

Chapter 12 and 11.1

Planar graphs, regular polyhedra, and graph colorings

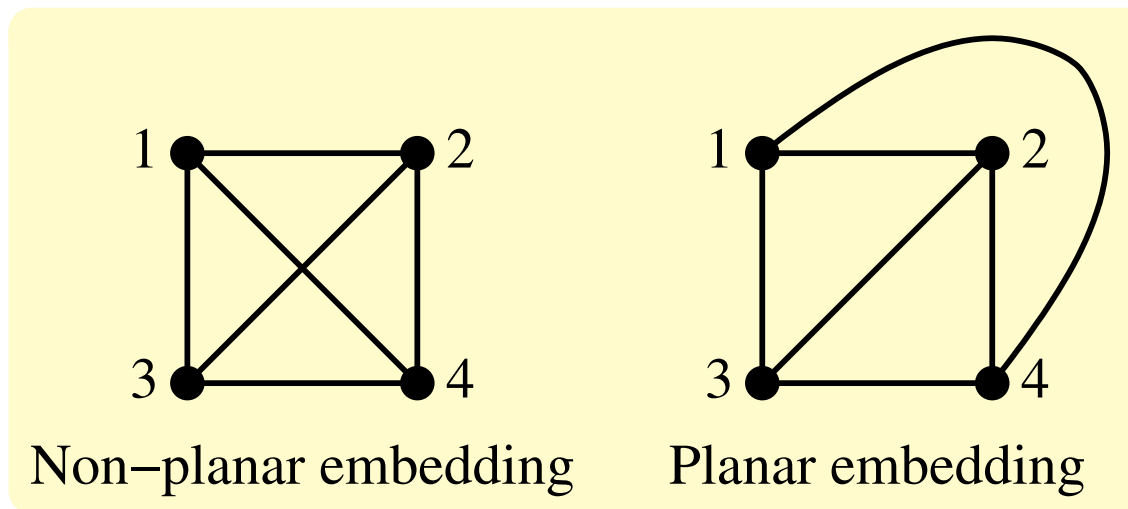
Prof. Tesler

Math 184A
Winter 2019

12.1–12.2. Planar graphs

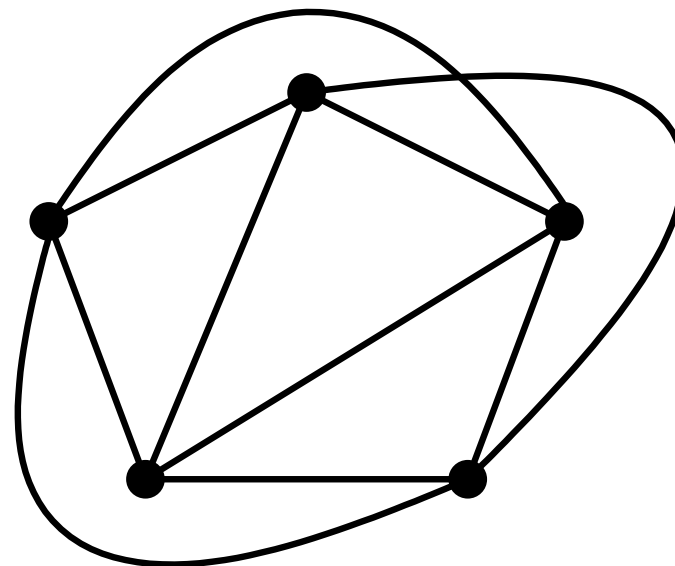
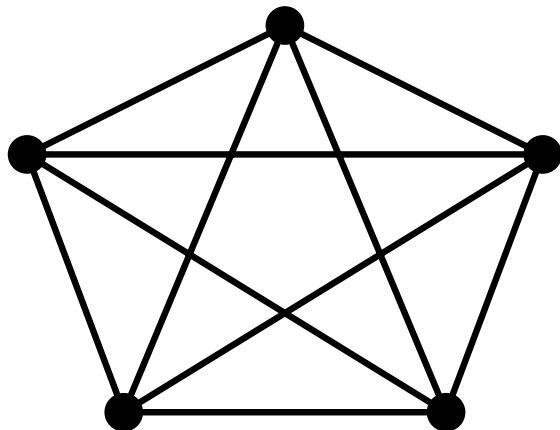
Definition

- A *planar embedding* of a graph is a drawing of the graph in the plane without edges crossing.
- A graph is *planar* if a planar embedding of it exists.
- Consider two drawings of the graph K_4 :
 $V = \{1, 2, 3, 4\}$ $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$



- The abstract graph K_4 is planar because it can be drawn in the plane without crossing edges.

How about K_5 ?



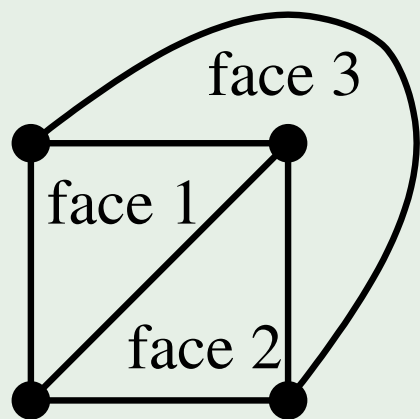
- Both of these drawings of K_5 have crossing edges.
- We will develop methods to prove that K_5 is not a planar graph, and to characterize what graphs are planar.

Euler's formula for planar graphs

Euler's Theorem on Planar Graphs

- Let G be a connected planar graph (drawn without crossing edges).
- Define V = number of vertices
 E = number of edges
 F = number of faces, including the “infinite” face
- Then $V - E + F = 2$.
- *Note:* this notation conflicts with standard graph theory notation
 V = set of vertices, E = set of edges.

Example



face 4 (infinite face)

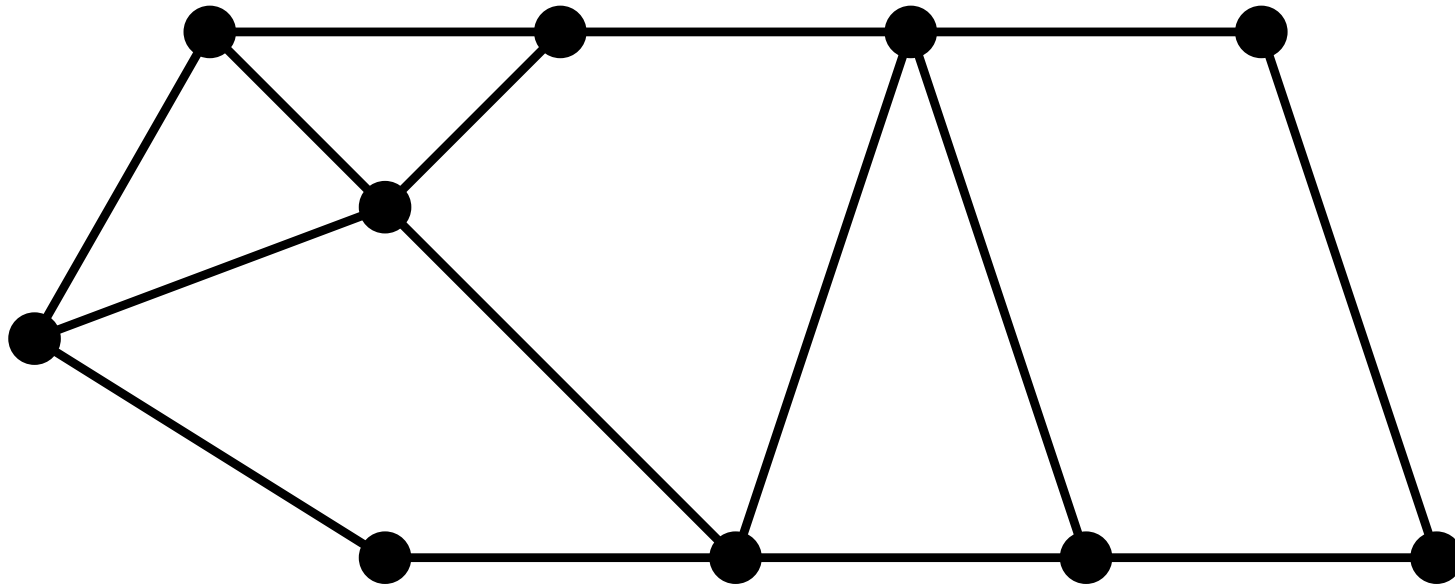
$$V = 4$$

$$E = 6$$

$$F = 4$$

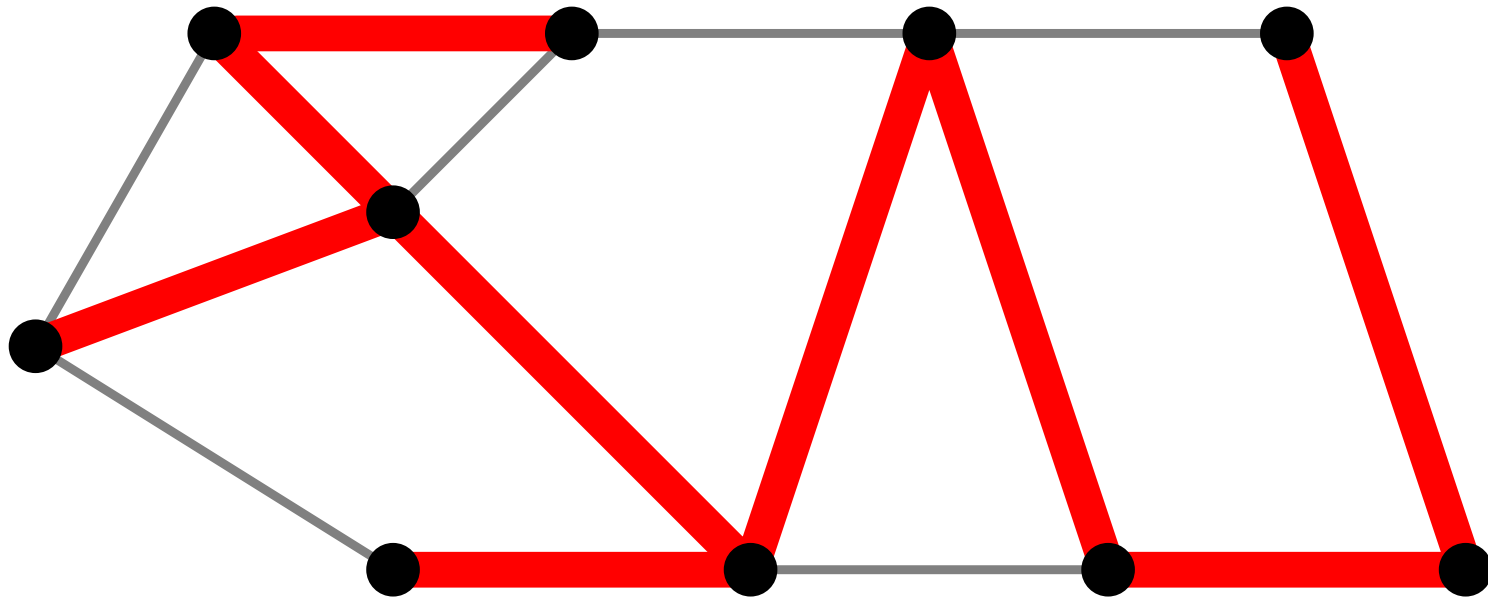
$$V - E + F = 4 - 6 + 4 = 2$$

Euler's formula for planar graphs



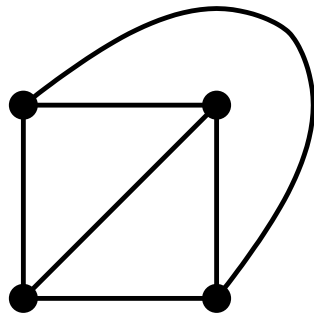
- $V = 10$
- $E = 15$
- $F = 7$
- $V - E + F = 10 - 15 + 7 = 2$

Spanning tree

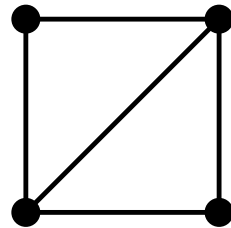


- A *spanning tree* of a connected graph is a subgraph that's a tree reaching all vertices. An example is highlighted in red.
- We previously saw we could obtain a spanning tree of any connected graph by repeatedly picking any cycle and removing an edge, until there are no cycles remaining.

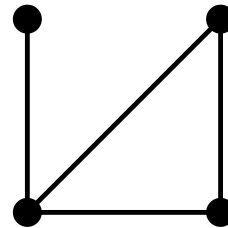
Proof of Euler's formula for planar graphs



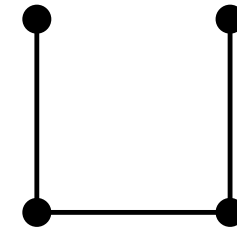
$$V - E + F = 4 - 6 + 4 = 2$$



$$4 - 5 + 3 = 2$$



$$4 - 4 + 2 = 2$$



$$4 - 3 + 1 = 2$$

We will do a proof by induction on the number of edges.

Motivation for the proof:

- Keep removing one edge at a time from the graph while keeping it connected, until we obtain a spanning tree.
- When we delete an edge:
 - V is unchanged.
 - E goes down by 1.
 - F also goes down by 1 since two faces are joined into one.
 - $V - E + F$ is unchanged.
- When we end at a tree, $E = V - 1$ and $F = 1$, so $V - E + F = 2$.

Proof of Euler's formula for planar graphs

Let G be a connected graph on n vertices, drawn without crossing edges. We will induct on the number of edges.

Base case: The smallest possible number of edges in a connected graph on n vertices is $n - 1$, in which case the graph is a tree:

$$V = n$$

$$E = n - 1$$

$$F = 1 \text{ (no cycles, so the only face is the infinite face)}$$

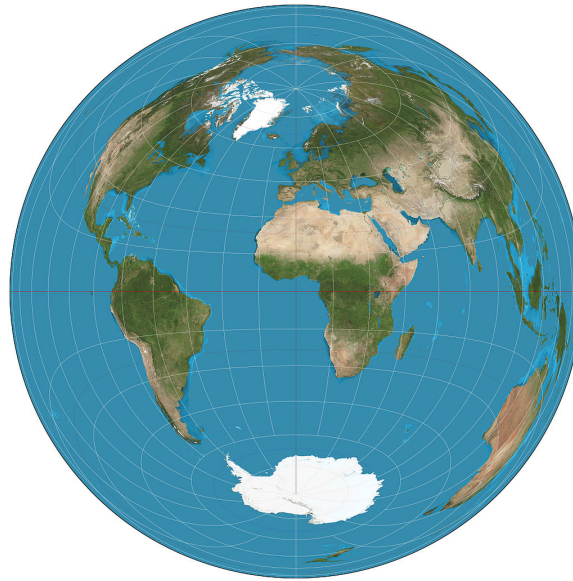
$$V - E + F = n - (n - 1) + 1 = 2$$

Proof of Euler's formula for planar graphs

Induction step:

- Let G be a connected planar graph on n vertices and k edges, drawn without edge crossings.
- The base case was $k = n - 1$. Now consider $k \geq n$.
- Assume Euler's formula holds for connected graphs with n vertices and $k - 1$ edges.
- Remove an edge from any cycle to get a connected subgraph G' .
- G' has V' vertices, E' edges, and F' faces:
 - $V' = V = n$
 - $E' = E - 1 = k - 1$ since we removed one edge.
 - $F' = F - 1$ since the faces on the two sides of the removed edge were different but have been merged together.
- Since $E' = k - 1$, by induction, G' satisfies $V' - E' + F' = 2$.
- Observe that $V' - E' + F' = V - (E - 1) + (F - 1) = V - E + F$, so $V - E + F = 2$ also.

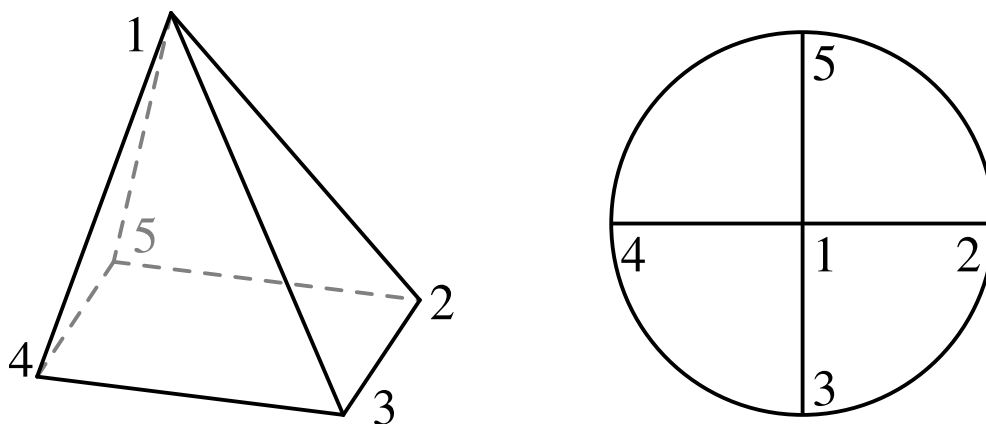
Graph on a sphere



- Consider a graph drawn on a sphere.
- Poke a hole inside a face, stretch it out from the hole and flatten it onto the plane. *(Demo)*
- The face with the hole becomes the *outside* or *infinite* face, while all the other faces are distorted but remain finite.
- If a connected graph can be drawn on a surface of a sphere without edges crossing, it's a planar graph.
- The values of V , E , F are the same whether it's drawn on a plane or the surface of a sphere, so $V - E + F = 2$ still applies.

Figure: http://en.wikipedia.org/wiki/File:Lambert_azimuthal_equal-area_projection_SW.jpg

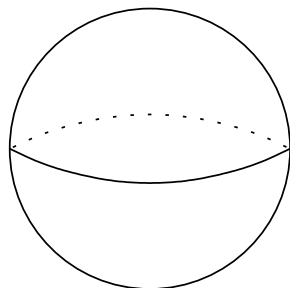
3D polyhedra w/o holes are topologically equivalent to spheres



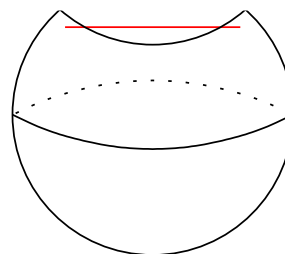
Pyramid with a square or rectangular base:

- Poke a pinhole in the base of the pyramid (left).
Stretch it out and flatten it into a planar embedding (right).
The pyramid base (left) corresponds to the infinite face (right).
- Euler's formula (and other formulas we'll derive for planar embeddings) apply to polyhedra without holes.
- $V = 5, \quad E = 8, \quad F = 5, \quad V - E + F = 5 - 8 + 5 = 2$

Convex polyhedra



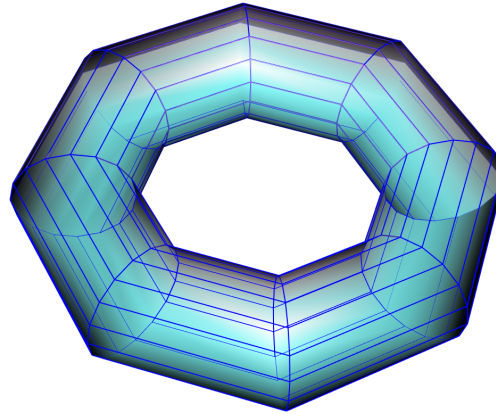
**Sphere
(convex)**



**Indented sphere
(not convex)**

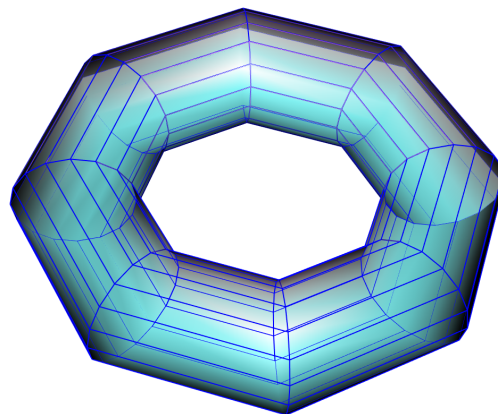
- A shape in 2D or 3D is *convex* when the line connecting any two points in it is completely contained in the shape.
- A sphere is convex. An indented sphere is not (red line above).
- But we can deform the indented sphere to an ordinary sphere, so the graphs that can be drawn on their surfaces are the same.
- Convex polyhedra are a special case of 3D polyhedra w/o holes.
- The book presents results about graphs on convex polyhedra; more generally, they also apply to 3D polyhedra without holes.

Beyond spheres – graphs on solids with holes



- A *torus* is a donut shape. It is not topologically equivalent to a sphere, due to a hole.
- Consider a graph drawn on a torus without crossing edges.
- Transforming a sphere to a torus requires cutting, stretching, and pasting. Edges on the torus through the cut can't be drawn there on the sphere. When redrawn on the sphere, they may cross.
- So, there may be graphs that can be drawn on the surface of a torus without crossing edges, but which cannot be drawn on the surface of a sphere without crossing edges.

Beyond spheres – graphs on solids with holes



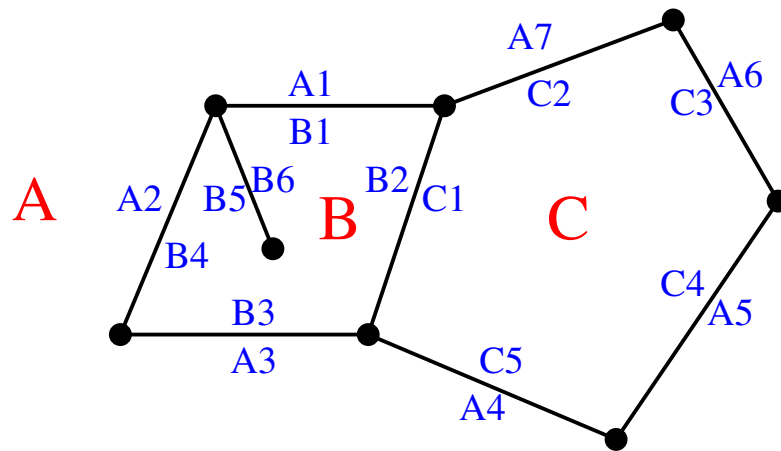
- An $m \times n$ grid on a torus has

$$V = mn, \quad E = 2mn, \quad F = mn$$
$$V - E + F = mn - 2mn + mn = 0$$

- **Theorem:** for a connected graph on a g -holed torus,
 $V - E + F = 2(1 - g)$. ($g = 0$ for sphere, 1 for donut, etc.)

More relations on V, E, F in planar graphs

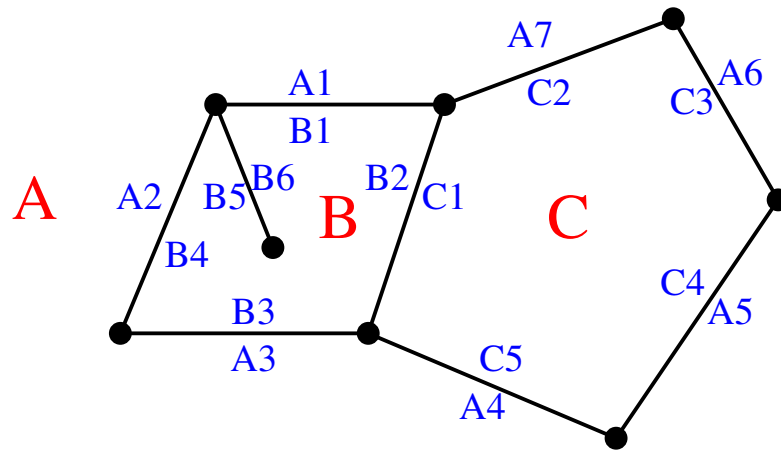
Face degrees



Face degrees

- Trace around a face, counting each encounter with an edge.
- Face A, has edge encounters $A1$ through $A7$, giving $\deg A = 7$.
- Face B has edge encounters $B1$ through $B6$, including two encounters with one edge ($B5$ and $B6$). So $\deg B = 6$.
- $\deg C = 5$.

Face degrees



Total degrees

- The sum of the face degrees is $2E$, since each edge is encountered twice:

$$S = \deg A + \deg B + \deg C = 7 + 6 + 5 = 18$$

$$2E = 2(9) = 18$$

- The sum of the vertex degrees is $2E$ for all graphs. Going clockwise from the upper left corner, we have

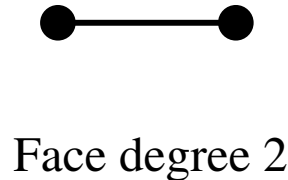
$$3 + 3 + 2 + 2 + 2 + 3 + 2 + 1 = 18.$$

Face degrees

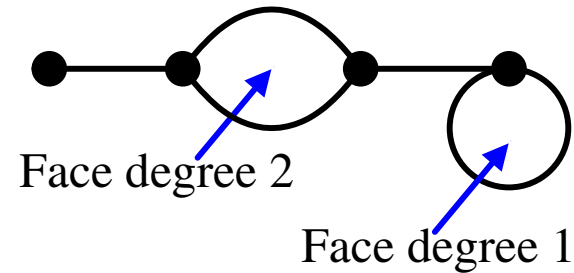
Empty graph

Face degree 0

One edge graph



Multigraph



- Faces usually have at least 3 sides, but it is possible to have fewer.
- In a simple (no loops, no multiedges) connected graph with at least three vertices, these cases don't arise, so all faces have face degree at least 3.
- Thus, the sum of the face degrees is $S \geq 3F$, so $2E \geq 3F$.
- In a bipartite graph, all cycles have even length, so all faces have even degree. Adding bipartite to the above conditions, each face has at least 4 sides. Thus, $2E \geq 4F$, which simplifies to $E \geq 2F$.

Inequalities between V, E, F

Theorem

In a connected graph drawn in the plane without crossing edges:

① $V - E + F = 2$

② *Additionally, if G is simple (no multiedges) and if $V \geq 3$, then*

(a) $3F \leq 2E$ (b) $E \leq 3V - 6$ (c) $F \leq 2V - 4$

③ *If G is simple and bipartite, these bounds improve to*

(a) $2F \leq E$ (b) $E \leq 2V - 4$ (c) $F \leq V - 2$

- Part 1 is Euler's formula. We just showed 2(a) and 3(a).
- We will prove the other parts, and use them to prove certain graphs are not planar.

Inequalities between V, E, F

(a) $3F \leq 2E$ (b) $E \leq 3V - 6$ (c) $F \leq 2V - 4$

Let G be a connected simple graph with $V \geq 3$, drawn in the plane without crossing edges.

(a) So far, we showed $V - E + F = 2$ and (a) $3F \leq 2E$.

(b) Thus, $F \leq 2E/3$ and

$$2 = V - E + F \leq V - E + (2E/3) = V - E/3$$

so $2 \leq V - E/3$, or $E \leq 3V - 6$, which is (b).

(c) $3F \leq 2E$ also gives $E \geq 3F/2$ and

$$2 = V - E + F \leq V - (3F/2) + F = V - F/2$$

so $2 \leq V - F/2$, or $F \leq 2V - 4$, which is (c).

Inequalities between V, E, F for a simple bipartite graph

(a) $2F \leq E$ (b) $E \leq 2V - 4$ (c) $F \leq V - 2$

Let G be a connected simple bipartite graph with $V \geq 3$, drawn in the plane without crossing edges.

(a) For this case, we showed $V - E + F = 2$ and (a) $2F \leq E$.

(b) Thus, $F \leq E/2$ and

$$2 = V - E + F \leq V - E + (E/2) = V - E/2$$

so $2 \leq V - E/2$, or $E \leq 2V - 4$, which is (b).

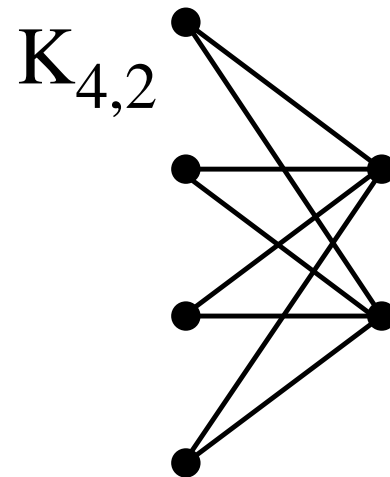
(c) $2F \leq E$ also gives

$$2 = V - E + F \leq V - 2F + F = V - F$$

so $2 \leq V - F$, or $F \leq V - 2$, which is (c).

Characterizing planar graphs

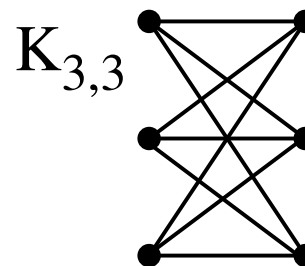
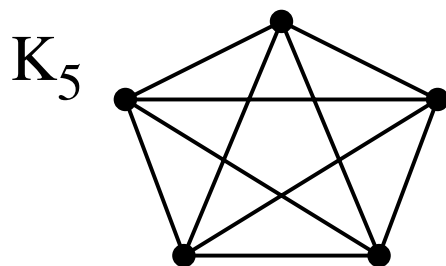
Complete bipartite graph $K_{m,n}$



The *complete bipartite graph* $K_{m,n}$ has

- Vertices $V = A \cup B$ where $|A| = m$ and $|B| = n$, and $A \cap B = \emptyset$.
- Edges $E = \{ \{a, b\} : a \in A \text{ and } b \in B \}$
All possible edges with one vertex in A and the other in B .
- In total, $m + n$ vertices and mn edges.

K_5 and $K_{3,3}$ are not planar



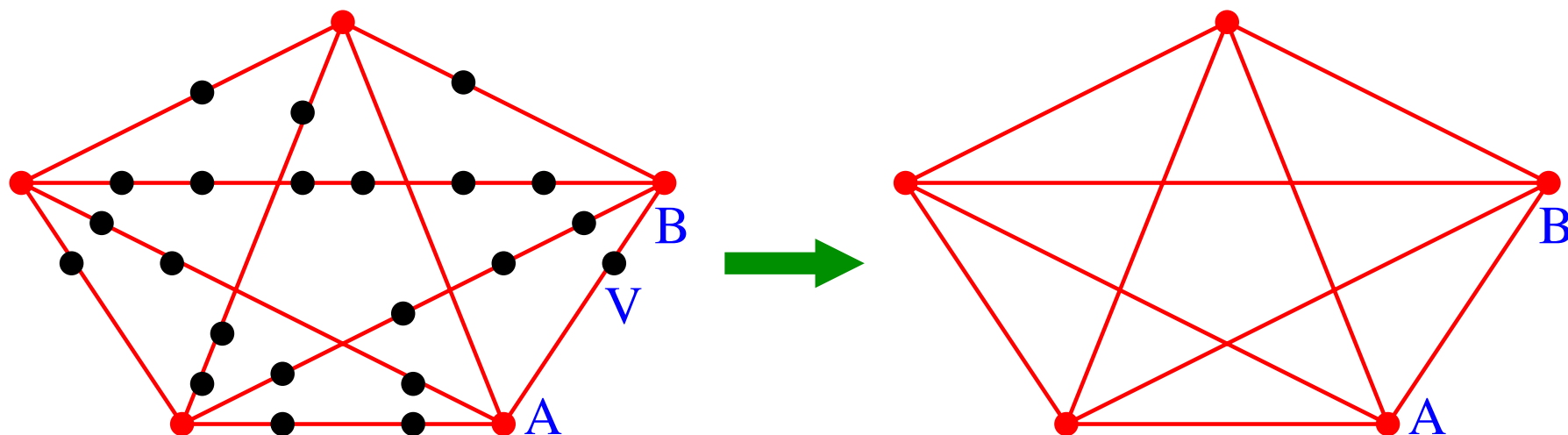
K_5 is not planar

- $V = 5$
- $E = \binom{5}{2} = 10$
- This violates $E \leq 3V - 6$ since $3V - 6 = 15 - 6 = 9$ and $10 \not\leq 9$.

$K_{3,3}$ is not planar

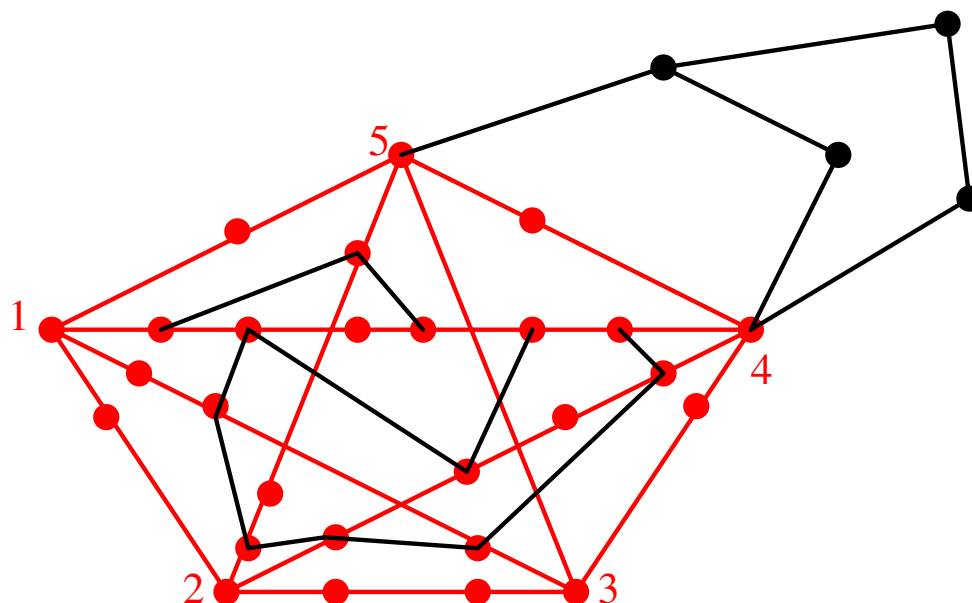
- $V = 6$
- $E = 3 \cdot 3 = 9$
- This is a bipartite graph, so if it has a planar embedding, it satisfies $E \leq 2V - 4$.
- However, $2V - 4 = 2(6) - 4 = 8$, and $9 \not\leq 8$.

Homeomorphisms (a.k.a. edge equivalency)



- Suppose that we can turn graph G into graph H by repeatedly applying these two operations:
 - Split an edge AB into two edges AV and VB by adding a vertex V somewhere in the middle (not incident with any other edge).
 - Let V be a vertex of degree 2.
Replace two edges AV and VB by one edge AB and delete vertex V .
- Then G and H are *homeomorphic* (a.k.a. *edge equivalent*).
- The left graph is homeomorphic to K_5 (on the right):
 - Apply the 2nd operation above to every black vertex to get K_5 .
 - Repeatedly apply the 1st operation to K_5 to get the left graph.

Characterizing planar graphs



Theorem (Kuratowski's Theorem)

G is planar iff it does not have a subgraph homeomorphic to K_5 or $K_{3,3}$.

- **Necessity:** If G is planar, so is every subgraph. But if G has a subgraph homeomorphic to K_5 or $K_{3,3}$, the subgraph is not planar.
- **Sufficiency:** The proof is too advanced.
- The graph shown above has a subgraph (shown in red) homeomorphic to K_5 , and thus, it is not a planar graph.

More properties of planar graphs

Possible degrees in a planar graph

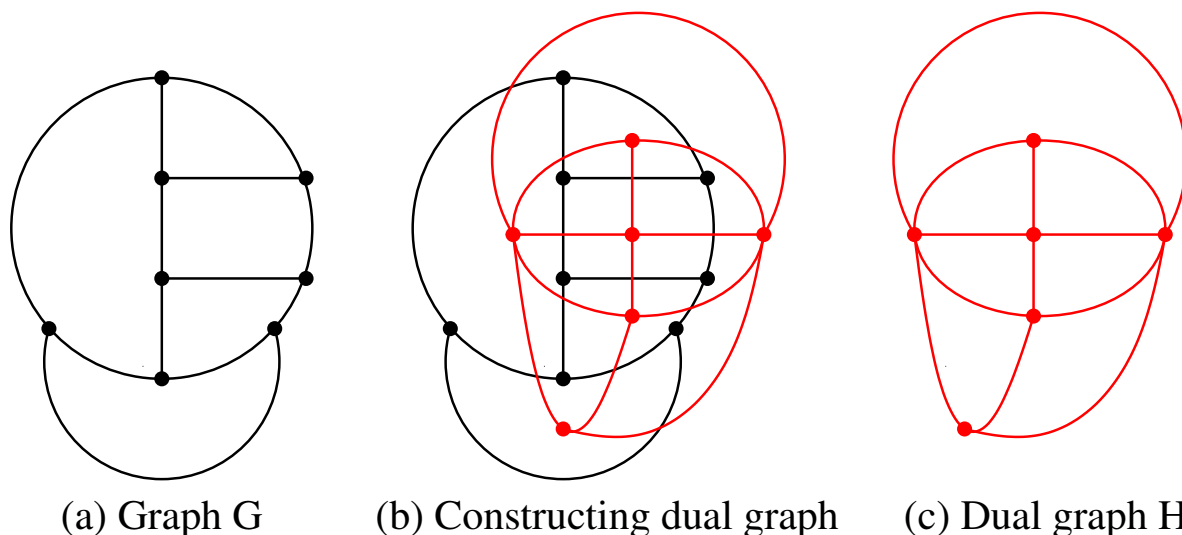
Theorem

A connected, simple, planar graph has a vertex with degree at most 5.

Proof:

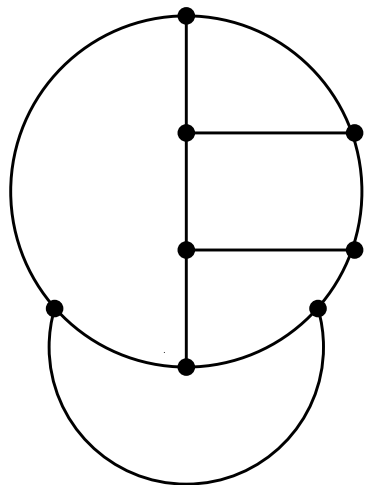
- The sum of vertex degrees in any graph equals $2E$.
- Assume by way of contradiction that all vertices have degree ≥ 6 . Then the sum of vertex degrees is $\geq 6V$.
- So $2E \geq 6V$, so $E \geq 3V$.
- This contradicts $E \leq 3V - 6$, so some vertex has degree ≤ 5 .

Dual graph

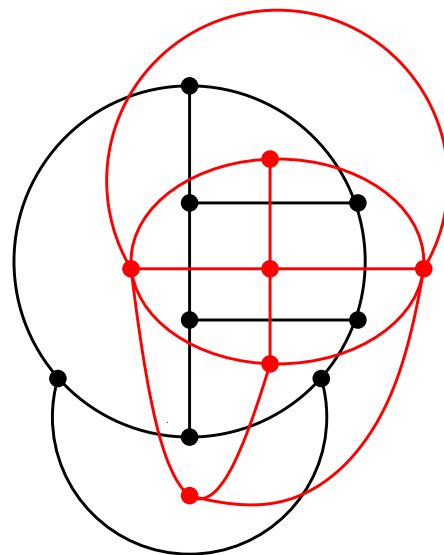


- Start with a planar embedding of a graph G (shown in black).
- Draw a red vertex inside each face, including the “infinite face.”
- For every edge e of G :
 - Let a, b be the red vertices in the faces on the two sides of e .
 - Draw a red edge $\{a, b\}$ crossing e .
- Remove the original graph G to obtain the red graph H .
- H is a *dual graph* of G . (It is “a” dual graph instead of “the” dual graph, since it may depend on how G is drawn.)
- **If G is connected**, then G is also a dual graph of H — just switch the roles of the colors!

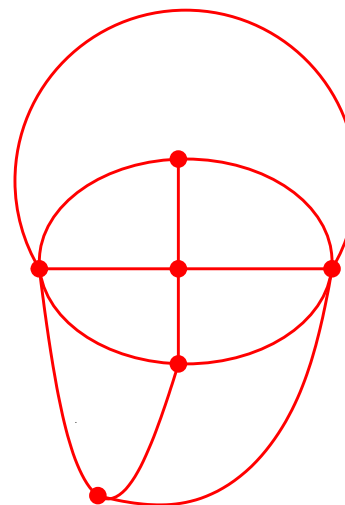
Dual graph



(a) Graph G



(b) Constructing dual graph



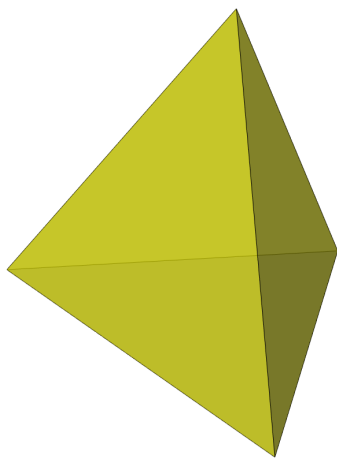
(c) Dual graph H

	G	H
V	8	6
E	12	12
F	6	8

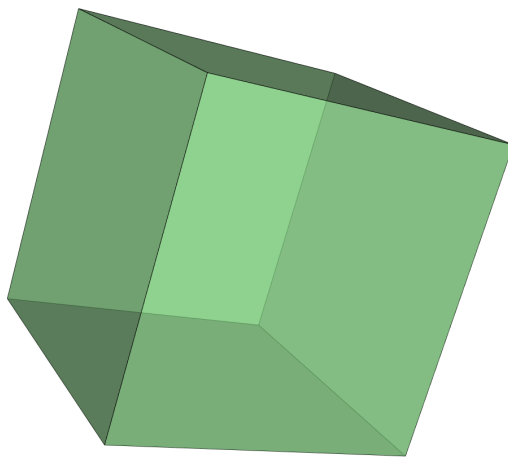
- G and H have the same number of edges:
 - Each edge of G crosses exactly one edge of H and vice-versa.
- # faces of G = # vertices of H and # faces of H = # vertices of G :
 - Bijections: vertices of either graph \leftrightarrow faces of the other.

12.2. Classifying regular polyhedra

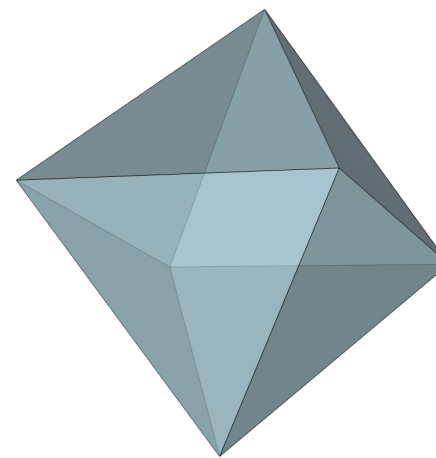
Classifying regular polyhedra



Tetrahedron



Cube



Octahedron

- A *polyhedron* is a 3D solid whose surface consists of polygons. As a graph, no loops and no multiple edges.
- All faces have ≥ 3 edges and all vertices are in ≥ 3 edges.
- To be 3D, there must be ≥ 4 vertices, ≥ 4 faces, and ≥ 6 edges.
- A *regular polyhedron* has these symmetries:
 - All faces are regular ℓ -gons for the same $\ell \geq 3$.
 - All vertices have the same degree ($r \geq 3$).
 - All edges have the same length.
 - All pairs of adjacent faces have the same angle between them.

Classifying regular polyhedra

- Suppose all vertices have the same degree $r \geq 3$ and all faces are ℓ -gons (same $\ell \geq 3$ for all faces).
- The sum of vertex degrees is $r \cdot V = 2E$, so $V = 2E/r$.
- The sum of face degrees is $\ell \cdot F = 2E$, so $F = 2E/\ell$.
- Plug these into $V - E + F = 2$:

$$\frac{2E}{r} - E + \frac{2E}{\ell} = 2 \quad E \cdot \left(\frac{2}{r} - 1 + \frac{2}{\ell} \right) = 2 \quad E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}$$

- We have to find all integers $r, \ell \geq 3$ for which V, E, F are positive integers, and then check if polyhedra with those parameters exist.

Classifying regular polyhedra

- Suppose all vertices have the same degree $r \geq 3$ and all faces are ℓ -gons (same $\ell \geq 3$ for all faces).
- Compute (V, E, F) using $E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}$, $V = \frac{2E}{r}$, $F = \frac{2E}{\ell}$:
- E.g., $r = 3$ and $\ell = 4$ gives

$$E = \frac{2}{\frac{2}{3} + \frac{2}{4} - 1} = \frac{2}{1/6} = 12$$

$$V = 2(12)/3 = 8$$

$$F = 2(12)/4 = 6$$

- What shape is it?

Classifying regular polyhedra

What range of vertex degree (r) and face degree (ℓ) are permitted?

First method

- We have $r \geq 3$.
Since some vertex has degree ≤ 5 , all do, so r is 3, 4, or 5.
- Vertices and faces are swapped in the dual graph, so ℓ is 3, 4, or 5.

Second method: analyze formula of E

- E is a positive integer, so its denominator must be positive:

$$\frac{2}{r} + \frac{2}{\ell} - 1 > 0$$

- We have $r, \ell \geq 3$.
- If both $r, \ell \geq 4$, the denominator of E is $\leq \frac{2}{4} + \frac{2}{4} - 1 = 0$, which is invalid. So r and/or ℓ is 3.
- If $r = 3$, then the denominator of E is $\frac{2}{3} + \frac{2}{\ell} - 1 = \frac{2}{\ell} - \frac{1}{3}$.
To be positive requires $\ell \leq 5$.
- Similarly, if $\ell = 3$ then $r \leq 5$.

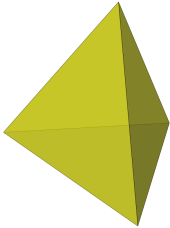
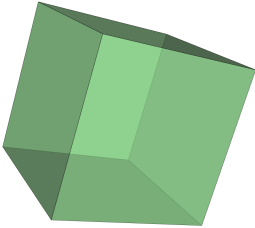
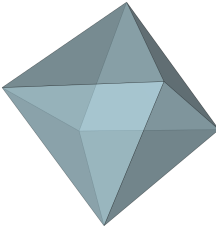
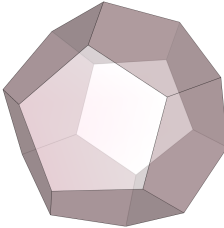
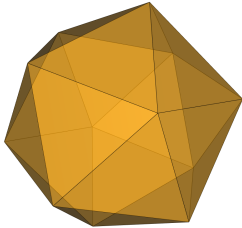
Classifying regular polyhedra

- Suppose all vertices have the same degree $r \in \{3, 4, 5\}$ and all faces are ℓ -gons (same $\ell \in \{3, 4, 5\}$ for all faces).
- Compute (V, E, F) using $E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}$, $V = \frac{2E}{r}$, $F = \frac{2E}{\ell}$:

(V, E, F)	$\ell = 3$	$\ell = 4$	$\ell = 5$
$r = 3$	(4, 6, 4)	(8, 12, 6)	(20, 30, 12)
$r = 4$	(6, 12, 8)	Division by 0	(-10, -20, -8)
$r = 5$	(12, 30, 20)	(-8, -20, -10)	(-4, -10, -4)

- If V, E, F are not all positive integers, it can't work (shown in pink).
- We found five possible values of (V, E, F) with graph theory. Use geometry to actually find the shapes (if they exist).

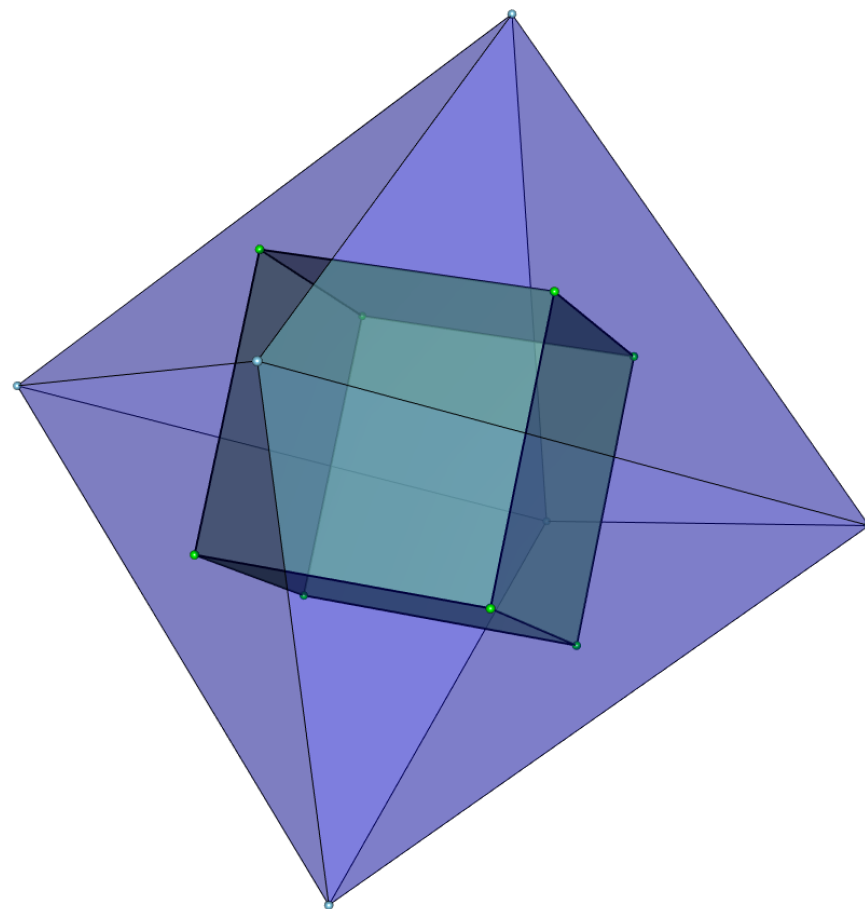
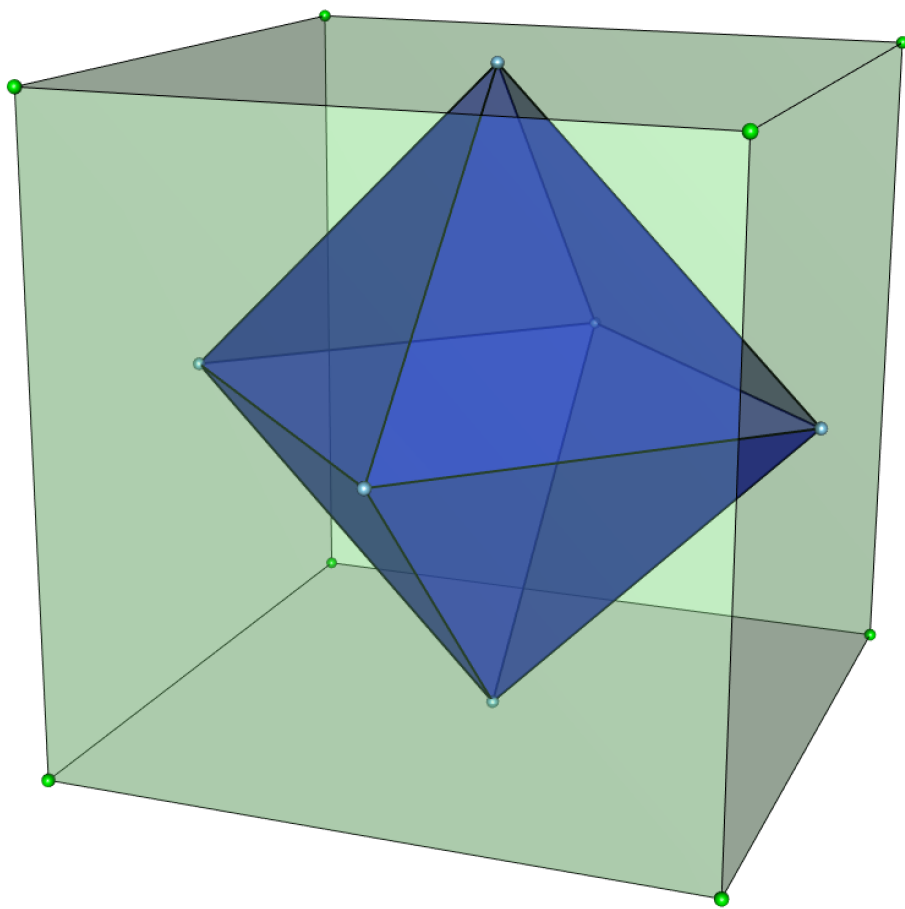
Classifying regular polyhedra

Shape					
	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
$r =$ vertex degree	3	3	4	3	5
$\ell =$ face degree	3	4	3	5	3
$V =$ # vertices	4	8	6	20	12
$E =$ # edges	6	12	12	30	30
$F =$ # faces	4	6	8	12	20

- These are known as the *Platonic solids*.
- The cube and octahedron are dual graphs.
The dodecahedron and icosahedron are dual graphs.
The tetrahedron is its own dual.

Octahedron and cube are dual

- Can draw either one inside the other.
Place a dual vertex at the center of each face.
- In 3D, this construction shrinks the dual, vs. in 2D, it did not.



11.1 and 12.3. Graph colorings

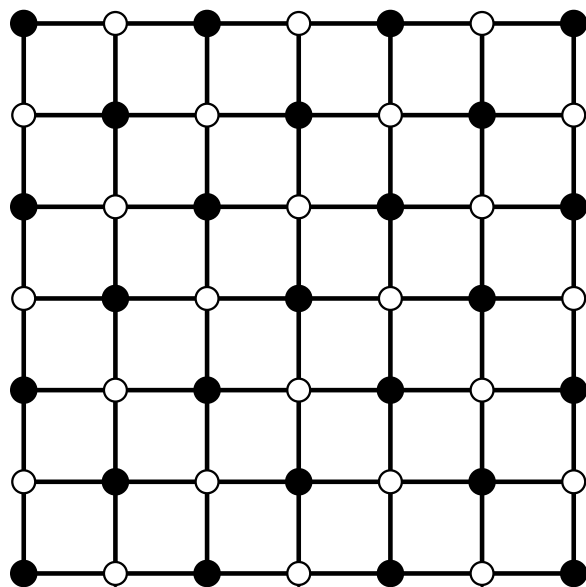
Graph Colorings

- Let G be a graph and C be a set of *colors*, e.g.,

$$C = \{\text{black, white}\} \quad C = \{a, b\} \quad C = \{1, 2\}$$

- A *proper coloring* of G by C is to assign a color from C to every vertex, such that in every edge $\{v, w\}$, the vertices v and w have different colors.
- G is *k -colorable* if it has a proper coloring with k colors (e.g., $C = [k]$).

Coloring a bipartite graph



A *bipartite graph* is a graph in which:

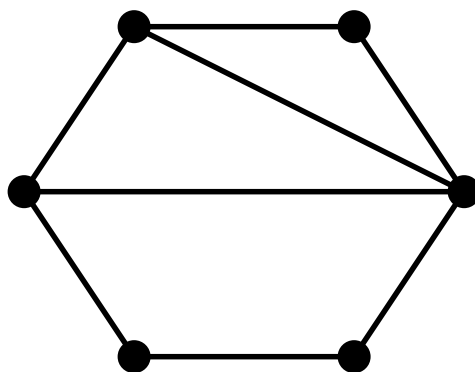
- The set of vertices can be split as $V = A \cup B$, where $A \cap B = \emptyset$.
- Every edge has the form $\{a, b\}$ where $a \in A$ and $b \in B$.

A graph is bipartite if and only if it is 2-colorable:
set $A =$ black vertices, $B =$ white vertices.

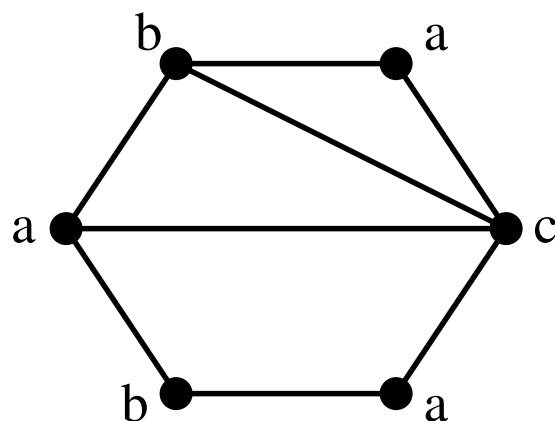
Color this with as few colors a, b, c, d, e, \dots as possible

The *chromatic number*, $\chi(G)$, of a graph G is the minimum number of colors needed for a proper coloring of G .

Color this with as few colors as possible:

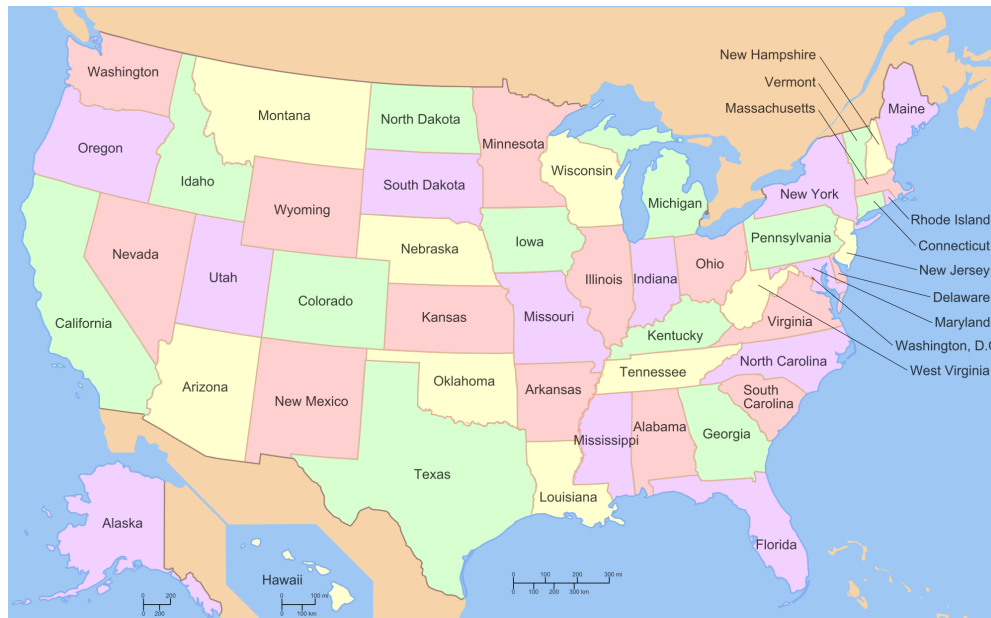


Color this with as few colors a, b, c, d, e, \dots as possible



- We've shown it's 3-colorable, so $\chi(G) \leq 3$.
- It has a triangle as a subgraph, which requires 3 colors. Other vertices may require additional colors, so $\chi(G) \geq 3$.
- Combining these gives $\chi(G) = 3$.
- A triangle is K_3 .
More generally, if G has K_m as a subgraph, then $\chi(G) \geq m$.

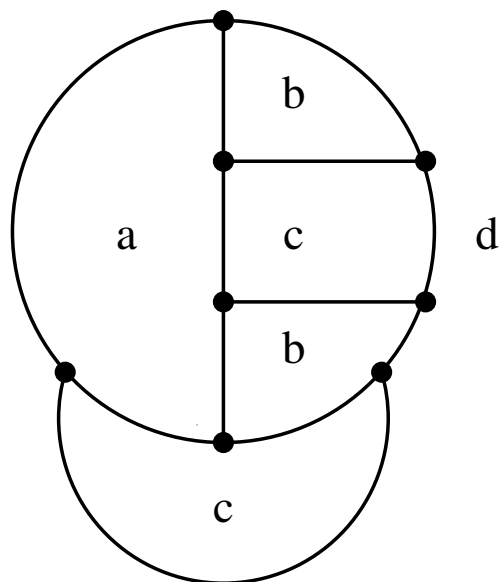
Coloring maps



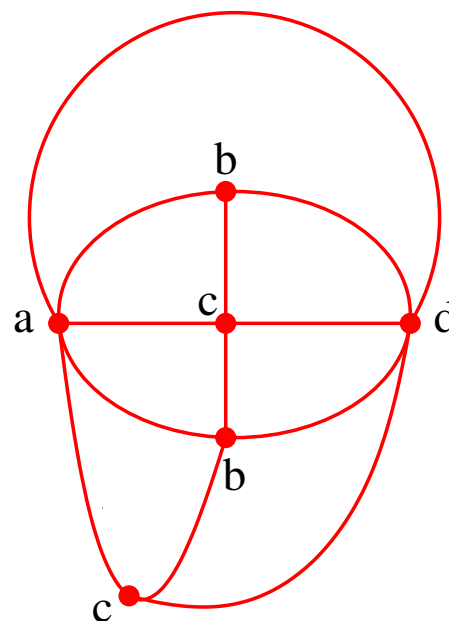
http://en.wikipedia.org/wiki/File:Map_of_USA_with_state_names_2.svg

- Color states so that neighboring states have different colors. This map uses 4 colors for the states.
- Assume each state is a contiguous region. Michigan isn't. All of its regions have to be colored the same, which could increase the number of colors required, but we can artificially fill in Lake Michigan to make Michigan contiguous.
- Also assume the states form a contiguous region. Alaska and Hawaii are isolated, and just added onto the map separately.

A proper coloring of the faces of a graph \leftrightarrow a proper coloring of the vertices of its dual graph



Coloring faces of G



Coloring vertices of H

- The regions/states/countries of the map are faces of a graph, G .
- Place a vertex inside each region and form the dual graph, H .
- A proper coloring of the vertices of H gives a proper coloring of the faces of G .

Four Color Theorem

Theorem (Four Color Theorem)

Every planar graph has a proper coloring with at most four colors.

- Map makers have believed this for centuries, but it was not proven.
- This was the first major theorem to be proved using a computer program (Kenneth Appel and Wolfgang Haken, 1976).
- The original proof had 1936 cases! Their program determined the cases and showed they are all 4-colorable.
- The proof was controversial because it was the first proof that was impractical for any human to verify.
- Over the years, people have found errors in the proof, but they have been fixed, and the result still stands. The number of cases has been cut down to 633.