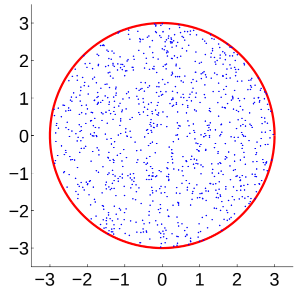
Math 180A, Fall 2005, Prof. Tesler – November 4, 2005
A continuous probability distribution: Throwing darts at a dartboard

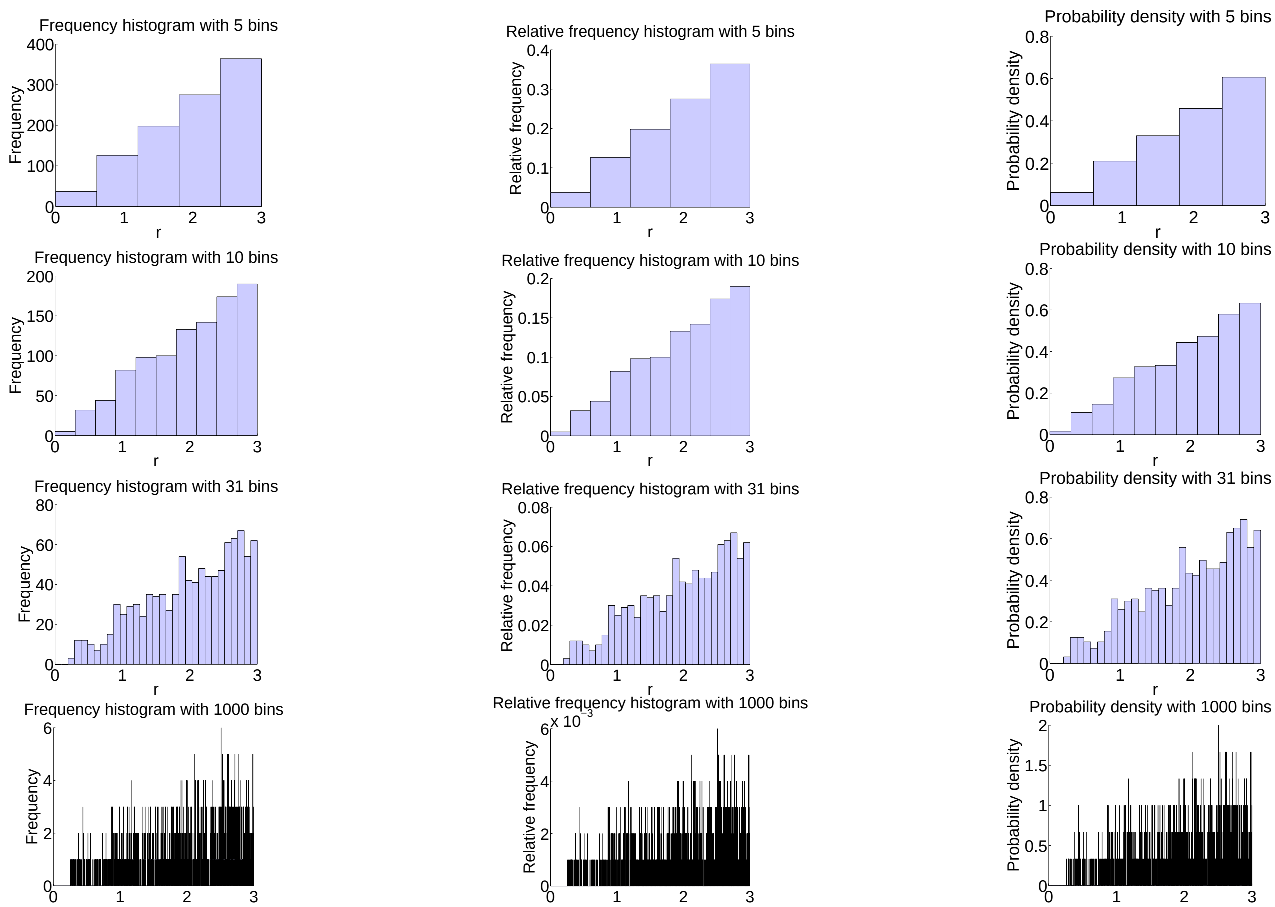
A dart is repeatedly thrown at a dartboard. The dartboard is a circle of radius 3, centered at the origin. If a dart misses, we ignore it. Otherwise, assume all points on the board can be hit with equal ("uniform") probability. (This may not be realistic, but just assume it's true for the purposes of this example.) For each hit $i = 1, 2, 3, \ldots$, we record the coordinates (x_i, y_i) and compute $r_i = \sqrt{{x_i}^2 + {y_i}^2}$, the distance of the hit from the center of the circle. Below is a plot of the points (x_i, y_i) points and histograms of r_i. What is the probability distribution of r?

Notation: $n = \#$ points $= 1000$; $w =$ bin width $= 3/(\#$ bins$)$; $n_j =$ number of points in bin j.



i	x	y	$r = \sqrt{x^2 + y^2}$
1	1.575	1.022	1.878
2	−1.640	1.265	2.071
3	−1.625	0.607	1.734
4	−1.143	−1.947	2.257
5	−1.054	−0.822	1.337
...	...	...	...
999	1.747	0.850	1.943
1000	1.519	−1.429	2.086

	Frequency	**Relative Frequency**	**Probability Density**
Bin j bar height	$= n_j$	n_j/n	$n_j/(nw)$
Total Area	$= nw$	$nw/n = w$	$nw/(nw) = 1$



PDF of continuous random variable: Let X, Y be random variables for the coordinates and $R = \sqrt{X^2 + Y^2}$. For each r between 0 and 3,

$$P(R \le r) = \text{Area of circle of radius } r \text{ (centered at origin) / Area of whole circle}$$
$$= (\pi r^2)/(\pi 3^2) = r^2/9.$$

But the area up to r in the probability density histogram is $P(R \le r) = \int_0^r f(t)\,dt$

so $f(r) = \frac{d}{dr} P(R \le r) = \frac{d}{dr} \frac{r^2}{9} = \frac{2r}{9}$ if $0 \le r \le 3$, and $f(r) = 0$ otherwise.