Problem H-2. Let $\alpha$ be a real number and $k \geq 0$ be an integer. The falling factorial and binomial coefficient notations generalize to

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha-1)(\alpha-2) \cdots (\alpha-k+1), \quad \binom{\alpha}{k} = 1, \quad \binom{\alpha}{k} = (\alpha)_k/k!$$

and the binomial series is $(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

The double factorial of a nonnegative integer $n$ is

$$n!! = n(n-2)(n-4) \cdots 2 \quad \text{if } n > 0 \text{ is even;}$$

$$n!! = n(n-2)(n-4) \cdots 1 \quad \text{if } n > 0 \text{ is odd;}$$

$$0!! = 1$$

E.g., $5!! = 5 \cdot 3 \cdot 1 = 15$ and $6!! = 6 \cdot 4 \cdot 2 = 48$.

(a) Give simple formulas for $n!!$ in both the even and odd cases, in terms of factorials and powers of 2.

$$(1/2)_k = -2 \left( -\frac{1}{4} \right)^k \frac{(2k-2)!}{(k-1)!}$$

Also fill in the values of $(1/2)_k$ for any $k \geq 0$ for which it is not valid.

$$(1/2)_5 = \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{5}{2} \right) \left( -\frac{7}{2} \right)$$

$$(1/2)_0 = \left( \frac{1}{2} \right)$$

(b) Prove the following and state which values of $k$ it holds for:

$$(1/2)_k = -2 \left( -\frac{1}{4} \right)^k \frac{(2k-2)!}{(k-1)!}$$

(c) Use the binomial series formula to compute the Taylor series for $\sqrt{1 + 8x}$ and $1/(1 + 3x)$. Simplify the series coefficients as much as possible.

(d) Prove this recursion, and state the values of $\alpha$ and $k$ for which it is valid:

$$\binom{\alpha}{k} = \binom{\alpha-1}{k} + \binom{\alpha-1}{k-1}$$