Chapter 1. Pigeonhole Principle

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Math 184A
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Pigeonhole principle

https://commons.wikimedia.org/wiki/File:TooManyPigeons.jpg

If you put 10 pigeons into 9 holes, at least one hole has at least two pigeons.

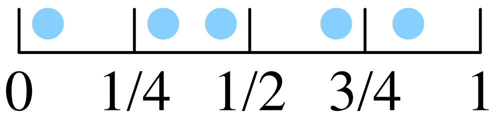
Pigeonhole Principle

If you put n items into k boxes, for integers $n > k > 0$, then at least one box has 2 or more items.

Example: Five real numbers in $[0, 1]$

- Pick five real numbers between 0 and 1: $x_1, \ldots, x_5 \in [0, 1]$.
  We will show that at least two of them differ by at most $1/4$.

- Split $[0, 1]$ into 4 bins of width $1/4$:



- Put 5 numbers into 4 bins. At least two numbers are in the same bin, and thus, are within $1/4$ of each other.

- *Technicality:* We didn't say how to handle $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.
  Is $\frac{1}{4}$ in the 1st or 2nd bin? However, it doesn't affect the conclusion.

Generalization

For an integer $n \geqslant 2$, pick n numbers between 0 and 1.
Then at least two of them are within $\frac{1}{n-1}$ of each other.

Example: Picking 6 numbers from 1 to 10

Pick 6 different integers from 1 to 10.
Claim: Two of them must add up to 11.

Example

1, 4, 7, 8, 9, 10 has $1 + 10 = 11$ and $4 + 7 = 11$.

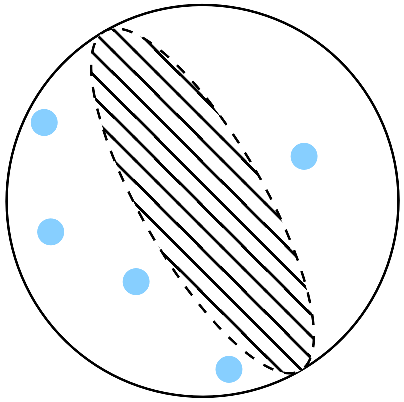
Proof.

- Pair up the numbers from 1 to 10 as follows:

$$(1, 10) \quad (2, 9) \quad (3, 8) \quad (4, 7) \quad (5, 6)$$

- There are 5 pairs, each summing to 11.
- At least two of the 6 numbers must be in the same pair. □

Example: 5 points on a sphere



Select any five points on a sphere. Then the sphere may be cut in half (both halves include the boundary) so that one of the two hemispheres has at least four of the points.

Proof:

- Pick any two of the five points.
- Form the great circle through those points, splitting it into two hemispheres.
- Three of the five points remain.
- One of the hemispheres contains at least two of these points, plus the original two points. Thus, it contains at least four of the points.

Terminology

- In the last two examples:
  - “pigeons” are numbers $1, \ldots, 10$, and “holes” are ordered pairs;
  - “pigeons” are points on a sphere and the “holes” are hemispheres.
- *“Pigeons”* and *“holes”* are abstractions, and may be replaced by other terms:
  - Place pigeons into holes.
  - Place balls into boxes.
  - Place items into bins.
- In any particular problem, be sure to
  - Properly identify what plays the roles of “pigeons” and “holes.”
  - Give the rule for placing pigeons into holes.

Notation

Let x be a real number.

- $\lfloor x \rfloor =$ *floor* of x = largest integer that's $\leqslant x$
- $\lceil x \rceil =$ *ceiling* of x = smallest integer that's $\geqslant x$

$$\begin{array}{lll} \lfloor 2.5 \rfloor = 2 & \lfloor -2.5 \rfloor = -3 & \lfloor 2 \rfloor = \lceil 2 \rceil = 2 \\ \lceil 2.5 \rceil = 3 & \lceil -2.5 \rceil = -2 & \lfloor -2 \rfloor = \lceil -2 \rceil = -2 \end{array}$$

- You have probably seen $[x]$ used for *greatest integer*, and you may encounter some differences in definitions for negative numbers. However, we will use the floor and ceiling notation above, and will use square brackets for

$$[n] = \{1, 2, 3, \ldots, n\} \text{ (for integers } n \geqslant 1) \quad \text{and} \quad [0] = \emptyset.$$

This notation is common in Combinatorics.

Generalized Pigeonhole Principle

Put n items into k boxes (integers $n \geqslant 0$ and $k \geqslant 1$). Then

- There is a box with at least $\lceil n/k \rceil$ items.
- There is a box with at most $\lfloor n/k \rfloor$ items.

Example of Generalized Pigeonhole Principle

150 pigeons are placed into 60 holes.

- At least one hole has $\lceil 150/60 \rceil = \lceil 2.5 \rceil = 3$ or more pigeons.
- At least one hole has $\lfloor 150/60 \rfloor = \lfloor 2.5 \rfloor = 2$ or fewer pigeons.

Relation to basic version of the pigeonhole principle

- The basic version of the pigeonhole principle is for $n > k > 0$.
- Then $\lceil n/k \rceil \geqslant n/k > 1$, so $\lceil n/k \rceil \geqslant 2$.
- So there is a box with at least 2 items.

Proof for this example

Example

150 pigeons are placed into 60 holes.

- At least one hole has $\lceil 150/60 \rceil = \lceil 2.5 \rceil = 3$ or more pigeons.
- At least one hole has $\lfloor 150/60 \rfloor = \lfloor 2.5 \rfloor = 2$ or fewer pigeons.

Proof that at least one hole has 3 or more.

- Suppose all holes have at most 2 pigeons.
- Then combined, the holes have at most $2(60) = 120$ pigeons. But they have 150 pigeons, and $120 < 150$, a contradiction.
- Thus, some hole has at least 3 pigeons. $\square$

Proof for this example

Example

150 pigeons are placed into 60 holes.

- At least one hole has $\lceil 150/60 \rceil = \lceil 2.5 \rceil = 3$ or more pigeons.
- At least one hole has $\lfloor 150/60 \rfloor = \lfloor 2.5 \rfloor = 2$ or fewer pigeons.

Proof that at least one hole has 2 or fewer.

- Suppose all holes have at least 3 pigeons.
- Then combined, the holes have at least $3(60) = 180$ pigeons. But they have 150 pigeons, and $180 > 150$, a contradiction.
- Thus, some hole has at most 2 pigeons. □

Proof of Generalized Pigeonhole Principle

Put n items into k boxes (with $n \geqslant 0$ and $k \geqslant 1$). Then

1. There is a box with at least $\lceil n/k \rceil$ items.
2. There is a box with at most $\lfloor n/k \rfloor$ items.

- Let a_i be the number of items in box i, for $i = 1, \ldots, k$.
- Each a_i is a nonnegative integer, and $a_1 + \cdots + a_k = n$.
- Assume (by way of contradiction) that all boxes have fewer than $\lceil n/k \rceil$ items: $a_i < \lceil n/k \rceil$ for all i.
  Thus, $a_i \leqslant \lceil n/k \rceil - 1$ for all i.
- Then the total number of items placed in the boxes is

$$\underbrace{a_1 + \cdots + a_k}_{=n} \leqslant k \cdot \left(\lceil n/k \rceil - 1 \right) .$$

Proof of Generalized Pigeonhole Principle

1. Show there is a box with at least $\lceil n/k \rceil$ items

- Assume that every box has fewer than $\lceil n/k \rceil$ items.
  Then the total number of items placed in the boxes is

$$(*) \qquad \underbrace{a_1 + \cdots + a_k}_{=n} \leqslant k \cdot (\lceil n/k \rceil - 1)$$

- Dividing n by k gives $n = kq + r$, where the quotient, q, is an integer and the remainder, r, is in the range $0 \leqslant r \leqslant k-1$.
  - If there is no remainder ($r = 0$), then $\lceil n/k \rceil = n/k = q$.
    Inequality $(*)$ becomes $\quad n \leqslant k(q-1) = kq - k = n - k$,
    so $\quad n \leqslant n - k$.
    But $n - k$ is smaller than n, a contradiction.
  - If there is a remainder ($1 \leqslant r \leqslant k-1$), then $\lceil n/k \rceil = q + 1$.
    Inequality $(*)$ becomes $\quad n \leqslant k((q+1)-1) = kq = n - r$,
    so $\quad n \leqslant n - r$.
    But $n - r$ is smaller than n, a contradiction.
- Thus, our assumption that all $a_i < \lceil n/k \rceil$ was incorrect, so some box must have $a_i \geqslant \lceil n/k \rceil$.

Proof of Generalized Pigeonhole Principle

2. Show there is a box with at most $\lfloor n/k \rfloor$ items

- Assume that every box has more than $\lfloor n/k \rfloor$ items.
  Then the total number of items placed in the boxes is

$$(*) \qquad \underbrace{a_1 + \cdots + a_k}_{=n} \geqslant k \cdot (\lfloor n/k \rfloor + 1)$$

- Dividing n by k gives $n = kq + r$, where the quotient, q, is an integer and the remainder, r, is in the range $0 \leqslant r \leqslant k - 1$.

- $\lfloor n/k \rfloor = q$, so inequality $(*)$ becomes

$$n \geqslant k(q+1) = kq + k = n - r + k,$$

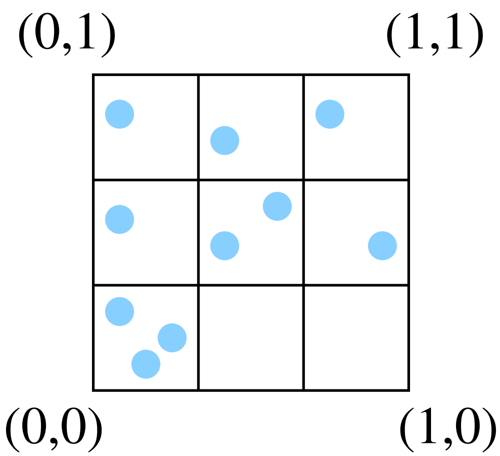
$$\text{so } n \geqslant n + (k - r).$$

  But $k - r > 0$, so n is smaller than $n + (k - r)$, a contradiction.

- Thus, our assumption that all $a_i > \lfloor n/k \rfloor$ was incorrect, so some box must have $a_i \leqslant \lfloor n/k \rfloor$.

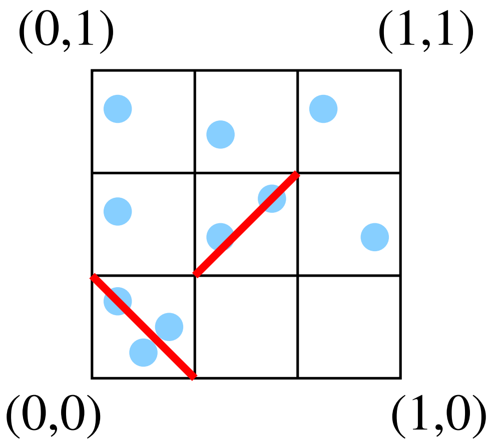
Example: Ten points in the unit square

2D generalization of the 5 points in $[0, 1]$ problem



- Pick 10 points $(x_1, y_1), \ldots, (x_{10}, y_{10})$ in $[0, 1] \times [0, 1]$.
- We'll show that two of them have to be "close" together, within a certain small distance.
- Split $[0, 1] \times [0, 1]$ into a 3×3 grid of $1/3 \times 1/3$ squares.
- 10 points placed into 9 squares, and $10 > 9$, so some square has at least two points.

Example: Ten points in the unit square

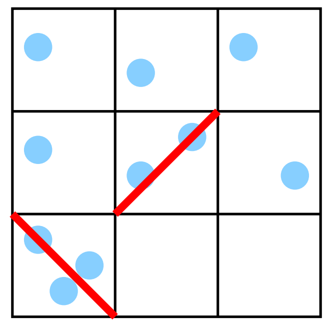


- The distance between two points in the same $1/3 \times 1/3$ square is

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} \leqslant \sqrt{(1/3)^2 + (1/3)^2} = \frac{\sqrt{2}}{3}$$

- So the distance is at most $\sqrt{2}/3$.
  $\sqrt{2}/3$ is achieved for opposite corners of a square (the red diagonals show some examples).

Example: Ten points in the unit square



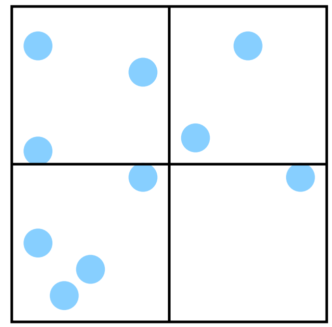
- Pick 10 points $(x_1, y_1), \ldots, (x_{10}, y_{10})$ in $[0, 1] \times [0, 1]$.
- Split $[0, 1] \times [0, 1]$ into a 3×3 grid of $1/3 \times 1/3$ squares.
- 10 points placed into 9 squares, and $10 > 9$, so some square has at least two points.
- Two points in the same square are at most $\frac{\sqrt{2}}{3}$ apart.

Result

If you pick 10 points in $[0, 1] \times [0, 1]$, there must be two of them at most $\sqrt{2}/3 \approx 0.4714045$ apart.

Example: Ten points in the unit square — 2nd version

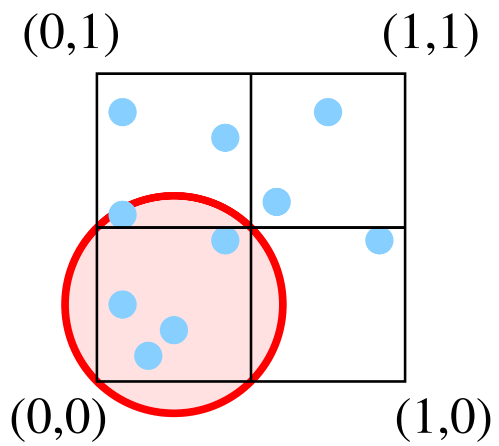
Using Generalized Pigeonhole Principle



- Pick 10 points $(x_1, y_1), \ldots, (x_{10}, y_{10})$ in $[0,1] \times [0,1]$.
- Split $[0,1] \times [0,1]$ into a 2×2 grid of $1/2 \times 1/2$ squares.
- One of the 4 squares must have at least $\lceil 10/4 \rceil = \lceil 2.5 \rceil = 3$ points.
- The farthest apart that two points can be in the same square is
$$\sqrt{(1/2)^2 + (1/2)^2} = \frac{\sqrt{2}}{2}.$$
- **Result:** For any 10 points in a unit square, you can find three of them where the distance between each two of the three is $\leqslant \frac{\sqrt{2}}{2}$.
But there's a more elegant way to formulate this.

Example: Ten points in the unit square — 2nd version

Using Generalized Pigeonhole Principle



Result

If you pick 10 points in $[0,1] \times [0,1]$, there is a disk of diameter $\sqrt{2}/2 \approx 0.7071068$ with at least three of the points.

- Select a $1/2 \times 1/2$ square with at least 3 of the points.
- Circumscribe a circle around it, by making either diagonal of the square a diameter of the circle. Fill in the circle to make a disk.
- In this example, the disk has the 4 points from the selected square, plus a 5^{th} point since the disk extends outside the square.

Caution: Common homework mistake

The Pigeonhole Principle doesn't say *which* pigeons will be in the same hole!

Don't do this!

- Number the pigeons and place them into k holes in order $1, 2, \ldots$.
- The $(k+1)^{\text{st}}$ pigeon must be in a hole with one of the first k. **NOPE!**

Placing points in the unit square in order $1, \ldots, 10$ does not guarantee that 10 will be in the same bin as one of $1, \ldots, 9$.
Here, 10 is in its own bin: