Chapter 9 # 1, 8*, 9, 12, 28, 34
and Problems H-24 through H-28 below (see both pages).

*Notes:
- Problem 9.8: The book’s solution is incomplete; small values of \( n \) work differently on some parts.

Drawings of a few of the graphs requested:

![Graphs](image)

Problem H-24. Define a directed graph on integers 1, \ldots, 6, in which \( i \to j \) is an edge iff \( j/i \) is a prime number.

(a) Write the graph in the form \( G = (V, E) \) where \( V \) is the set of vertices and \( E \) is the set of edges.

(b) Draw the graph.

(c) Write the adjacency matrix.

(d) Compute the indegree and outdegree of each vertex.

Problem H-25.

(a) How many simple graphs are there on vertex set [6]?

(b) How many graphs are there on vertex set [6] if we allow loops and allow up to three edges connecting each pair of vertices (and also up to three loops on each vertex)?

(c) How many simple directed graphs are there on vertex set [6]?

(d) How many simple graphs (undirected) on vertex set [6] have exactly \( k \) edges?

Problem H-26. In (a–b), let \( C \) be the set of courses at UCSD and \( S \) be the set of students. Let \( V = C \cup S \) and let \( \{s, c\} \in E \) if and only if student \( s \) is enrolled in course \( c \).

(a) Prove that \( G = (V, E) \) is a simple graph.

(b) Prove that every cycle of \( G \) has an even number of edges.

(c) \( S = \{Alice, Bob, Cindy, Dan, Emily\} \) is a set of students.

\( C = \{\text{Math 20A}, \text{Math 20B}, \text{Math 20C}, \text{Math 20D}, \text{Math 20E}, \text{Math 20F}\} \) is a set of classes.

A graph is formed in which \( \{s, c\} \) is an edge iff student \( s \) is enrolled in course \( c \).

In a given quarter, a student can only be enrolled in at most one of Math 20A,B,C,D because each is a prerequisite for the next one. However, a student who has already passed Math 20C can take Math 20D,E,F in any order, and can even take two or three of them simultaneously. (Although not recommended, it’s not forbidden.)

(i) Suppose Alice is in Math 20A, Bob is in Math 20B, Cindy is in Math 20C, and Dan is in Math 20D and Math 20F. Emily is not taking math this quarter. Draw the graph representing this.

(ii) Suppose Alice and Bob are currently in Math 20A, and we know that Cindy and Dan are currently in Math 20D but do not know whether or not they are in the other classes. We also know that Emily already passed Math 20C but do not know her current classes.

What are the minimum and maximum number of edges possible in the graph based on the given rules and partial enrollment information?

Draw the graph with definite enrollments shown by solid edges and possible enrollments (which we don’t know either way) shown by dotted edges.
Problem H-27. A graph $G = (V, E)$ is bipartite if $V$ can be partitioned $V = A \cup B$ (with $A \cap B = \emptyset$) and every edge has one endpoint in $A$ and the other in $B$. The graphs on problem H-26 are examples of this.

(a) Determine the formula for the number of simple bipartite graphs on two given sets of vertices, $A$ and $B$, with sizes $|A| = n$ and $|B| = m$.

Note that a bipartite graph can have multiple edges between two vertices, but cannot have loops. A simple graph cannot have multiple edges and also cannot have loops.

(b) For the same setup as (a), determine the formula for the number of simple bipartite graphs with exactly $k$ edges.

Problem H-28. In each part below, information about a graph is given. Decide if a graph, without loops, exists that satisfies the specified conditions or not. Justify your answers.

(a) A simple graph with degree sequence $(1, 1, 2, 3, 3, 5)$ (that is, six vertices, where two vertices have degree 1, one has degree 2, two have degree 3, and one has degree 5).

(b) A multigraph with degree sequence $(1, 2, 2, 3, 3, 5)$.

(c) A simple graph with degree sequence $(1, 2, 2, 3, 3, 5)$.

(d) A simple graph with degree sequence $(3, 3, 3, 3)$.

(e) A tree with six vertices and six edges.

(f) A tree with three or more vertices, where two vertices have degree 1 and the rest have degrees $\geq 3$.

(g) A disconnected simple graph with 10 vertices, 8 edges, and a cycle.