3. Elementary Counting Problems
4.1,4.2. Binomial and Multinomial Theorems
2. Mathematical Induction

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Multiplication rule

Combinatorics is a branch of Mathematics that deals with systematic methods of counting things.

Example

- How many outcomes (x, y, z) are possible, where
$$\begin{aligned} x &= \text{roll of a 6-sided die;} \\ y &= \text{value of a coin flip;} \\ z &= \text{card drawn from a 52 card deck?} \end{aligned}$$
- $(6$ choices of $x) \times (2$ choices of $y) \times (52$ choices of $z) = \boxed{\mathbf{624}}$

Multiplication rule

The number of sequences $(x_1, x_2, \ldots, x_k)$ where there are
$$n_1 \text{ choices of } x_1, \quad n_2 \text{ choices of } x_2, \quad \ldots, \quad n_k \text{ choices of } x_k$$
is $n_1 \cdot n_2 \cdots n_k$.

This assumes the number of choices of x_i is a constant n_i that doesn't depend on the other choices.

Cartesian product

- The Cartesian Product of sets A and B is
$$A \times B = \{\, (x, y) \;:\; x \in A,\;\; y \in B \,\}$$
By the Multiplication Rule, this has size $|A \times B| = |A| \cdot |B|$.

- **Example:** $\{1, 2\} \times \{3, 4, 5\} = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

- The Cartesian product of sets A, B, and C is
$$A \times B \times C = \{\, (x, y, z) \;:\; x \in A,\;\; y \in B,\;\; z \in C \,\}$$
This has size $|A \times B \times C| = |A| \cdot |B| \cdot |C|$.

- This extends to any number of sets.

Example

Roll of a 6-sided die	$A = \{1, 2, 3, 4, 5, 6\}$	$\lvert A\rvert = 6$
Value of a coin flip	$B = \{H, T\}$	$\lvert B\rvert = 2$
Cards	$C = \{A\heartsuit, 2\heartsuit, \ldots\}$	$\lvert C\rvert = 52$

The example on the previous slide becomes
$|A \times B \times C| = 6 \cdot 2 \cdot 52 = 624$.

Notation

- We often will need an n-element set. For $n \geqslant 1$, define
$$[n] = \{1, 2, 3, \ldots, n\}$$
and also $[0] = \emptyset$. E.g.,
$$\begin{aligned} [0] &= \emptyset \\ [1] &= \{1\} \\ [2] &= \{1, 2\} \\ [3] &= \{1, 2, 3\} \\ &\vdots \end{aligned}$$

- Again, you may have seen $[x]$ used for *greatest integer*, but we instead use $\lfloor x \rfloor$ for floor and $\lceil x \rceil$ for ceiling.

Powers

Let A be a set.

- $A^k = A \times A \times \cdots \times A \qquad$ (k times)
- $|A^k| = |A|^k$

Example

- $[2] = \{1, 2\}$, with size $|[2]| = |\{1, 2\}| = 2$.
- $[2]^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$
- $|[2]^3| = 2^3 = 8$

Example

How many k letter strings are there over an n letter alphabet?

- 3-letter strings over the alphabet $\{a, b, \ldots, z\}$:
$$aaa, aab, aac, \ldots, zzy, zzz$$
There are 26^3 of them.
- In general, there are n^k strings.

Power set

- The *power set* of a set is the set of all of its subsets:

$$\mathcal{P}(S) = \{ A \ : \ A \subseteq S \}$$

$$\begin{aligned} \mathcal{P}([3]) &= \mathcal{P}(\{1,2,3\}) \\ &= \{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}\} \end{aligned}$$

- Be careful on use of $\in$ vs. $\subset$:

$$1 \notin \mathcal{P}([3])$$

$$\begin{array}{ccc} \emptyset \in \mathcal{P}([3]) & \{1\} \in \mathcal{P}([3]) & \{1\},\{2\},\{1,2\} \in \mathcal{P}([3]) \\ \{\emptyset\} \subset \mathcal{P}([3]) & \{\{1\}\} \subset \mathcal{P}([3]) & \{\{1\},\{2\},\{1,2\}\} \subset \mathcal{P}([3]) \end{array}$$

- How big is $|\mathcal{P}(S)|$?
  Equivalently, how many subsets does a set have?

Number of subsets of an n-element set

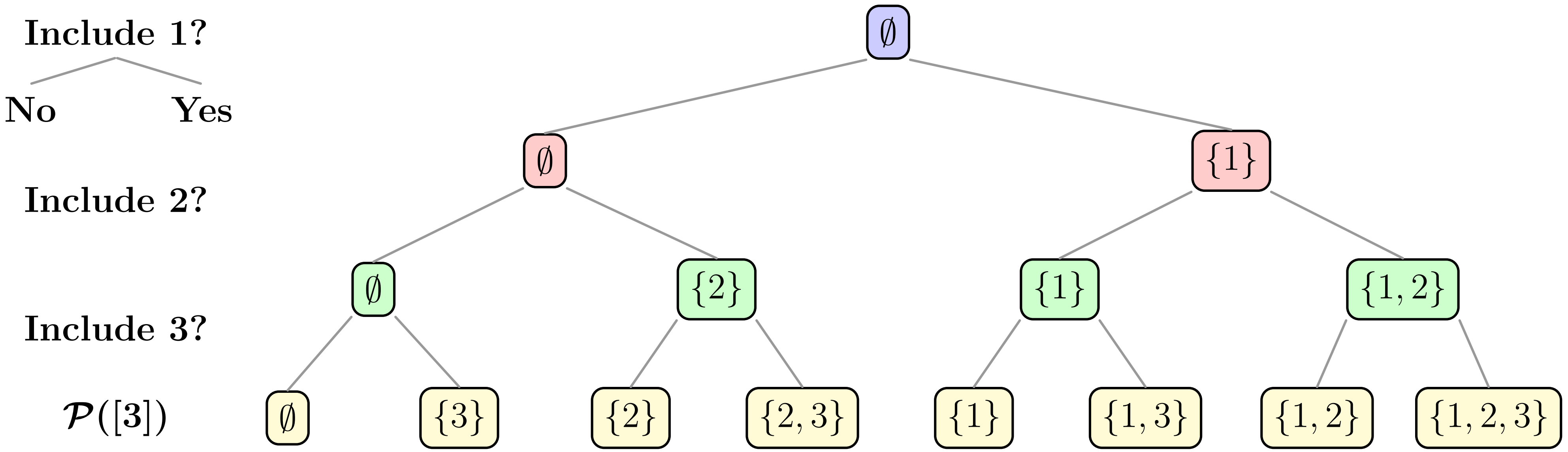
First solution

- How many subsets does an n element set have?
- We'll use $[n]$; the solution will work for any set of size n, but it's easier to work with a specific set.
- Make a sequence of decisions:
  - Include 1 or not?
  - Include 2 or not?
  - $\cdots$
  - Include n or not?

  - Total: $(\text{2 choices})(\text{2 choices})\cdots(\text{2 choices}) = 2^n$

Number of subsets of an n-element set

First solution



Number of subsets of an n-element set

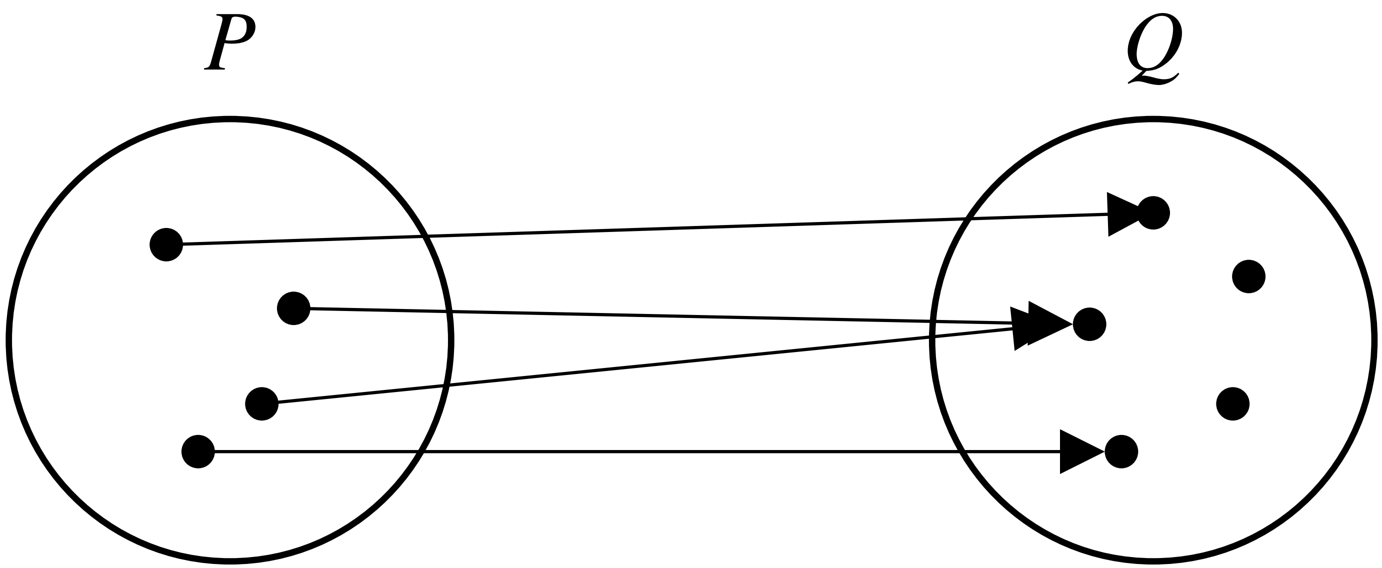
Second solution

- Consider a subset $S \subseteq [n]$.
- Form the *word* $w_1 \cdots w_n$ or a sequence $(w_1, \ldots, w_n)$

$$w_i = \begin{cases} 1 & \text{if } i \in S; \\ 0 & \text{otherwise.} \end{cases}$$

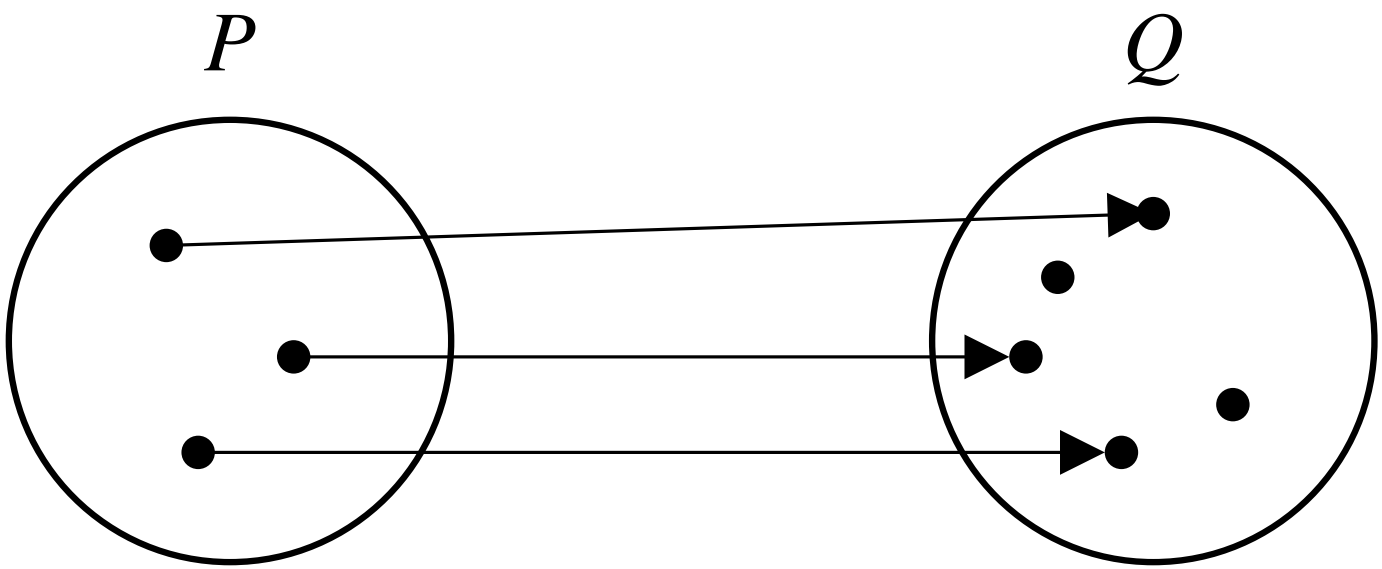
- **Example:** The subset $S = \{1, 3, 4\}$ of $[5]$ is encoded as a word 10110 or as a sequence $(1, 0, 1, 1, 0)$.
- Each subset of $[n]$ gives a unique word in $\{0, 1\}^n$ and vice-versa.
- $|\{0, 1\}^n| = 2^n$, so there are 2^n words and thus 2^n subsets.
- This is called a *bijective proof*.

Function terminology



- Consider a function $f : P \to Q$.
- For each element x in the set P, the function assigns a value $f(x)$ in the set Q.
- In the diagram,
  - Each element of P has exactly one arrow going out;
  - A variable number ($\geqslant 0$) of arrows goes into each element of Q. We'll consider some special cases of this.

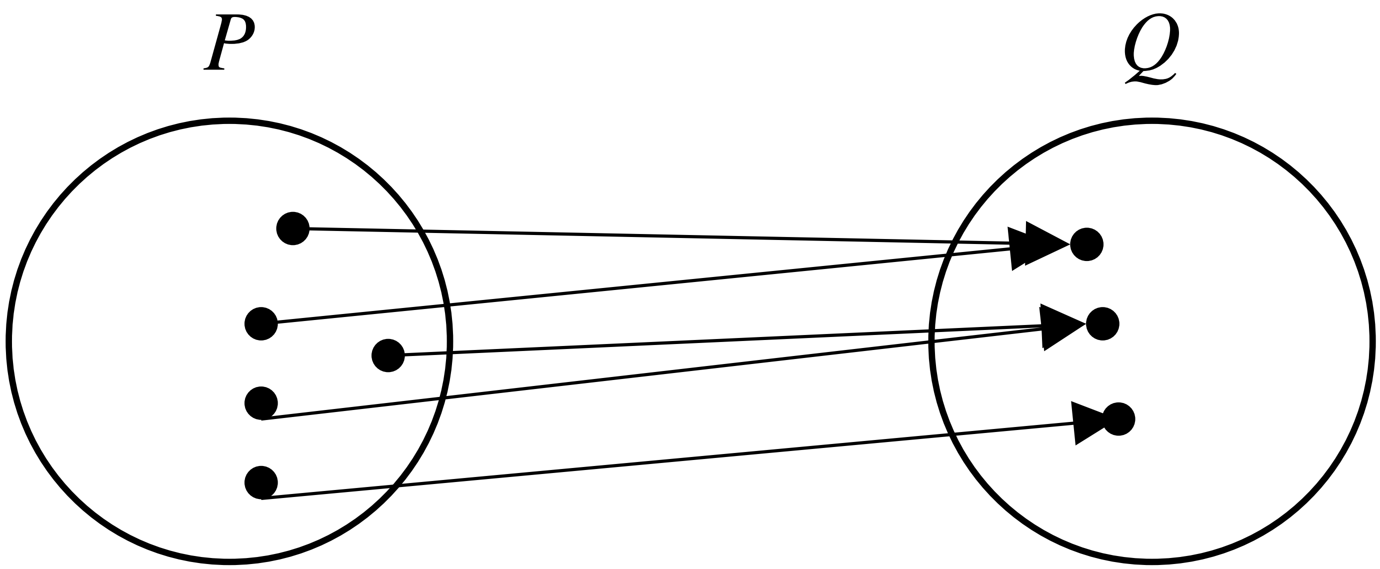
Function terminology



f is *one-to-one* iff for all $x, y \in P$, when $f(x) = f(y)$ then $x = y$.

- This is also called an *injection*.
- This means every element of Q either has exactly one inverse, or has no inverse.
- Each element of Q has $\leqslant 1$ arrows coming into it.
- If f is one-to-one then $|P| \leqslant |Q|$.

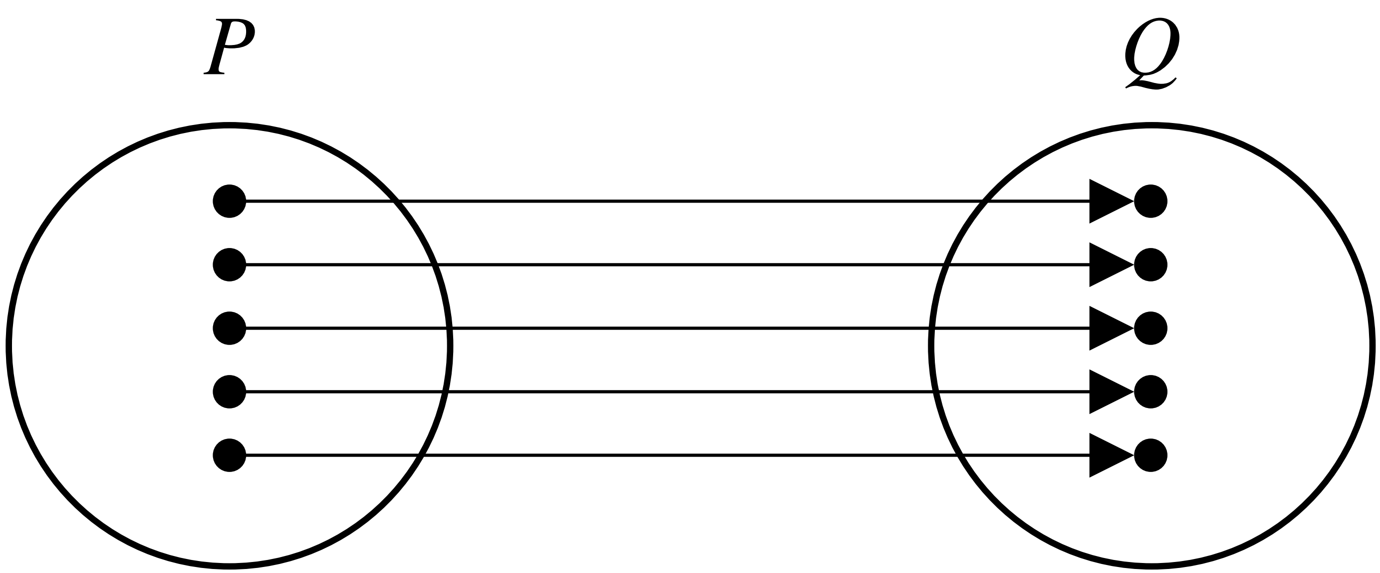
Function terminology



f is *onto* iff for every $z \in Q$, there is at least one $x \in P$ with $f(x) = z$.

- This is also called a *surjection*.
- This means every element of Q has at least one inverse.
- Each element of Q has $\geqslant 1$ arrows coming into it.
- If f is onto then $|P| \geqslant |Q|$.

Function terminology



f is a *bijection* iff it is one-to-one and onto.

- This means every element of Q has exactly one inverse.
- Each element of Q has exactly one arrow coming into it.
- If f is a bijection then $|P| = |Q|$.

Number of subsets of an n-element set

Second solution, continued

- Define $f : \mathcal{P}([n]) \to \{0, 1\}^n$ as follows:
  for $S \subseteq [n]$, form the word $f(S) = w = w_1 \cdots w_n$, where
  $$w_i = \begin{cases} 1 & \text{if } i \in S; \\ 0 & \text{otherwise.} \end{cases}$$
- f is one-to-one:
  - Suppose $f(S) = f(T) = w$. We need to show this requires $S = T$.
  - Both S and T are subsets of $[n]$. For each $i = 1, \ldots, n$,
    if $w_i = 1$ then $i \in S$ and $i \in T$,
    while if $w_i = 0$ then $i \notin S$ and $i \notin T$.
  - Thus, S and T have the exact same elements, so $S = T$.
- f is onto:
  - Given $w \in \{0, 1\}^n$, we must construct an inverse in the domain.
    There may be more than one inverse; we just have to construct one.
  - $S = \{\, i \in [n] \;:\; w_i = 1 \,\}$ is in the domain and satisfies $f(S) = w$.
- Thus, f is a bijection. So $|\mathcal{P}([n])|$ (the number of subsets of an n-element set) equals $|\{0, 1\}^n| = 2^n$.

Number of subsets of an n-element set

Third solution

We will use *Mathematical Induction* (Chapter 2) to prove that the number of subsets of $[n]$ is 2^n, for all integers $n \geqslant 0$:

- In general, the goal is to prove that a statement is true for all integers $n \geqslant n_0$. Often, n_0 is 0 or 1, but that's not required.
- *Base case:* Show that the statement is true for $n = n_0$. Sometimes it's necessary to prove it specially for several other small values of n.
- *Induction step:* Assume that the statement holds for n. Use that to prove that it holds true for $n + 1$.
- *Conclusion:* the statement holds for all integers $n \geqslant n_0$.

Number of subsets of an n-element set

Third solution

For all integers $n \geqslant 0$, the number of subsets of $[n]$ is 2^n.

Base case

First we show the statement is true for the smallest value of n (in this case, $n = 0$).

- When $n = 0$, $[n] = [0] = \emptyset$ has just one subset, which is $\emptyset$.
- The formula gives $2^n = 2^0 = 1$.
- These agree, so the statement holds for the base case.

Number of subsets of an n-element set

Third solution

Induction step

For some $n \geqslant 0$, assume that the number of subsets of $[n]$ is 2^n.
Use that to prove the number of subsets of $[n+1]$ is 2^{n+1}

- Split the subsets of $[n+1]$ into $P \cup Q$, where
  - P is the set of subsets of $[n+1]$ that don't have $n+1$, and
  - Q is the set of subsets of $[n+1]$ that do have $n+1$.
- P is the set of subsets of $[n]$. By the induction hypothesis, $|P| = 2^n$.
- Insert $n+1$ into each set in P to form Q. Thus, $|P| = |Q|$.

  E.g., for $n = 2$:
  $$\begin{aligned} P &= \{ \quad \emptyset, \quad \{1\}, \quad \{2\}, \quad \{1,2\} \quad \} \\ Q &= \{ \quad \{3\}, \quad \{1,3\}, \quad \{2,3\}, \quad \{1,2,3\} \quad \} \end{aligned}$$
- The total number of subsets of $[n+1]$ is $|P| + |Q| = 2(2^n) = 2^{n+1}$.
  (This is an example of the Addition Rule, to be covered next.)

Conclusion: For all integers $n \geqslant 0$, the number of subsets of $[n]$ is 2^n.

Addition rule

- Count the number of days in a year, as follows.
- Assume it's not a leap year.
- How many pairs (m, d) are there where

$$m = \text{month } 1, \ldots, 12;$$
$$d = \text{day of the month?}$$

- 12 choices of m, but the number of choices of d depends on m (and if it's a leap year), so the total is not "$12 \times$ __"
- Split dates into $A_m = \{\,(m, d) \;:\; d \text{ is a valid day in month } m\,\}$:

$$A = A_1 \cup \cdots \cup A_{12} = \text{whole year}$$
$$|A| = |A_1| + \cdots + |A_{12}|$$
$$= 31 + 28 + \cdots + 31 = 365$$

Addition rule

- Two sets A and B are *disjoint* when $A \cap B = \emptyset$.
- The set of even integers and set of odd integers are disjoint. But the set of even integers and the set of positive prime integers are not disjoint, since their intersection is $\{2\}$.
- Multiple sets $A_1, A_2, \ldots, A_n$ are *pairwise disjoint* (also called *mutually exclusive*) if $A_i \cap A_j = \emptyset$ when $i \neq j$.
- **Addition Rule:** If $A_1, \ldots, A_n$ are pairwise disjoint, then
$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|$$
- The left side is a generalization of $\sum$ notation. It means:
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

Set partitions

Let S be a set. A *partition* of S is $\{A_1, \ldots, A_n\}$ where

- Each A_i is a nonempty subset of S.
- $A_1, \ldots, A_n$ are pairwise disjoint.
- $S = \bigcup_{i=1}^{n} A_i$.

This is also called a *set partition*, to distinguish it from an *integer partition*, which we will learn about soon.

Each A_i is called a *block* or a *part*.

Examples

- We just partitioned the days of a year into 12 sets by month.
- Partition integers into even integers and odd integers.
- Partition integers into positive integers, negative integers, and $\{0\}$.

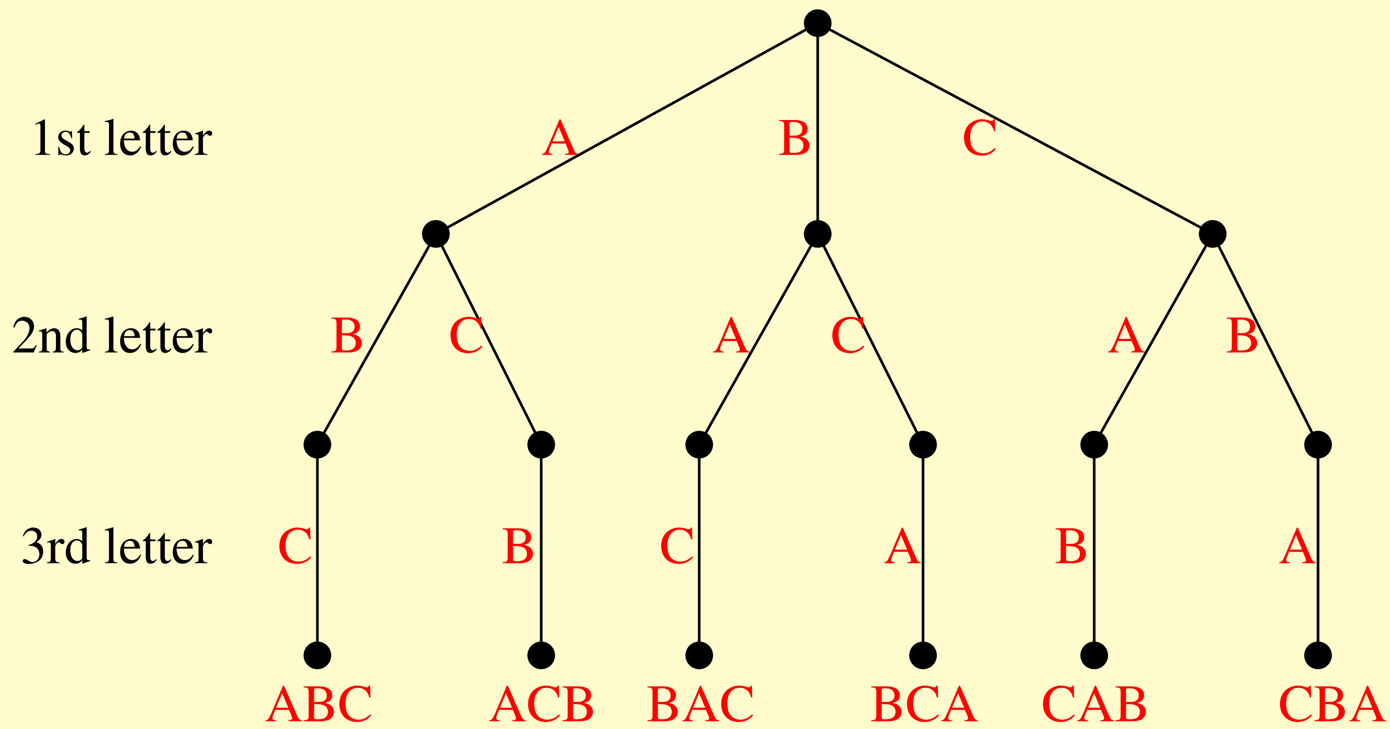
Permutations

- Here are all the *permutations* of A, B, C:

$$ABC \quad ACB \quad BAC \quad BCA \quad CAB \quad CBA$$

- There are 6 of them. We'll see how to count them systematically.

Permutations of distinct objects
Decision tree



- There are 3 items: A, B, C.
- There are 3 choices for which item to put first.
- There are 2 choices remaining to put second.
- There is 1 choice remaining to put third.
- Thus, the total number of permutations is $3 \cdot 2 \cdot 1 = 6$.

Permutations of distinct objects

- Notice that the specific choices available at each step depend on the previous steps, but the number of choices does not, so the multiplication rule applies.
- The number of permutations of n distinct items is *"n-factorial":* $n! = n(n-1)(n-2)\cdots 1$ for integers $n = 1, 2, \ldots$

Convention: $0! = 1$

- For integer $n > 1$, $$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 1 \\ &= n \cdot (n-1)! \end{aligned}$$
  so $(n-1)! = n!/n$.
- E.g., $2! = 3!/3 = 6/3 = 2$.
- Extend it to $0! = 1!/1 = 1/1 = 1$.
- Doesn't extend to negative integers: $(-1)! = \frac{0!}{0} = \frac{1}{0} =$ undefined.

Stirling's Approximation

- In how many orders can a deck of 52 cards be shuffled?
- $52! = 80658175170943878571660636856403766975289505440883277824000000000000$

  (a 68 digit integer when computed exactly)

  $52! \approx 8.0658 \cdot 10^{67}$

- *Stirling's Approximation:* For large n,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

- Stirling's approximation gives $52! \approx 8.0529 \cdot 10^{67}$

Partial permutations of distinct objects

- How many ways can you deal out 3 cards from a 52 card deck, where the order in which the cards are dealt matters?
E.g., dealing the cards in order $(A\clubsuit, 9\heartsuit, 2\diamondsuit)$ is counted differently than the order $(2\diamondsuit, A\clubsuit, 9\heartsuit)$.

- This is called an *ordered* 3-card hand, because we keep track of the order in which the cards are dealt.

- $52 \cdot 51 \cdot 50 = 132600.$

- This is also $52!/49!$:

$$\frac{52!}{49!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \dots \cdot 2 \cdot 1}{49 \cdot 48 \cdot \dots \cdot 2 \cdot 1} = 52 \cdot 51 \cdot 50$$

Partial permutations of distinct objects

- How many ordered k-card hands can be dealt from an n-card deck?

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = {}_nP_k = (n)_k$$

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = {}_{52}P_3 = (52)_3$$

- **General problem:** Instead of cards, more generally, this is "permutations of length k taken from n objects."
- **Notation:** Our book uses $(n)_k$. Many calculators use ${}_nP_k$.
- $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$ is called a *falling factorial*.

Combinations

- In an *unordered* hand, the order in which the cards are dealt does not matter; only the set of cards matters. E.g., dealing in order $(A\clubsuit, 9\heartsuit, 2\diamondsuit)$ or $(2\diamondsuit, A\clubsuit, 9\heartsuit)$ both give the same hand. This is usually represented by a set: $\{A\clubsuit, 9\heartsuit, 2\diamondsuit\}$.
- How many 3 card hands can be dealt from a 52-card deck if the order in which the cards are dealt does not matter?
- The 3-card hand $\{A\clubsuit, 9\heartsuit, 2\diamondsuit\}$ can be dealt in $3! = 6$ different orders:

$$\begin{array}{ccc} (A\clubsuit, 9\heartsuit, 2\diamondsuit) & (9\heartsuit, A\clubsuit, 2\diamondsuit) & (2\diamondsuit, 9\heartsuit, A\clubsuit) \\ (A\clubsuit, 2\diamondsuit, 9\heartsuit) & (9\heartsuit, 2\diamondsuit, A\clubsuit) & (2\diamondsuit, A\clubsuit, 9\heartsuit) \end{array}$$

- Every unordered 3-card hand arises from 6 different orders. So $52 \cdot 51 \cdot 50$ counts each unordered hand $3!$ times. Thus, the number of unordered hands is

$$\frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = \frac{52!/49!}{3!} = \frac{(52)_3}{3!} = 22100$$

Combinations

- The # of unordered k-card hands taken from an n-card deck is
$$\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k\cdot(k-1)\cdots 2\cdot 1}=\frac{(n)_k}{k!}=\frac{n!}{k!\,(n-k)!}$$
- This is denoted $\binom{n}{k}=\frac{n!}{k!\,(n-k)!}$ (or ${}_nC_k$, mostly on calculators).
- $\binom{n}{k}$ is the "binomial coefficient" and is pronounced "n choose k."
- The number of unordered 3-card hands is
$$\binom{52}{3}={}_{52}C_3=\text{“52 choose 3”}=\frac{52\cdot 51\cdot 50}{3\cdot 2\cdot 1}=\frac{52!}{3!\,49!}=22100$$
- **General problem:** Let S be a set with n elements. The number of k-element subsets of S is $\binom{n}{k}$.
- **Special cases:** $\binom{n}{0}=\binom{n}{n}=1$ $\quad\binom{n}{k}=\binom{n}{n-k}$ $\quad\binom{n}{1}=\binom{n}{n-1}=n$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- For $n = 4$: $\qquad (x+y)^4 = (x+y)(x+y)(x+y)(x+y)$
- On expanding, each factor contributes an x or a y.
  After expanding, we group, simplify, and collect like terms:

$$\begin{aligned}
(x+y)^4 = {} & yyyy \\
& + yyyx + yyxy + yxyy + xyyy \\
& + yyxx + yxyx + yxxy + xyyx + xyxy + xxyy \\
& + yxxx + xyxx + xxyx + xxxy \\
& + xxxx \\
= {} & y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + x^4
\end{aligned}$$

- Exponents of x and y must add up to n (which is 4 here).
- For the coefficient of $x^k y^{n-k}$, there are $\binom{n}{k}$ ways to choose k factors to contribute x's. The other $n-k$ factors contribute y's.
- Thus, $\binom{n}{k}$ unsimplified terms simplify to $x^k y^{n-k}$, giving $\binom{n}{k} x^k y^{n-k}$.

Permutations with repetitions

Here are all the permutations of the letters of ALLELE:

EEALLL	*EELALL*	*EELLAL*	*EELLLA*	*EAELLL*	*EALELL*
EALLEL	*EALLLE*	*ELEALL*	*ELELAL*	*ELELLA*	*ELAELL*
ELALEL	*ELALLE*	*ELLEAL*	*ELLELA*	*ELLAEL*	*ELLALE*
ELLLEA	*ELLLAE*	*AEELLL*	*AELELL*	*AELLEL*	*AELLLE*
ALEELL	*ALELEL*	*ALELLE*	*ALLEEL*	*ALLELE*	*ALLLEE*
LEEALL	*LEELAL*	*LEELLA*	*LEAELL*	*LEALEL*	*LEALLE*
LELEAL	*LELELA*	*LELAEL*	*LELALE*	*LELLEA*	*LELLAE*
LAEELL	*LAELEL*	*LAELLE*	*LALEEL*	*LALELE*	*LALLEE*
LLEEAL	*LLEELA*	*LLEAEL*	*LLEALE*	*LLELEA*	*LLELAE*
LLAEEL	*LLAELE*	*LLALEE*	*LLLEEA*	*LLLEAE*	*LLLAEE*

There are 60 of them, not $6! = 720$, due to repeated letters.

Permutations with repetitions

- There are $6! = 720$ ways to permute the subscripted letters $A_1, L_1, L_2, E_1, L_3, E_2$.
- Here are all the ways to put subscripts on EALLEL:

$$\begin{array}{cccc}
E_1A_1L_1L_2E_2L_3 & E_1A_1L_1L_3E_2L_2 & E_2A_1L_1L_2E_1L_3 & E_2A_1L_1L_3E_1L_2 \\
E_1A_1L_2L_1E_2L_3 & E_1A_1L_2L_3E_2L_1 & E_2A_1L_2L_1E_1L_3 & E_2A_1L_2L_3E_1L_1 \\
E_1A_1L_3L_1E_2L_2 & E_1A_1L_3L_2E_2L_1 & E_2A_1L_3L_1E_1L_2 & E_2A_1L_3L_2E_1L_1
\end{array}$$

- Each rearrangement of ALLELE has
  - $1! = 1$ way to subscript the A's;
  - $2! = 2$ ways to subscript the E's; and
  - $3! = 6$ ways to subscript the L's,

  giving $1! \cdot 2! \cdot 3! = 1 \cdot 2 \cdot 6 = 12$ ways to assign subscripts.
- Since each permutation of ALLELE is represented 12 different ways in permutations of $A_1L_1L_2E_1L_3E_2$, the number of permutations of ALLELE is

$$\frac{6!}{1!\,2!\,3!} = \frac{720}{12} = 60.$$

Multinomial coefficients

- For a word of length n with k_1 of one letter, k_2 of a 2nd letter, . . . , the number of permutations is given by the *multinomial coefficient:*
$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!\,k_2!\,\cdots\,k_r!}$$
where $n, k_1, k_2, \ldots, k_r$ are integers $\geqslant 0$ and $n = k_1 + \cdots + k_r$.

- For ALLELE, it's $\binom{6}{1,2,3} = 60$. Read $\binom{6}{1,2,3}$ as "6 choose 1, 2, 3."

- For a multinomial coefficient, the numbers on the bottom must add up to the number on the top ($n = k_1 + \cdots + k_r$), vs. for a binomial coefficient $\binom{n}{k}$, instead it's $0 \leqslant k \leqslant n$.

Binomial coefficient as special case of multinomial coefficient

$$\text{Binomial coefficient } \binom{n}{k} = \text{Multinomial coefficient } \binom{n}{k,\ n-k}$$
$$= \frac{n!}{k!\,(n-k)!}$$

- **Binomial coefficient:**
  $\binom{n}{k}$ = number of k element subsets of an n-element set.

- **Multinomial coefficient:**
  $\binom{n}{k,\ n-k}$ = number of permutations of k 1's and $n-k$ 0's.

  In any such permutation, the positions of the 1's determine a k-element subset of an n-element set.

Counting permutations with repetitions — 2nd method

- Make a template with n blanks:

  ___ ___ ___ ___ ___ ___

- Choose k_1 of the n positions to fill in with the 1st letter:

  L ___ L ___ ___ L $\quad\binom{n}{k_1}$ ways

- Choose k_2 of the remaining $n - k_1$ positions for the 2nd letter.

  L ___ L ___ A L $\quad\binom{n-k_1}{k_2}$ ways

- Choose k_3 of the remaining $n - k_1 - k_2$ positions for the 3rd letter.

  L E L E A L $\quad\binom{n-k_1-k_2}{k_3}$ ways

- Continue in the same way for all letters. Total:

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots = \frac{n!}{k_1!\,(n-k_1)!}\cdot\frac{(n-k_1)!}{k_2!\,(n-k_1-k_2)!}\cdot\frac{(n-k_1-k_2)!}{k_3!\,(n-k_1-k_2-k_3)!}\cdots$$

$$= \frac{n!}{k_1!\,k_2!\cdots k_r!\,(n-k_1-k_2-\cdots-k_r)!} = \frac{n!}{k_1!\,k_2!\cdots k_r!}$$

Since $k_1 + \cdots + k_r = n$, the factor $(n - k_1 - k_2 - \cdots - k_r)! = 0! = 1$.

Multinomial Theorem

- Expand and simplify $(x+y+z)^4$:

$$\begin{aligned}(x+y+z)^4 &= \quad (x+y+z)\ (x+y+z)\ (x+y+z)\ (x+y+z)\\ &= \quad x \cdot x \cdot x \cdot x\\ &\quad + x \cdot x \cdot x \cdot y\\ &\quad + x \cdot x \cdot x \cdot z\\ &\quad + \cdots\end{aligned}$$

- Each line simplifies to $x^i y^j z^k$ with exponents i, j, k that are nonnegative integers adding up to 4.
  After collecting like terms, we get a coefficient times this.

- The coefficient of x^2yz is the number of lines that simplify to x^2yz.
  - It's the number of rearrangements of $xxyz$ (variables contributed by the 4 factors), which is $\binom{4}{2,1,1} = \frac{4!}{2!\,1!\,1!} = 12$.
  - Equivalently, split the 4 factors as follows:
    choose 2 to contribute x's; 1 to contribute y; and 1 to contribute z.
    This can be done in $\binom{4}{2,1,1} = 12$ ways.

Multinomial Theorem

- **Binomial theorem:** For integers $n \geqslant 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y)^3 = \tbinom{3}{0}x^0y^3 + \tbinom{3}{1}x^1y^2 + \tbinom{3}{2}x^2y^1 + \tbinom{3}{3}x^3y^0 = y^3 + 3xy^2 + 3x^2y + x^3$$

- **Multinomial theorem:** For integers $n \geqslant 0$,

$$(x+y+z)^n = \underbrace{\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}}_{i+j+k=n} \binom{n}{i,j,k} x^i y^j z^k$$

$$\begin{aligned}(x+y+z)^2 \quad &= \quad \tbinom{2}{2,0,0}x^2y^0z^0 + \tbinom{2}{0,2,0}x^0y^2z^0 + \tbinom{2}{0,0,2}x^0y^0z^2 \\ &\qquad + \tbinom{2}{1,1,0}x^1y^1z^0 + \tbinom{2}{1,0,1}x^1y^0z^1 + \tbinom{2}{0,1,1}x^0y^1z^1 \\ &= \quad x^2 + y^2 + z^2 + 2xy + 2xz + 2yz\end{aligned}$$

  $(x_1 + \cdots + x_m)^n$ works similarly with m iterated sums.

- In $(x+y+z)^{10}$, the coefficient of $x^2y^3z^5$ is $\binom{10}{2,3,5} = \frac{10!}{2!\,3!\,5!} = 2520$.

Multisets

- A *set* is an unordered collection of different objects. Each object is either in the set or not in the set. For example,

$$\begin{aligned} \{1,5,10\} &= \{1,10,5\} && \text{Order irrelevant} \\ &= \{1,10,5,1\} && \text{Repetitions collapse into one element} \end{aligned}$$

- A *list*, *sequence*, or *tuple* allows repeats, and is ordered.
  - $(1,5,10)$, $(5,1,10)$, $(5,1,5,10)$ are different lists.
- A *multiset* allows repeated objects, but still is not ordered.
  - Multiset $\{x,y,z,z,y\}$ has one x, two y's, two z's.
  - As a multiset, $\{x,y,z,z,y\} = \{x,y,y,z,z\}$ (order irrelevant) but $\neq \{x,x,y,z\}$ (wrong multiplicities).
  - You need to say it's a multiset, since the notation looks the same.
  - This is informal notation for small multisets. It's better to give a function/table listing multiplicities.

Element	Multiplicity
x	1
y	2
z	2

Compositions of an integer

- Let n be a nonnegative integer.
- A *strict composition* of n into k parts is $(i_1, \ldots, i_k)$ where $i_1, \ldots, i_k$ are positive integers that add up to n.
  - The strict compositions of 4 into 3 parts are

$$(2,1,1),(1,2,1),(1,1,2)$$

- A *weak composition* uses nonnegative integers instead.
  - The weak compositions of 4 into 3 parts are

$$(4,0,0),(3,1,0),(3,0,1),(2,2,0),(2,1,1),(2,0,2),(1,3,0),(1,2,1),\\(1,1,2),(1,0,3),(0,4,0),(0,3,1),(0,2,2),(0,1,3),(0,0,4)$$

  - In the Multinomial Theorem, the exponents form a weak composition.
  - The terms of $(x+y+z)^4 = x^4 + 4x^3y + 4x^3z + 6x^2y^2 + \cdots$ correspond to the weak compositions listed above.

Dots and bars diagram of a weak composition

- The *dots and bars diagram* of a composition $(i_1, i_2, \ldots, i_k)$ of n:

$$(5,2,3) = \cdot\cdot\cdot\cdot\cdot \,|\, \cdot\cdot \,|\, \cdot\cdot\cdot$$

$$(3,0,3,1,0) = \cdot\cdot\cdot \,|\,|\, \cdot\cdot\cdot \,|\, \cdot \,|$$

- There are n dots and $k-1$ bars. This is $n+k-1$ characters.
- For weak compositions, the dots and bars may go in any order.
- There are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ orders.
- Thus, the number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$.
  - The # of weak compositions of 4 into 3 parts is $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$.
- In the Multinomial Theorem, $(x_1 + \cdots + x_m)^\ell$ has $\binom{\ell+m-1}{m-1}$ terms.
- The # of ℓ-element multisets formed from a set of size m is $\binom{\ell+m-1}{m-1}$.
  E.g., the 3-element multisets over the set $\{1,2\}$ are
  $\{1,1,1\}$, $\{1,1,2\}$, $\{1,2,2\}$, $\{2,2,2\}$. Total: $\binom{3+2-1}{2-1} = \binom{4}{1} = 4$.

Dots and bars diagram of a strict composition

For strict compositions, the parts have size at least 1.

- There are $n-1$ spaces between the n dots; these are the possible places to place bars in order to ensure that all parts have size $\geqslant 1$:

$$\cdot \mid \cdot \mid \cdot \mid \cdot$$

- Choose $k-1$ of these spaces for the bars, in one of $\binom{n-1}{k-1}$ ways.
- Thus, there are $\binom{n-1}{k-1}$ strict compositions of n into k parts.
- There are $\binom{4-1}{3-1} = \binom{3}{2} = 3$ strict compositions of 4 into 3 parts.

2.2. Strong induction

Theorem

Every integer $n \geqslant 2$ may be written as a product of primes, $n = p_1 \ldots p_k$.

- This is more complicated than going from n to $n+1$; we need to use multiple prior values.
- We will use *Strong Mathematical Induction*:
  - To prove a statement holds for all integers $n \geqslant n_0$:
  - *Base case:* Prove it holds for $n = n_0$.
  - *Induction step:* For $n > n_0$, assume it holds for all values $n_0, n_0+1, \ldots, n-1$ and use that to prove it holds for n.

Strong induction

Theorem

Every integer $n \geqslant 2$ may be written as a product of primes, $n = p_1 \ldots p_k$.

Base case: $n = 2$

- 2 is a product of primes with just one factor, $n = p_1 = 2$.

Induction step:

- Let $n \geqslant 3$.
- Assume $2, 3, \ldots, n-1$ may each be written as a product of primes. Use that to prove n may also be written as a product of primes.
- If n is prime, then it is a product of one prime factor (itself, $p_1 = n$).
- Otherwise, $n = ab$ where $1 < a, b < n$.
- By the induction hypothesis, $a = p_1 \cdots p_k$ and $b = q_1 \cdots q_\ell$ are products of primes.
- Then $n = p_1 \cdots p_k \, q_1 \cdots q_\ell$ is a product of primes.

- In Number Theory, there is a stronger result:

Theorem (The Fundamental Theorem of Arithmetic)

Every integer $n \geqslant 1$ has a unique factorization into primes, in the format

$$n = {p_1}^{a_1} \cdots {p_k}^{a_k}$$

where $p_1 < p_2 < \cdots < p_k$ are primes and a_i are positive integers.

- This collects the prime factors (easy) and proves uniqueness (which is beyond the scope of our proof).
- Note that ± 1 are *units* (divisors of 1), not primes, since that would violate uniqueness ($10 = 2 \cdot 5 = 1 \cdot 2 \cdot 5 = (-1)^2 \cdot 2 \cdot 5$, etc.).