Chapter 7
Planar graphs

In full: 7.1–7.3
Parts of: 7.4, 7.6–7.8
Skip: 7.5

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Planar graphs

**Definition**

- A *planar embedding* of a graph is a drawing of the graph in the plane without edges crossing.
- A graph is *planar* if a planar embedding of it exists.

Consider two drawings of the graph $K_4$:

- $V = \{1, 2, 3, 4\}$
- $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$

The abstract graph $K_4$ is planar because it can be drawn in the plane without crossing edges.
How about $K_5$?

- Both of these drawings of $K_5$ have crossing edges.
- We will develop methods to prove that $K_5$ is not a planar graph, and to characterize what graphs are planar.
Euler’s Theorem on Planar Graphs

- Let $G$ be a connected planar graph (drawn w/o crossing edges).
- Define $V =$ number of vertices
  $E =$ number of edges
  $F =$ number of faces, including the “infinite” face
- Then $V - E + F = 2$.
- **Note:** This notation conflicts with standard graph theory notation $V$ and $E$ for the sets of vertices and edges. Alternately, use $|V(G)| - |E(G)| + |F(G)| = 2$.

Example

- $V = 4$
- $E = 6$
- $F = 4$
- $V - E + F = 4 - 6 + 4 = 2$
Euler’s formula for planar graphs

- $V = 10$
- $E = 15$
- $F = 7$

$V - E + F = 10 - 15 + 7 = 2$
A *spanning tree* of a connected graph is a subgraph that’s a tree reaching all vertices. An example is highlighted in red.

**Algorithm to get a spanning tree of any connected graph:**
Repeatedly pick a cycle and remove an edge, until there aren’t any cycles.

We also had other algorithms (DFS and BFS), but the one we need now is removing one edge at a time.
Proof of Euler’s formula for planar graphs

We will do a proof by induction on the number of edges.

Motivation for the proof:

- Keep removing one edge at a time from the graph while keeping it connected, until we obtain a spanning tree.
- When we delete an edge:
  - \( V \) is unchanged.
  - \( E \) goes down by 1.
  - \( F \) also goes down by 1 since two faces are joined into one.
  - \( V - E + F \) is unchanged.
- When we end at a tree, \( E = V - 1 \) and \( F = 1 \), so \( V - E + F = 2 \).
Proof of Euler’s formula for planar graphs

Let $G$ be a connected graph on $n$ vertices, drawn without crossing edges. We will induct on the number of edges.

**Base case:** The smallest possible number of edges in a connected graph on $n$ vertices is $n - 1$, in which case the graph is a tree:

\[ V = n \]
\[ E = n - 1 \]
\[ F = 1 \text{ (no cycles, so the only face is the infinite face)} \]

\[ V - E + F = n - (n - 1) + 1 = 2 \]
Proof of Euler’s formula for planar graphs

**Induction step:**

- Let $G$ be a connected planar graph on $n$ vertices and $k$ edges, drawn without edge crossings.
- The base case was $k = n - 1$. Now consider $k \geq n$.
- **Induction hypothesis:** Assume Euler’s formula holds for connected graphs with $n$ vertices and $k - 1$ edges.
- Remove an edge from any cycle to get a connected subgraph $G'$. $G'$ has $V'$ vertices, $E'$ edges, and $F'$ faces:
  - $V' = V = n$
  - $E' = E - 1 = k - 1$ since we removed one edge.
  - $F' = F - 1$ since the faces on both sides of the removed edge were different but have been merged together.
- Since $E' = k - 1$, by induction, $G'$ satisfies $V' - E' + F' = 2$.
- $V' - E' + F' = V - (E - 1) + (F - 1) = V - E + F$, so $V - E + F = 2$ also.
Consider a graph drawn on a sphere.

Poke a hole inside a face, stretch it out from the hole, and flatten it onto the plane. \textit{(Demo)}

The face with the hole becomes the \textit{outside} or \textit{infinite} face. All other faces are distorted but remain finite.

If a connected graph can be drawn on a sphere without edges crossing, it’s a planar graph.

The values of $V, E, F$ are the same whether it’s drawn on a plane or sphere, so $V - E + F = 2$ still applies.
Pyramid with a square or rectangular base:

- Poke a pinhole in the base of the pyramid (left). Stretch it out and flatten it into a planar embedding (right). The pyramid base (left) corresponds to the infinite face (right).

- Euler’s formula (and other formulas we’ll derive for planar embeddings) apply to polyhedra without holes.

\[ V = 5, \quad E = 8, \quad F = 5, \quad V - E + F = 5 - 8 + 5 = 2 \]
Convex polyhedra

- A shape in 2D or 3D is *convex* when the line connecting any two points in it is completely contained in the shape.
- A sphere is convex. An indented sphere is not (red line above).
- But we can deform the indented sphere to an ordinary sphere, so the graphs that can be drawn on their surfaces are the same.
- Convex polyhedra are a special case of 3D polyhedra w/o holes.
- The book presents results about graphs on convex polyhedra; more generally, they also apply to 3D polyhedra without holes.
A torus is a donut shape. It is not topologically equivalent to a sphere, due to a hole.

Consider a graph drawn on a torus without crossing edges.

Transforming a sphere to a torus requires cutting, stretching, and pasting. Edges on the torus through the cut can’t be drawn there on the sphere. When redrawn on the sphere, they may cross.

So, there are graphs that can be drawn on a torus w/o crossing edges, but which can’t be drawn on a sphere w/o crossing edges.
Beyond spheres – graphs on solids with holes

- An $m \times n$ grid on a torus has
  \[ V = mn, \quad E = 2mn, \quad F = mn \]
  \[ V - E + F = mn - 2mn + mn = 0 \]

- **Theorem:** Let $G$ be a connected graph drawn on a $\gamma$-holed torus without edge crossings, and with all faces homeomorphic to discs. ($\gamma = 0$ for sphere, 1 for donut, etc.) Then
  \[ V - E + F = 2(1 - \gamma). \]

- **Note:** The quantity $2(1 - \gamma)$ is the *Euler characteristic*. It’s usually denoted $\chi$, which conflicts using $\chi(G)$ for chromatic number.
More relations on $V, E, F$ in planar graphs
Trace around a face, counting each encounter with an edge.

Face A has edge encounters $A_1$ through $A_7$, giving $\deg(A) = 7$.

Face B has edge encounters $B_1$ through $B_6$, including two encounters with one edge ($B_5$ and $B_6$). So $\deg(B) = 6$.

$\deg(C) = 5$. 
The sum of the face degrees is \(2E\), since each edge is used twice:

\[
S = \text{deg}(A) + \text{deg}(B) + \text{deg}(C) = 7 + 6 + 5 = 18
\]
\[
2E = 2(9) = 18
\]

This is an analogue of the Handshaking Lemma.

The sum of the vertex degrees is \(2E\) for all graphs.

Going clockwise from the upper left corner, we have

\[
3 + 3 + 2 + 2 + 2 + 3 + 2 + 1 = 18.
\]
<table>
<thead>
<tr>
<th>Face degree</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Empty graph</td>
</tr>
<tr>
<td>1</td>
<td>Face degree 1</td>
</tr>
<tr>
<td>2</td>
<td>Face degree 2</td>
</tr>
</tbody>
</table>

Faces usually have at least 3 sides, but it is possible to have fewer.

In a simple (no loops, no multiedges) connected graph with at least three vertices, these cases don’t arise, so all faces have face degree at least 3.

Thus, the sum of the face degrees is $S \geq 3F$, so $2E \geq 3F$.

In a bipartite graph, all cycles have even length, so all faces have even degree. Adding bipartite to the above conditions, each face has at least 4 sides. Thus, $2E \geq 4F$, which simplifies to $E \geq 2F$. 
Inequalities between $V, E, F$

**Theorem**

In a connected graph drawn in the plane without crossing edges:

1. $V - E + F = 2$

2. Additionally, if $G$ is simple (no multiedges) and if $V \geq 3$, then
   - (a) $3F \leq 2E$
   - (b) $E \leq 3V - 6$
   - (c) $F \leq 2V - 4$

3. If $G$ is simple and bipartite, these bounds improve to
   - (a) $2F \leq E$
   - (b) $E \leq 2V - 4$
   - (c) $F \leq V - 2$

- Part 1 is Euler’s formula. We just showed 2(a) and 3(a).
- We will prove the other parts, and use them to prove certain graphs are not planar.
Inequalities between $V, E, F$

(a) $3F \leq 2E$  
(b) $E \leq 3V - 6$  
(c) $F \leq 2V - 4$

Let $G$ be a connected simple graph with $V \geq 3$, drawn in the plane without crossing edges.

(a) So far, we showed $V - E + F = 2$ and (a) $3F \leq 2E$.

(b) Thus, $F \leq 2E/3$ and

$$2 = V - E + F \leq V - E + (2E/3) = V - E/3$$

so $2 \leq V - E/3$, or $E \leq 3V - 6$, which is (b).

(c) $3F \leq 2E$ also gives $E \geq 3F/2$ and

$$2 = V - E + F \leq V - (3F/2) + F = V - F/2$$

so $2 \leq V - F/2$, or $F \leq 2V - 4$, which is (c).
Inequalities between $V, E, F$ for a simple bipartite graph

(a) $2F \leq E$  
(b) $E \leq 2V - 4$  
(c) $F \leq V - 2$

Let $G$ be a connected simple bipartite graph with $V \geq 3$, drawn in the plane without crossing edges.

(a) For this case, we showed $V - E + F = 2$ and (a) $2F \leq E$.

(b) Thus, $F \leq E/2$ and
\[
2 = V - E + F \leq V - E + \frac{E}{2} = V - E/2
\]
so $2 \leq V - E/2$, or $E \leq 2V - 4$, which is (b).

(c) $2F \leq E$ also gives
\[
2 = V - E + F \leq V - 2F + F = V - F
\]
so $2 \leq V - F$, or $F \leq V - 2$, which is (c).
Let $G$ be a connected graph with $V \geq 3$, drawn in the plane without crossing edges. Suppose all cycles have length $\geq g$, with $g \geq 3$.

(a) Sum of face degrees: $S = 2E$ and $S \geq g \cdot F$, so $F \leq \frac{2}{g}E$.

(b) Thus, $2 = V - E + F \leq V - E + \frac{2}{g}E = V - (1 - \frac{2}{g})E = V - \frac{g-2}{g}E$ so $E \leq \frac{g}{g-2}(V - 2)$.

(c) $F \leq \frac{2}{g}E$ also gives $E \geq \frac{g}{2}F$ and $2 = V - E + F \leq V - \frac{g}{2}F + F = V - (\frac{g}{2} - 1)F$ so $(\frac{g}{2} - 1)F \leq (V - 2)$ so $F \leq \frac{2}{g-2}(V - 2)$.
Characterizing planar graphs
**K₅ and K₃,₃ are not planar**

![K₅ and K₃,₃](image)

**K₅ is not planar**
- \( V = 5 \)
- \( E = \binom{5}{2} = 10 \)
- This violates \( E \leq 3V - 6 \) since \( 3V - 6 = 15 - 6 = 9 \) and \( 10 \not\leq 9 \).

**K₃,₃ is not planar**
- \( V = 6 \)
- \( E = 3 \cdot 3 = 9 \)
- This is bipartite, so if it has a planar embedding, \( E \leq 2V - 4 \).
- However, \( 2V - 4 = 2(6) - 4 = 8 \), and \( 9 \not\leq 8 \).
Suppose that we can turn graph $G$ into graph $H$ by repeatedly applying these two operations:

- **Subdividing**: Split an edge $AB$ into two edges $AV$ and $VB$ by adding a vertex $V$ somewhere in the middle (not incident with any other edge).
- **Smoothing**: Let $V$ be a vertex of degree 2. Replace two edges $AV$ and $VB$ by one edge $AB$ and delete vertex $V$.

Then $G$ and $H$ are *homeomorphic* (a.k.a. *edge equivalent*).

The left graph is homeomorphic to $K_5$ (on the right):

- Smooth out every black vertex (left graph) to get $K_5$ (right graph).
- Repeatedly subdivide edges of $K_5$ (right) to get the left graph.
Theorem (Kuratowski’s Theorem)

\( G \) is planar iff it does not have a subgraph homeomorphic to \( K_5 \) or \( K_{3,3} \).

- **Necessity**: If \( G \) is planar, so is every subgraph. But if \( G \) has a subgraph homeomorphic to \( K_5 \) or \( K_{3,3} \), the subgraph is not planar.

- **Sufficiency**: The proof is too advanced, but it’s in the book.

The graph shown above has a subgraph (shown in red) homeomorphic to \( K_5 \), and thus, it is not a planar graph.
Dual graphs
Start with a planar embedding of a graph $G$ (shown in black).

Draw a red vertex inside each face, including the “infinite face.”

For every edge $e$ of $G$:
- Let $a, b$ be the red vertices in the faces on the two sides of $e$.
- Draw a red edge $\{a, b\}$ crossing $e$.

Remove the original graph $G$ to obtain the red graph $H$.

$H$ is the dual graph of this drawing of $G$.
(Also called plane dual or combinatorial dual.)

The dual graph depends on how $G$ is drawn.
If $G$ is connected, then $G$ is also a dual graph of $H$ — just switch the roles of the colors!
$G$ and $H$ have the same number of edges:
- Each edge of $G$ crosses exactly one edge of $H$ and vice-versa.

# faces of $G = \#$ vertices of $H$ and
# faces of $H = \#$ vertices of $G$:
- Bijections: vertices of either graph $\leftrightarrow$ faces of the other.

The fact that the sum of face degrees is $2E$ becomes the Handshaking Lemma applied to the dual graph!
Coloring maps
Coloring maps

Color states so that neighboring states have different colors. This map uses 4 colors for the states.

Assume each state is a contiguous region.
  - Michigan isn’t.
    - Its parts all have to be the same color, which could increase the # colors required. Artificially fill in Lake Michigan to make it contiguous.

Also assume the states form a contiguous region.
  - Alaska and Hawaii are isolated, and just added on separately.
A proper coloring of the faces of a planar graph ↔ a proper coloring of the vertices of its dual graph

The regions/states/countries of the map are faces of a graph, $G$.
Place a vertex inside each region and form the dual graph, $H$.
A proper coloring of the vertices of $H$ gives a proper coloring of the faces of $G$ (aside from a technicality on the next slide).
A proper coloring of the faces of a planar graph \( \leftrightarrow \) a proper coloring of the vertices of its dual graph

**Technicality:** A vertex of degree 1 in \( G \) gives an edge sticking out into a face, resulting in a loop in \( H \). (See dashed edges).

- A graph with a loop can’t have a proper coloring!
- The edge sticking out in \( G \) doesn’t separate faces of \( G \).
- Delete vertices of degree 1 in \( G \), and loops in \( H \), to get an equivalent problem in terms of coloring the faces of \( G \).
Coloring planar graphs
Theorem

*Every simple planar graph has a vertex with degree at most 5.*

Proof:

- If the graph isn’t connected, just restrict to one component of it.

- The sum of vertex degrees in any graph equals $2E$.

- Assume by way of contradiction that all vertices have degree $\geq 6$. Then the sum of vertex degrees is $2E \geq 6V$.

- So $2E \geq 6V$, so $E \geq 3V$.

- This contradicts $E \leq 3V - 6$ in any planar graph, so some vertex has degree $\leq 5$. 
Possible degrees in a planar graph

Theorem

Every simple planar graph is 5-degenerate.

Proof:

- Recall that a graph is $k$-degenerate when all subgraphs have minimum degree $\leq k$.

- Every subgraph of a simple planar graph is also simple and planar, and thus has minimum degree $\leq 5$.

- So every simple planar graph is 5-degenerate.
Easy: Six Color Theorem

Theorem

*Every simple planar graph is 6-colorable.*

Proof:

- We showed that any $k$-degenerate graph is $(k + 1)$-colorable.
- Every simple planar graph is 5-degenerate, and thus, 6-colorable.
Moderate difficulty: Five Color Theorem

Theorem

Every simple planar graph is 5-colorable.

Proof:

- We will induct on $|V(G)|$.

- **Base case:** If $|V(G)| \leq 5$, just assign all vertices different colors.
Proof, continued — Induction step:

- Assume the theorem holds for all graphs with fewer vertices.

- If $G$ has a vertex $v$ of degree $\leq 4$, then $G - \{v\}$ is 5-colorable by induction.

- When we add $v$ back in, since it has $\leq 4$ neighbors, at least one of the 5 colors is available, so we can complete the 5-coloring.

- So, we will have to consider $\delta(G) \geq 5$. Since all simple planar graphs have $\delta(G) \leq 5$, this gives $\delta(G) = 5$. 
Proof, continued — Induction step:

- Assume the theorem holds for all graphs with fewer vertices, and assume \( \delta(G) = 5 \).

- Let \( v \) be a vertex of degree 5.

- If all neighbors of \( v \) are adjacent to each other, they form a \( K_5 \). But then the graph isn’t planar — a contradiction.
- So there are neighbors \( a \) and \( b \) of \( v \) with \( ab \notin E(G) \).
Five Color Theorem
Every simple planar graph is 5-colorable.

Proof, continued — Induction step:

- **Recall:** \(d(v) = 5\), and \(a, b\) are neighbors of \(v\) with \(ab \notin E(G)\)
- Let \(H = G/\{a, b, v\}\) (graph contraction).
  Vertices \(a, b, v\) are contracted to a new vertex \(w\).
- \(H\) is still planar:
  - Slide \(a\) and \(v\) together along edge \(av\). Same for \(b\) and \(v\).
  - Merge \(a, b, v\) into one vertex \(w\).
  - Remove edges \(av, bv\), and reduce any multiedges just created.
Five Color Theorem
Every simple planar graph is 5-colorable.

Proof, continued — Induction step:
- By induction, \( H \) has a 5-coloring \( c_H : V(H) \to \{1, \ldots, 5\} \).
- Extend to a 5-coloring \( c_G \) of \( G \):
  - For all vertices \( u \) except \( a, b, v \), set \( c_G(u) = c_H(u) \).
  - Set \( c_G(a) = c_G(b) = c_H(w) \). This is fine since \( ab \) isn’t an edge in \( G \).
  - The 5 neighbors of \( v \) in \( G \) use at most 4 colors (since \( a \) and \( b \) use the same color). So there is a color available to assign to \( c_G(v) \).
Theorem (Four Color Theorem)

Every simple planar graph is 4-colorable.

- Map makers have believed this for centuries empirically, but it wasn’t proven mathematically.
- This was the first major theorem to be proved using a computer program (Kenneth Appel and Wolfgang Haken, 1976).
- The original proof had 1936 cases! Their program determined the cases and showed they are all 4-colorable.
- The proof was controversial because
  - It was the first proof that was impractical for any human to verify.
  - There could be bugs in the software, hardware, compiler, O/S, etc.
- Over the years, people have found errors in the proof, but they have been fixed, and the result still stands. The number of cases has been cut down to 633.
Classifying regular polyhedra
A **polyhedron** is a 3D solid whose surface consists of polygons. As a graph, no loops and no multiple edges.

All faces have \( \geq 3 \) edges and all vertices are in \( \geq 3 \) edges.

To be 3D, there must be \( \geq 4 \) vertices, \( \geq 4 \) faces, and \( \geq 6 \) edges.

A **regular polyhedron** has these symmetries:

- All faces are regular \( \ell \)-gons for the same \( \ell \geq 3 \).
- All vertices have the same degree (\( r \geq 3 \)).
- All edges have the same length.
- All pairs of adjacent faces have the same angle between them.
Classifying regular polyhedra

- Suppose all vertices have the same degree \( r \geq 3 \) and all faces are \( \ell \)-gons (same \( \ell \geq 3 \) for all faces).

- The sum of vertex degrees is \( r \cdot V = 2E \), so \( V = \frac{2E}{r} \).

- The sum of face degrees is \( \ell \cdot F = 2E \), so \( F = \frac{2E}{\ell} \).

- Plug these into \( V - E + F = 2 \):

  \[
  \frac{2E}{r} - E + \frac{2E}{\ell} = 2 \quad E \cdot \left( \frac{2}{r} - 1 + \frac{2}{\ell} \right) = 2 \quad E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}
  \]

- We have to find all integers \( r, \ell \geq 3 \) for which \( V, E, F \) are positive integers, and then check if polyhedra with those parameters exist.
Classifying regular polyhedra

- Suppose all vertices have the same degree $r \geq 3$ and all faces are $\ell$-gons (same $\ell \geq 3$ for all faces).

- Compute $(V, E, F)$ using $E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}$, $V = \frac{2E}{r}$, $F = \frac{2E}{\ell}$:

- E.g., $r = 3$ and $\ell = 4$ gives

$$E = \frac{2}{\frac{2}{3} + \frac{2}{4} - 1} = \frac{2}{1/6} = 12$$

$$V = \frac{2(12)}{3} = 8$$

$$F = \frac{2(12)}{4} = 6$$

- What shape is it?
Classifying regular polyhedra
What range of vertex degree \( (r) \) and face degree \( (\ell) \) are permitted?

First method
- We have \( r \geq 3 \).
  Since some vertex has degree \( \leq 5 \), all do, so \( r \) is 3, 4, or 5.
- Vertices and faces are swapped in the dual graph, so \( \ell \) is 3, 4, or 5.

Second method: Analyze formula \( E = 2/(\frac{2}{r} + \frac{2}{\ell} - 1) \)
- \( E \) is a positive integer, so its denominator must be positive:
  \( \frac{2}{r} + \frac{2}{\ell} - 1 > 0 \)
- We have \( r, \ell \geq 3 \).
- If both \( r, \ell \geq 4 \), the denominator of \( E \) is \( \leq \frac{2}{4} + \frac{2}{4} - 1 = 0 \), which is invalid. So \( r \) and/or \( \ell \) is 3.
- If \( r = 3 \), then the denominator of \( E \) is \( \frac{2}{3} + \frac{2}{\ell} - 1 = \frac{2}{\ell} - \frac{1}{3} \).
  To be positive requires \( \ell \leq 5 \).
- Similarly, if \( \ell = 3 \) then \( r \leq 5 \).
Classifying regular polyhedra

- Suppose all vertices have the same degree \( r \in \{3, 4, 5\} \) and all faces are \( \ell \)-gons (same \( \ell \in \{3, 4, 5\} \) for all faces).

- Compute \((V, E, F)\) using \( E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}, \ V = \frac{2E}{r}, \ F = \frac{2E}{\ell} \):

  \[
  \begin{array}{ccc}
  (V, E, F) & \ell = 3 & \ell = 4 & \ell = 5 \\
  r = 3 & (4, 6, 4) & (8, 12, 6) & (20, 30, 12) \\
  r = 4 & (6, 12, 8) & \text{Division by 0} & (20, 30, 12) \\
  r = 5 & (12, 30, 20) & (-8, -20, -10) & (-4, -10, -4) \\
  \end{array}
  \]

- If \(V, E, F\) are not all positive integers, it can’t work (shown in pink).

- We found five possible values of \((V, E, F)\) with graph theory. Use geometry to actually find the shapes (if they exist).
### Classifying regular polyhedra

<table>
<thead>
<tr>
<th>Shape</th>
<th>Tetrahedron</th>
<th>Cube</th>
<th>Octahedron</th>
<th>Dodecahedron</th>
<th>Icosahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = \text{vertex degree}$</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$\ell = \text{face degree}$</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$V = # \text{vertices}$</td>
<td>4</td>
<td>8</td>
<td>6</td>
<td>20</td>
<td>12</td>
</tr>
<tr>
<td>$E = # \text{edges}$</td>
<td>6</td>
<td>12</td>
<td>12</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$F = # \text{faces}$</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>20</td>
</tr>
</tbody>
</table>

- These are known as the *Platonic solids*.
- The cube and octahedron are dual graphs.
- The dodecahedron and icosahedron are dual graphs.
- The tetrahedron is its own dual.
Octahedron and cube are dual

- Can draw either one inside the other.
  Place a dual vertex at the center of each face.
- In 3D, this construction shrinks the dual, vs. in 2D, it did not.