

Chapter 1. Pigeonhole Principle

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Pigeonhole principle



<https://commons.wikimedia.org/wiki/File:TooManyPigeons.jpg>

If you put 10 pigeons into 9 holes, at least one hole has at least two pigeons.

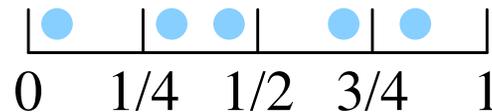
Pigeonhole Principle

If you put n items into k boxes, for integers $n > k > 0$, then at least one box has 2 or more items.

Example: Five real numbers in $[0, 1]$

- Pick five real numbers between 0 and 1: $x_1, \dots, x_5 \in [0, 1]$. We will show that at least two of them differ by at most $1/4$.

- Split $[0, 1]$ into 4 bins of width $1/4$:



- Put 5 numbers into 4 bins. At least two numbers are in the same bin, and thus, are within $1/4$ of each other.
- *Technicality:* We didn't say how to handle $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. Is $\frac{1}{4}$ in the 1st or 2nd bin? However, it doesn't affect the conclusion.

Generalization

For an integer $n \geq 2$, pick n numbers between 0 and 1. Then at least two of them are within $\frac{1}{n-1}$ of each other.

Example: Picking 6 numbers from 1 to 10

Pick 6 different integers from 1 to 10.

Claim: Two of them must add up to 11.

Example

1, 4, 7, 8, 9, 10 has $1 + 10 = 11$ and $4 + 7 = 11$.

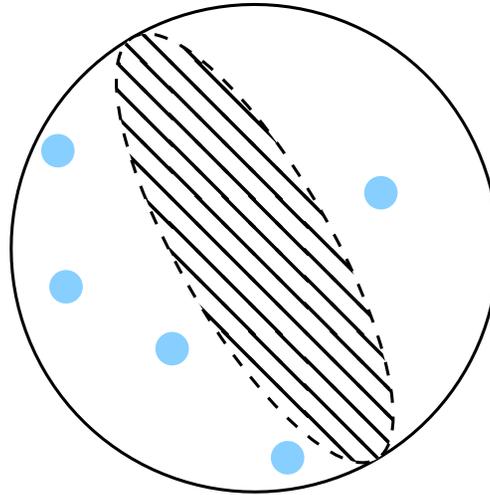
Proof.

- Pair up the numbers from 1 to 10 as follows:

$$(1, 10) \quad (2, 9) \quad (3, 8) \quad (4, 7) \quad (5, 6)$$

- There are 5 pairs, each summing to 11.
- At least two of the 6 numbers must be in the same pair. □

Example: 5 points on a sphere



Select any five points on a sphere. The sphere may be cut in half (both halves include the boundary) so that one of the two hemispheres has at least four of the points.

- Pick any two of the five points.
- Form the great circle through those points, splitting it into two hemispheres.
- Three of the five points remain.
- One of the two hemispheres must contain at least two of these points, plus the original two points. Thus, it contains at least four of the points.

Terminology

- In the last two examples:
 - “pigeons” are numbers $1, \dots, 10$, and “holes” are ordered pairs;
 - “pigeons” are points on a sphere and the “holes” are hemispheres.
- As abstract terms, “*pigeons*” and “*holes*” may be replaced by other terms. Some examples (but by no means a complete list):
 - Place pigeons into holes.
 - Place balls into boxes.
 - Place items into bins.
- In any particular problem, be sure to
 - Properly identify what plays the roles of “pigeons” and “holes.”
 - Give the rule for placing pigeons into holes.

Notation

Let x be a real number.

- $\lfloor x \rfloor = \text{floor of } x = \text{largest integer that's } \leq x$
- $\lceil x \rceil = \text{ceiling of } x = \text{smallest integer that's } \geq x$

$$\begin{array}{lll} \lfloor 2.5 \rfloor = 2 & \lfloor -2.5 \rfloor = -3 & \lfloor 2 \rfloor = \lceil 2 \rceil = 2 \\ \lceil 2.5 \rceil = 3 & \lceil -2.5 \rceil = -2 & \lceil -2 \rceil = \lfloor -2 \rfloor = -2 \end{array}$$

- You have probably seen $[x]$ used for *greatest integer*, and you may encounter some differences in definitions for negative numbers. However, we will use the floor and ceiling notation above, and will use square brackets for

$$[n] = \{1, 2, 3, \dots, n\} \text{ (for integers } n \geq 1) \quad \text{and} \quad [0] = \emptyset.$$

This notation is common in Combinatorics.

Generalized Pigeonhole Principle

Put n items into k boxes. Then

- There is a box with at least $\lceil n/k \rceil$ items.
- There is a box with at most $\lfloor n/k \rfloor$ items.

Example

150 pigeons are placed into 60 holes.

- At least one hole has $\lceil 150/60 \rceil = \lceil 2.5 \rceil = 3$ or more pigeons.
- At least one hole has $\lfloor 150/60 \rfloor = \lfloor 2.5 \rfloor = 2$ or fewer pigeons.

Proof for this example

Example

150 pigeons are placed into 60 holes.

- At least one hole has $\lceil 150/60 \rceil = \lceil 2.5 \rceil = 3$ or more pigeons.
- At least one hole has $\lfloor 150/60 \rfloor = \lfloor 2.5 \rfloor = 2$ or fewer pigeons.

Proof that at least one hole has 3 or more.

- Suppose all holes have at most 2 pigeons.
- Then combined, the holes have at most $2(60) = 120$ pigeons. But they have 150 pigeons, and $120 < 150$, a contradiction.
- Thus, some hole has at least 3 pigeons. □

Proof for this example

Example

150 pigeons are placed into 60 holes.

- At least one hole has $\lceil 150/60 \rceil = \lceil 2.5 \rceil = 3$ or more pigeons.
- At least one hole has $\lfloor 150/60 \rfloor = \lfloor 2.5 \rfloor = 2$ or fewer pigeons.

Proof that at least one hole has 2 or fewer.

- Suppose all holes have at least 3 pigeons.
- Then combined, the holes have at least $3(60) = 180$ pigeons. But they have 150 pigeons, and $180 > 150$, a contradiction.
- Thus, some hole has at most 2 pigeons. □

Proof of Generalized Pigeonhole Principle

Put n items into k boxes. Then

- 1 There is a box with at least $\lceil n/k \rceil$ items.
- 2 There is a box with at most $\lfloor n/k \rfloor$ items.

- Let a_i be the number of items in box i , for $i = 1, \dots, k$.
- Each a_i is a nonnegative integer, and $a_1 + \dots + a_k = n$.
- Assume (by way of contradiction) that all boxes have fewer than $\lceil n/k \rceil$ items: $a_i < \lceil n/k \rceil$ for all i .
Thus, $a_i \leq \lceil n/k \rceil - 1$ for all i .
- Then the total number of items placed in the boxes is

$$\underbrace{a_1 + \dots + a_k}_{=n} \leq k \cdot (\lceil n/k \rceil - 1) .$$

Proof of Generalized Pigeonhole Principle

1. Show there is a box with at least $\lceil n/k \rceil$ items

- Assume that every box has fewer than $\lceil n/k \rceil$ items. Then the total number of items placed in the boxes is

$$(*) \quad \underbrace{a_1 + \cdots + a_k}_{=n} \leq k \cdot (\lceil n/k \rceil - 1)$$

- Dividing n by k gives $n = kq + r$, where the quotient, q , is an integer and the remainder, r , is in the range $0 \leq r \leq k - 1$.
 - If there is no remainder ($r = 0$), then $\lceil n/k \rceil = n/k = q$. Inequality (*) becomes $n \leq k(q - 1) = n - k$. But $n - k$ is smaller than n , a contradiction.
 - If there is a remainder ($1 \leq r \leq k - 1$), then $\lceil n/k \rceil = q + 1$. Inequality (*) becomes $n \leq k((q + 1) - 1) = kq = n - r$, so $n \leq n - r$. But $n - r$ is smaller than n , a contradiction.
- Thus, our assumption that all $a_i < \lceil n/k \rceil$ was incorrect, so some box must have $a_i \geq \lceil n/k \rceil$.

Proof of Generalized Pigeonhole Principle

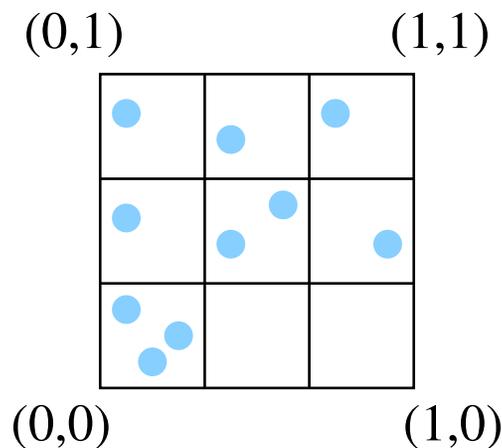
2. Show there is a box with at most $\lfloor n/k \rfloor$ items

- Assume that every box has more than $\lfloor n/k \rfloor$ items. Then the total number of items placed in the boxes is

$$(*) \quad \underbrace{a_1 + \cdots + a_k}_{=n} \geq k \cdot (\lfloor n/k \rfloor + 1)$$

- Dividing n by k gives $n = kq + r$, where the quotient, q , is an integer and the remainder, r , is in the range $0 \leq r \leq k - 1$.
- $\lfloor n/k \rfloor = q$, so inequality $(*)$ becomes $n \geq k(q + 1) = kq + k = n - r + k$, so $n \geq n + (k - r)$. But $k - r > 0$, so n is smaller than $n + (k - r)$, a contradiction.
- Thus, our assumption that all $a_i > \lfloor n/k \rfloor$ was incorrect, so some box must have $a_i \leq \lfloor n/k \rfloor$.

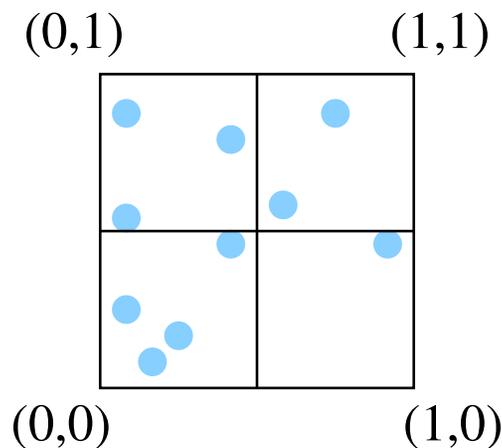
Example: Ten points in the unit square



- Pick 10 points $(x_1, y_1), \dots, (x_{10}, y_{10})$ in $[0, 1] \times [0, 1]$.
- Split $[0, 1] \times [0, 1]$ into a 3×3 grid of $1/3 \times 1/3$ squares.
- 10 points placed into 9 squares, and $10 > 9$, so some square has at least two points.
- The farthest apart that points can be within the same square is $\sqrt{(1/3)^2 + (1/3)^2} = \frac{\sqrt{2}}{3}$.
- Thus, at least two points are at most $\frac{\sqrt{2}}{3}$ apart.

If you pick 10 points in $[0, 1] \times [0, 1]$, there must be two of them at most $\frac{\sqrt{2}}{3} \approx 0.4714045$ apart.

Example: Ten points in the unit square



- Pick 10 points $(x_1, y_1), \dots, (x_{10}, y_{10})$ in $[0, 1] \times [0, 1]$.
- Split $[0, 1] \times [0, 1]$ into a 2×2 grid of $1/2 \times 1/2$ squares.
- One of the 4 squares must have at least $\lceil 10/4 \rceil = \lceil 2.5 \rceil = 3$ points.
- The farthest apart that two points can be within the same square is $\sqrt{(1/2)^2 + (1/2)^2} = \frac{\sqrt{2}}{2}$.
- Thus, there is a disk (filled-in circle) of diameter $\sqrt{2}/2$ with at least three of the points.

If you pick 10 points in $[0, 1] \times [0, 1]$, there is a disk of radius $\sqrt{2}/4 \approx 0.3535534$ with at least three of the points.

Caution: Common homework mistake

Don't do this!

- Number the pigeons and place them into k holes in order $1, 2, \dots$
- The $(k + 1)^{\text{st}}$ pigeon must be in the same hole as one of the first k .

(NOPE)

- **But the Pigeonhole Principle doesn't say *which* pigeons will be in the same hole!**
- Placing points in the unit square in order $1, \dots, 10$ does not guarantee that 10 will be in the same bin as one of $1, \dots, 9$. Here, 10 is in its own bin:

