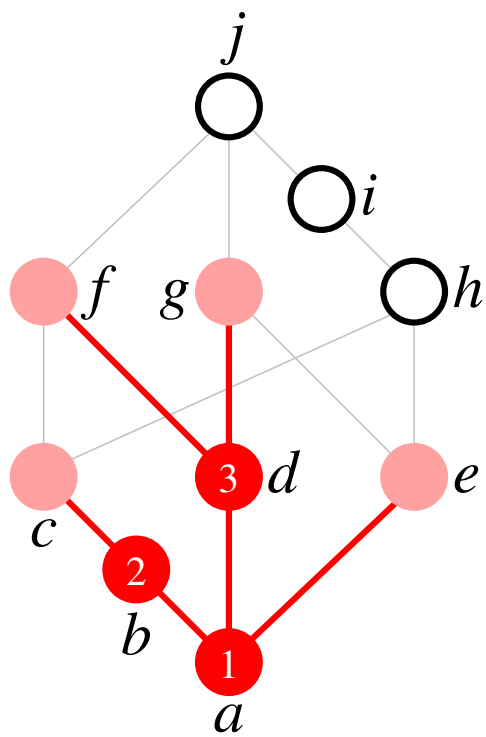

$\boldsymbol{Q}$: ~~a~~, ~~b~~, ~~d~~, e, c, f, g
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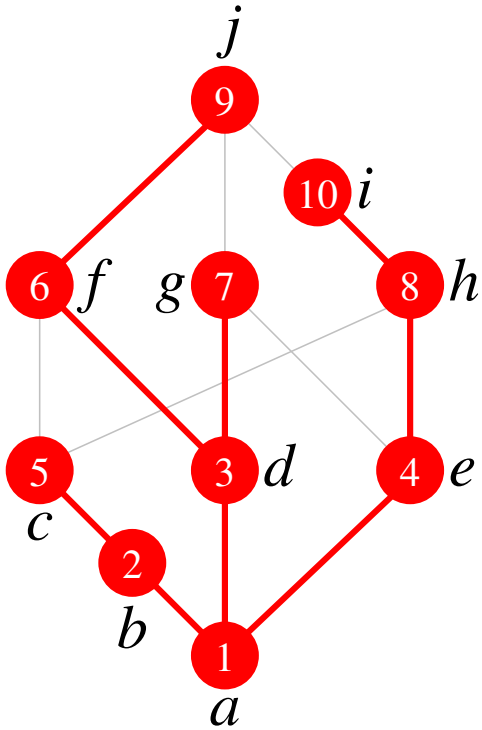
- **Time stamp u:** $T(d) = \text{time} = 3$
- **Undiscovered neighbors:** f, g
- **Append them to queue, mark them discovered (pink), and add tree edges to them (red)**
- **Mark u finished (red)**

Breadth first search (BFS) in a connected graph

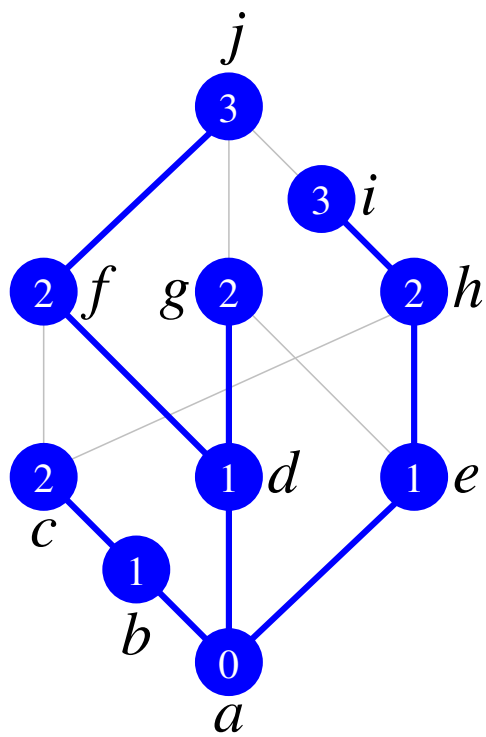


Q: ~~a~~, ~~b~~, ~~d~~, ~~e~~, ~~c~~, ~~f~~, ~~g~~, ~~h~~, ~~j~~, ~~i~~

u:

- Continue until forced to stop (no element to dequeue).
- All vertices reachable from a are included in the tree. If it's a connected graph, then it's a spanning tree reaching all vertices.

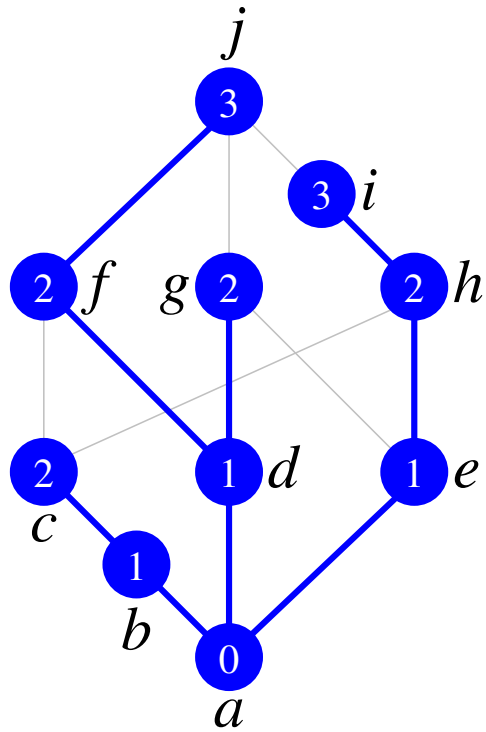
Distance (in edges) from a vertex to all other vertices



***Q*:**	a	b	d	e	c	f	g	h	j	i
Depth:	0	1	1	1	2	2	2	2	3	3

- The BFS queue is in weakly increasing order of distance from a.
- Instead of marking time stamps, we could have marked the distance from a ($=$ parent's distance $+ 1 =$ depth in tree).
- Processing vertices in order by layer assures we get shortest paths from a, although there may be ties for the shortest path to each vertex, e.g., a, d, g and a, e, g.

Distance (in edges) from a vertex to all other vertices



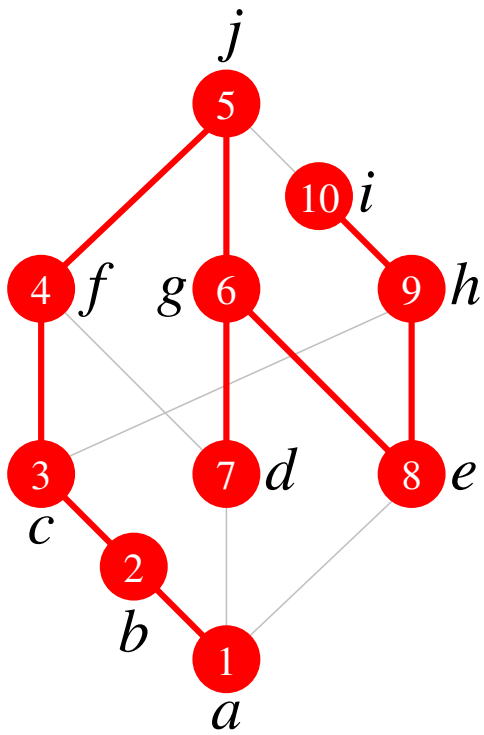
- Let $N_i(v)$ be the set of all vertices of G a distance i from v:

$$N_0(a) = \{a\} \qquad N_2(a) = \{c, f, g, h\}$$
$$N_1(a) = \{b, d, e\} \qquad N_3(a) = \{i, j\}$$

- We used breadth first search to compute that for $v = a$.

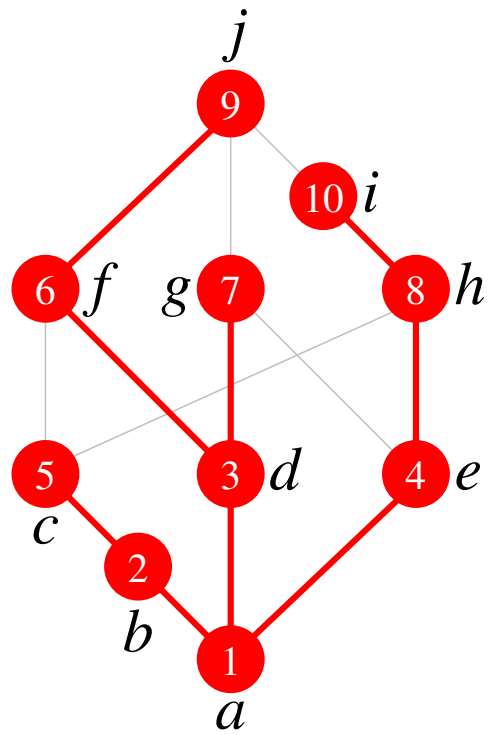
DFS vs. BFS

DFS from a



depth $= 8$

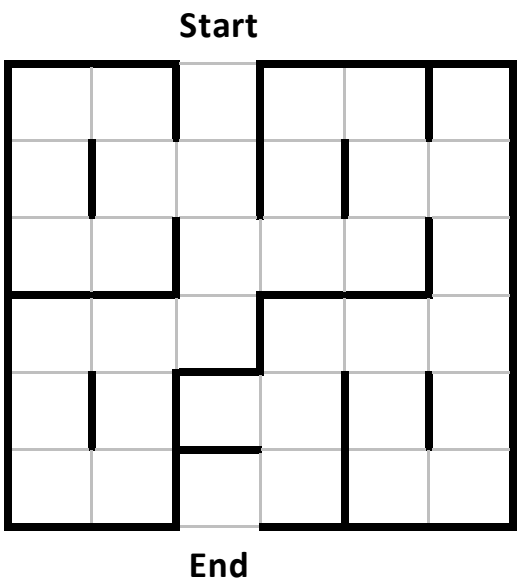
BFS from a



depth $= 3$

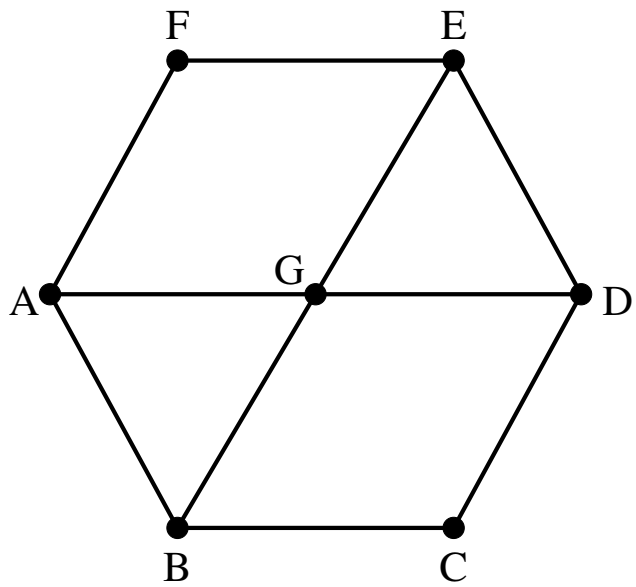
- DFS tends to give longer paths, and to branch out less.
- BFS tends to give shorter paths, and to branch out more.

Maze

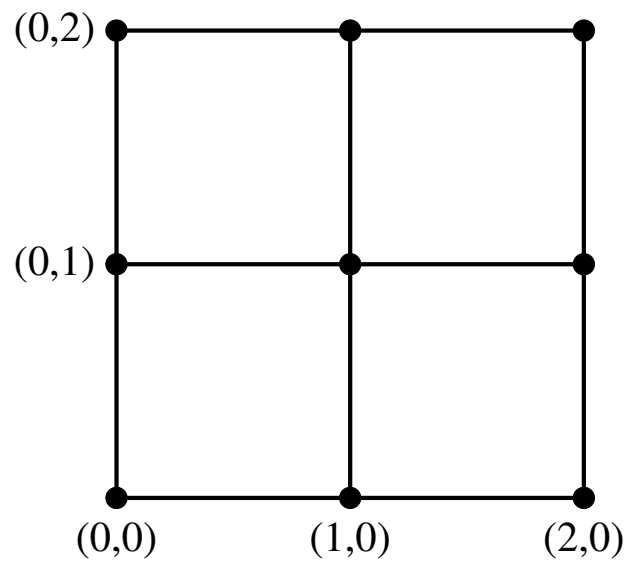


- A maze can be represented by a graph.
- Vertex for each cell and for **Start** and **End**.
- Edge between adjacent cells w/o wall in-between.
- Pick an ordering of neighbors of the cells:
  - Could go up, left, down, or right one cell.
  - Or, *best first:* Use a heuristic to guess best neighbor (may not be right). Order neighbors by distance to **End**, using Manhattan distance
    $d((x,y),(x',y')) = |x-x'| + |y-y'|$
    Break ties with a rule like down, left, right, up.
- Use DFS or BFS. But instead of exploring all cells, you can stop when you reach the goal.
- DFS may be better since you need to go deep rather than to find the shortest solution.

Diameter and radius of an undirected graph



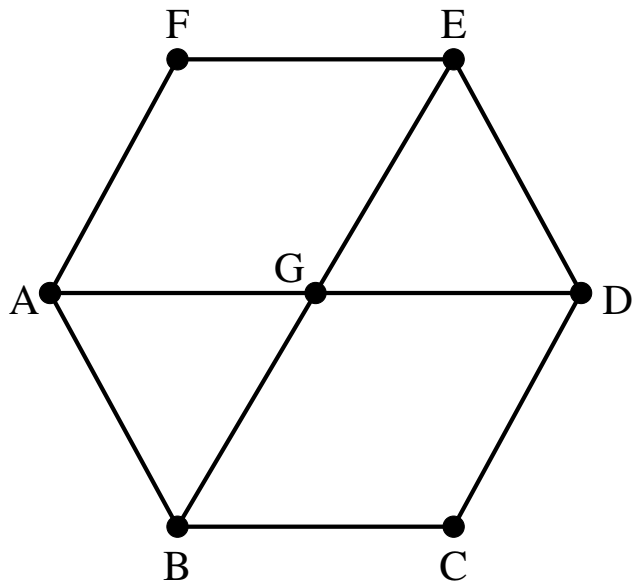
diameter 3
radius 2



diameter 4
radius 2

- The *eccentricity* of a vertex is the largest distance from the vertex.
  - Eccentricity of A is 2.
  - Eccentricity of F is 3.
- The BFS algorithm we just used gives all the distances from a vertex to the other vertices, so it can be used to compute this.

Diameter and radius of an undirected graph



diameter 3
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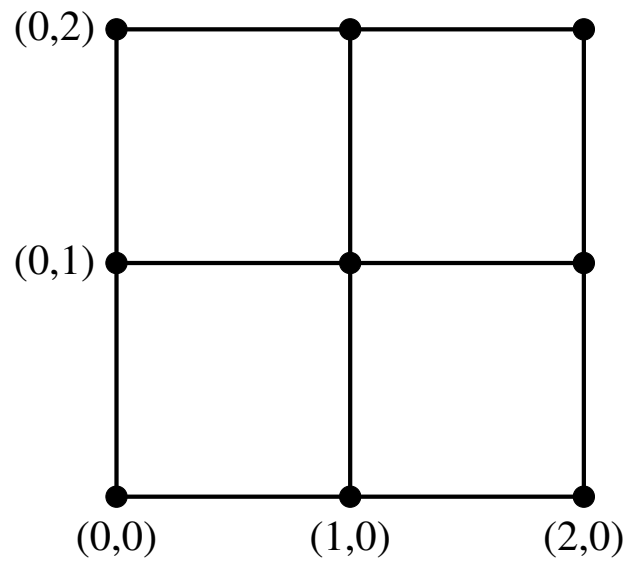


diameter 4
radius 2

- The *diameter* of a graph is maximum eccentricity: it's the largest distance between two vertices.
- The *radius* of a graph is the minimum eccentricity.

- **Left graph:** diameter 3 (from C to F); radius 2 (using G)
- **Right graph:** diameter 4 (from $(0,0)$ to $(2,2)$); radius 2 (using $(1,1)$).

Is a connected graph bipartite?

First solution

- We'll label all vertices as being in part A or part B.
- Pick a starting vertex and label it A.
- Use DFS, BFS, or any other traversal that adds on one edge at a time to form a spanning tree.
- As you add new vertices, label them A/B opposite of their parent.
- As you explore neighbors of u, if any neighbor v was already discovered and has the same label A/B as u, then it's not bipartite.
- If it is bipartite, then A and B are its two parts.

Is a connected graph bipartite?

Second solution

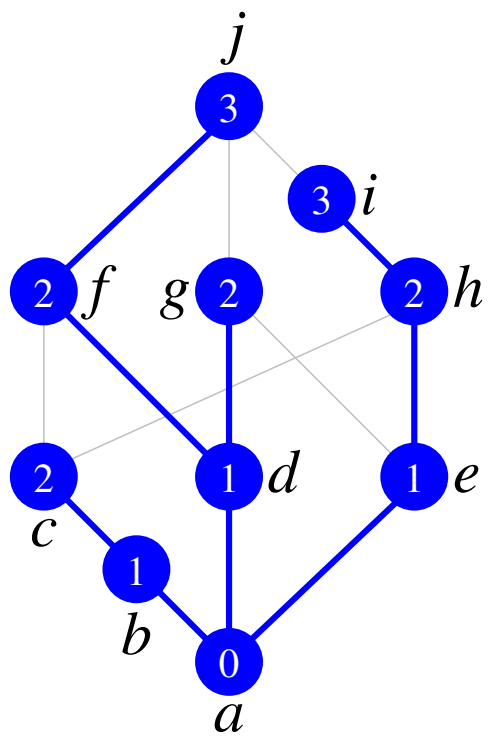
- Pick any starting vertex, u.
- Do BFS starting at u, and set

$$A = N_0(u) \cup N_2(u) \cup N_4(u) \cup \cdots \quad \text{(even distance)}$$
$$B = N_1(u) \cup N_3(u) \cup N_5(u) \cup \cdots \quad \text{(odd distance)}$$

- In BFS, all edges either connect vertices at two consecutive layers, or two vertices in the same layer.
  E.g., if $d(u, v) = 8$ and $\{v, w\}$ is an edge, then $d(u, w) \in \{7, 8, 9\}$.
- If there is any edge between two vertices of A (or two vertices of B), then there is an odd-length cycle, so it's not bipartite.
- Otherwise, the graph is bipartite and A and B are its two parts.

Is a connected graph bipartite?

Second solution



- Vertices are marked with their distance from a, found from BFS.
- There are edges between vertices at the same level (such as $\{c,f\}$ in $N_2(a)$, as well as others), so it's not bipartite.

Is a disconnected graph bipartite?

- Apply either procedure starting from any vertex in each component.
- If any component isn't bipartite, then the graph isn't bipartite.
- Otherwise, A and B gives a bipartition of the graph.
- Inverting A/B in any component gives other solutions.

Does a graph have a cycle?

- **Undirected simple graph:** In BFS or DFS, when checking the neighbors of a vertex, if any neighbor besides the parent was already discovered, then there's a cycle.
- **Multigraphs/pseudographs:** Loops are cycles of length 1. Multiple edges give cycles of length 2. If there are no loops or multiple edges, use the test for a simple graph.
- **Directed graph:** DFS can be used to determine if there's a directed cycle, but it's more involved.
  - Use DFS with three vertex colors.
  - Initialize all vertices to white (not discovered).
  - When entering a vertex, color it pink (discovered).
  - It stays pink while recursing to its out-neighbors.
  - After returning from the recursion, color it red (finished).
  - When checking out-neighbors of a vertex, if any is pink (discovered), it must be an ancestor in the tree, so there's a cycle.