Chapter 3.3, 4.1, 4.3. Binomial Coefficient Identities

Prof. Tesler

Math 184A
Winter 2017
### Table of binomial coefficients

<table>
<thead>
<tr>
<th>(_\binom{n}{k})</th>
<th>(k = 0)</th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 3)</th>
<th>(k = 4)</th>
<th>(k = 5)</th>
<th>(k = 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 0)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(n = 1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(n = 2)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(n = 3)</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(n = 4)</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(n = 5)</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(n = 6)</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

- **Compute a table of binomial coefficients using** \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\).

- **We’ll look at several patterns.** First, the nonzero entries of each row are symmetric; e.g., row \(n = 4\) is

\[
\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4} = 1, 4, 6, 4, 1,
\]

which reads the same in reverse. **Conjecture:** \(\binom{n}{k} = \binom{n}{n-k}\).
A binomial coefficient identity

Theorem

For nonegative integers \( k \leq n \),

\[
\binom{n}{k} = \binom{n}{n-k} \quad \text{including} \quad \binom{n}{0} = \binom{n}{n} = 1
\]

First proof: Expand using factorials:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \binom{n}{n-k} = \frac{n!}{(n-k)!k!}
\]

These are equal.
Theorem

For nonnegative integers $k \leq n$,

\[
\binom{n}{k} = \binom{n}{n-k}
\]

including \(\binom{n}{0} = \binom{n}{n} = 1\)

Second proof: A bijective proof.

- We’ll give a bijection between two sets, one counted by the left side, \(\binom{n}{k}\), and the other by the right side, \(\binom{n}{n-k}\). Since there’s a bijection, the sets have the same size, giving \(\binom{n}{k} = \binom{n}{n-k}\).
- Let \(\mathcal{P}\) be the set of \(k\)-element subsets of \([n]\). Note that \(|\mathcal{P}| = \binom{n}{k}\).
- For example, with \(n = 4\) and \(k = 2\), we have
  \[
  \mathcal{P} = \big\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\big\} \quad \text{and} \quad |\mathcal{P}| = \binom{4}{2} = 6
  \]
- We’ll use complements in \([n]\). For example, with subsets of \([4]\),
  \[
  \{1, 4\}^c = \{2, 3\}\quad \text{and}\quad \{3\}^c = \{1, 2, 4\}.
  \]
  Note that for any \(A \subset [n]\), we have \(|A^c| = n - |A|\) and \((A^c)^c = A\).
Theorem

For nonnegative integers \( k \leq n \),

\[
\binom{n}{k} = \binom{n}{n-k} \quad \text{including } \binom{n}{0} = \binom{n}{n} = 1
\]

Second proof: A bijective proof.

- Let \( \mathcal{P} \) be the set of \( k \)-element subsets of \([n]\). \( |\mathcal{P}| = \binom{n}{k} \).
- Let \( \mathcal{Q} \) be the set of \((n-k)\)-element subsets of \([n]\). \( |\mathcal{Q}| = \binom{n}{n-k} \).
- Define \( f: \mathcal{P} \rightarrow \mathcal{Q} \) by \( f(S) = S^c \) (complement of set \( S \) in \([n]\)).
- Show that this is a bijection:
  - \( f \) is onto: Given \( T \in \mathcal{Q} \), then \( S = T^c \) satisfies \( f(S) = (T^c)^c = T \).
    Note that \( S \subset [n] \) and \( |S| = n - |T| = n - (n - k) = k \), so \( S \in \mathcal{P} \).
  - \( f \) is one-to-one: If \( f(R) = f(S) \) then \( R^c = S^c \).
    The complement of that is \( (R^c)^c = (S^c)^c \), which simplifies to \( R = S \).
- Thus, \( f \) is a bijection, so \( |\mathcal{P}| = |\mathcal{Q}| \). Thus, \( \binom{n}{k} = \binom{n}{n-k} \).
## Sum of binomial coefficients

The sum of binomial coefficients for a given row $n \geq 0$ is computed as follows:

<table>
<thead>
<tr>
<th>$(n)_k$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
<th>$k = 5$</th>
<th>$k = 6$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>32</td>
</tr>
<tr>
<td>$n = 6$</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
<td>64</td>
</tr>
</tbody>
</table>

- Compute the total in each row.
- Any conjecture on the formula?
- The sum in row $n$ seems to be $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. 
Theorem

For integers $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

First proof: Based on the Binomial Theorem.

- The Binomial Theorem gives $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

- Plug in $x = y = 1$:
  - $(1 + 1)^n = 2^n$
  - $(1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$.
Theorem

For integers \( n \geq 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Second proof: Counting in two ways (also called “double counting”)

- How many subsets are there of \([n]\)?
- We’ll compute this in two ways. The two ways give different formulas, but since they count the same thing, they must be equal.
- Right side (we already showed this method):
  - Choose whether or not to include 1 (2 choices).
  - Choose whether or not to include 2 (2 choices).
  - Continue that way up to \( n \). In total, there are \( 2^n \) combinations of choices, leading to \( 2^n \) subsets.
Sum of binomial coefficients

Theorem

For integers \( n \geq 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Second proof, continued: Left side:
- Subsets of \([n]\) have sizes between 0 and \(n\).
- There are \(\binom{n}{k}\) subsets of size \(k\) for each \(k = 0, 1, \ldots, n\).
- The total number of subsets is \(\sum_{k=0}^{n} \binom{n}{k}\).
- Equating the two ways of counting gives \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\).

Partition \(\mathcal{P}([3])\) as \(\{A_0, A_1, A_2, A_3\}\), where \(A_k\) is the set of subsets of \([3]\) of size \(k\):
- \(A_0 = \{\emptyset\}\)
- \(A_1 = \\{\{1\}, \{2\}, \{3\}\}\)
- \(A_2 = \\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}\)
- \(A_3 = \\{\{1, 2, 3\}\}\)

Recall that parts of a partition should be nonempty. Note \(A_0\) is not equal to \(\emptyset\), but rather has \(\emptyset\) as an element.
A recursion involves solving a problem in terms of smaller instances of the same type of problem.

**Example:** Consider 3-element subsets of \([5]\):

<table>
<thead>
<tr>
<th>Subsets without 5</th>
<th>Subsets with 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3}</td>
<td>{1, 2, 5}</td>
</tr>
<tr>
<td>{1, 2, 4}</td>
<td>{1, 3, 5}</td>
</tr>
<tr>
<td>{1, 3, 4}</td>
<td>{1, 4, 5}</td>
</tr>
<tr>
<td>{2, 3, 4}</td>
<td>{2, 3, 5}</td>
</tr>
<tr>
<td></td>
<td>{2, 4, 5}</td>
</tr>
<tr>
<td></td>
<td>{3, 4, 5}</td>
</tr>
</tbody>
</table>

**Subsets without 5:** These are actually 3-element subsets of \([4]\), so there are \(\binom{4}{3} = 4\) of them.

**Subsets with 5:** Take all 2-element subsets of \([4]\) and insert a 5 into them. So \(\binom{4}{2} = 6\) of them.

Thus, \(\binom{5}{3} = \binom{4}{3} + \binom{4}{2}\).
Recursion for binomial coefficients

**Theorem**

For nonnegative integers \( n, k \):
\[
\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}
\]

- We will prove this by counting in two ways. It can also be done by expressing binomial coefficients in terms of factorials.
- How many \( k + 1 \) element subsets are there of \([n + 1]\)?
  - 1\(^{st}\) way: There are \( \binom{n+1}{k+1} \) subsets of \([n + 1]\) of size \( k + 1 \).
  - 2\(^{nd}\) way: Split the subsets into those that do / do not contain \( n + 1 \):
    - Subsets without \( n + 1 \) are actually \( (k + 1) \)-element subsets of \([n]\), so there are \( \binom{n}{k+1} \) of them.
    - Subsets with \( n + 1 \) are obtained by taking \( k \)-element subsets of \([n]\) and inserting \( n + 1 \) into them. There are \( \binom{n}{k} \) of these.
    - In total, there are \( \binom{n}{k+1} + \binom{n}{k} \) subsets of \([n + 1]\) with \( k + 1 \) elements.
- Equating the two counts gives the theorem.
For nonnegative integers $n, k$:

\[
\binom{n + 1}{k + 1} = \binom{n}{k} + \binom{n}{k + 1}
\]

- However, we can’t compute $\binom{5}{0}$ from this (uses $k = -1$), nor $\binom{0}{5}$ (uses $n = -1$). We must handle those separately.
- The *initial conditions* are

\[
\binom{n}{0} = 1 \text{ for } n \geq 0, \quad \binom{0}{k} = 0 \text{ for } k \geq 1
\]

- For $n \geq 0$, the only 0-element subset of $[n]$ is $\emptyset$, so $\binom{n}{0} = 1$.
- For $k \geq 1$, there are no $k$-element subsets of $[0] = \emptyset$, so $\binom{0}{k} = 0$. 
Recursion for binomial coefficients

Initial conditions:

<table>
<thead>
<tr>
<th>( \binom{n}{k} )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>( k = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Recursion: \( \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \); these are positioned like this:

\[
\begin{pmatrix}
\binom{n}{k} & \binom{n}{k+1} \\
\binom{n+1}{k} & \binom{n+1}{k+1}
\end{pmatrix}
\]
Use the recursion to fill in the table

<table>
<thead>
<tr>
<th>( \binom{n}{k} )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>( k = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
Pascal’s triangle

Alternate way to present the table of binomial coefficients $\binom{n}{k}$

- **Initial conditions:** Each row starts with $\binom{n}{0}=1$ and ends with $\binom{n}{n}=1$.
- **Recursion:** For the rest, each entry is the sum of the two numbers it’s in-between on the row above. E.g., $6 + 4 = 10$:

\[
\begin{array}{cccccc}
\binom{n}{k} & \binom{n}{k+1} & & & & \\
\binom{n+1}{k+1} & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccc}
\binom{4}{2} & \binom{4}{3} & & & & \\
\binom{5}{3} & & & & & \\
\end{array}
\]
Form a diagonal from a 1 on the right edge, in direction ↙ as shown, for any number of cells, and then turn ↘ for one cell.

Yellow: \(1 + 2 + 3 + 4 = 10\)

\[\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} = \binom{5}{2}\]

Pink: \(1 + 4 + 10 = 15\)

\[\binom{3}{3} + \binom{4}{3} + \binom{5}{3} = \binom{6}{4}\]

Pattern:
\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}
\]
Diagonal sums

**Theorem**

For integers $0 \leq k \leq n$,

\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}
\]

- Prove by counting $(k+1)$-element subsets of $[n+1]$ in two ways.
- **First way**: The number of such subsets is $\binom{n+1}{k+1}$.
Diagonal sums

Theorem

For integers $0 \leq k \leq n$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

- **Second way:** Categorize subsets by their largest element.
  - For $k = 2$ and $n = 4$, the 3-element subsets of $[5]$ are

<table>
<thead>
<tr>
<th>Largest element 3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 2, 3}$</td>
<td>${1, 2, 4}$</td>
<td>${1, 2, 5}$</td>
</tr>
<tr>
<td></td>
<td>${1, 3, 4}$</td>
<td>${1, 3, 5}$</td>
</tr>
<tr>
<td></td>
<td>${2, 3, 4}$</td>
<td>${2, 3, 5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>${1, 4, 5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>${3, 4, 5}$</td>
</tr>
</tbody>
</table>

- Thus, $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = \binom{5}{3}$.  

- **Binomial Coefficient Identities**

- Math 184A / Winter 2017
Diagonal sums

**Theorem**

For integers $0 \leq k \leq n$,\

\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}
\]

Partition the $(k+1)$-element subsets of $[n+1]$ by their largest element:

- The largest element ranges from $k+1$ to $n+1$.
- For largest element $j \in \{k+1, k+2, \ldots, n+1\}$, take any $k$-element subset of $[j-1]$ and insert $j$. There are $\binom{j-1}{k}$ subsets to choose.
- This gives the left side:

\[
\sum_{j=k+1}^{n+1} \binom{j-1}{k} = \sum_{j=k}^{n} \binom{j}{k}.
\]
Special cases of \((x + 1)^n\)

By the Binomial Theorem,

\[
(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

We will give combinatorial interpretations of these special cases:

- For \(n \geq 0\), \(2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k}\): We already did this.

- For \(n > 0\), \(0^n = (-1 + 1)^n\) gives \(0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\)

- For \(n \geq 0\), \(3^n = (2 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k\)
Alternating sums

Alternating sum

\[
\begin{align*}
1 & = 1 \\
1 - 1 & = 0 \\
1 - 2 + 1 & = 0 \\
1 - 3 + 3 - 1 & = 0 \\
1 - 4 + 6 - 4 + 1 & = 0 \\
1 - 5 + 10 - 10 + 5 - 1 & = 0
\end{align*}
\]

- Form an alternating sum in each row of Pascal’s Triangle. It appears that

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 
1 & \text{for } n = 0; \\
0 & \text{for } n > 0.
\end{cases}
\]

- This is the Binomial Theorem expansion of \((-1 + 1)^n\):
  - For \(n > 0\): it’s \(0^n = 0\).
  - For \(n = 0\): In general, \(0^0\) is not well-defined. Here, it arises from \((x + y)^0 = 1\), and then setting \(x = -1, y = 1\).

- We’ll also do a bijective proof.
Notation for set differences

Let $A$ and $B$ be sets.

- The *set difference* $A \setminus B$ is the set of elements that are in $A$ but not in $B$:
  \[
  A \setminus B = A \cap B^c = \{ x : x \in A \text{ and } x \notin B \}
  \]
  \[
  \{1, 3, 5, 7\} \setminus \{2, 3, 4, 5, 6\} = \{1, 7\}
  \]

- The *symmetric difference* $A \Delta B$ is the set of elements that are in $A$ or in $B$ but not in both:
  \[
  A \Delta B = \{ x : x \in A \cup B \text{ and } x \notin A \cap B \} = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)
  \]
  \[
  \{1, 3, 5, 7\} \Delta \{2, 3, 4, 5, 6\} = \{1, 2, 4, 6, 7\}
  \]
Alternating sums (bijective proof)

**Theorem**

For $n \geq 1$,

\[
\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}
\]

- Subtracting the right side from the left gives $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.
- Let $P$ be the even-sized subsets of $[n]$.
- Let $Q$ be the odd-sized subsets of $[n]$.
- This is a bijection $f : P \to Q$. For $A \in P$,

\[
f(A) = A \Delta \{1\} = \begin{cases} A \cup \{1\} & \text{if } 1 \notin A; \\ A \setminus \{1\} & \text{if } 1 \in A. \end{cases}
\]

The inverse $f^{-1} : Q \to P$ is $f^{-1}(B) = B \Delta \{1\}$.

- Thus, $|P| = |Q|$. Substitute $|P| = \sum_{k=0}^{n} \binom{n}{k}$ and $|Q| = \sum_{k=0}^{n} \binom{n}{k}$.
Example: Subsets of $[4]$

$\mathcal{P}$ (even): $\emptyset$ {1,2} {1,3} {1,4} {2,3} {2,4} {3,4} {1,2,3,4}

$\mathcal{Q}$ (odd): {1} {2} {3} {4} {1,2,3} {1,2,4} {1,3,4} {2,3,4}
Example: $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$

Prove by counting in two ways

Let $n \geq 0$. How many pairs of sets $(A, B)$ are there with $A \subset B \subset [n]$?

**First count**

- Choose the size of $B$: $k = 0, \ldots, n$.
- Choose $B \subset [n]$ of size $k$ in one of $\binom{n}{k}$ ways.
- Choose $A \subset B$ in one of $2^k$ ways.
- Total: $\sum_{k=0}^{n} \binom{n}{k} 2^k$

**Second count**

- For each element $i = 1, \ldots, n$, choose one of:
  - $i$ is in both $A$ and $B$
  - $i$ is in $B$ only
  - $i$ is in neither
- Cannot have $i$ in $A$ only, since then $A \not\subset B$.
- There are 3 choices for each $i$, giving a total $3^n$.

Thus, $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$
Example: $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ (for $n \geq 1$)

- Example of the above identity at $n = 4$:

$$1\binom{4}{1} + 2\binom{4}{2} + 3\binom{4}{3} + 4\binom{4}{4}$$

$$= 1 \cdot 4 + 2 \cdot 6 + 3 \cdot 4 + 4 \cdot 1$$

$$= 4 + 12 + 12 + 4$$

$$= 32 = 4 \cdot 2^{4-1}$$

- We will prove the identity using “Counting in two ways” and also using Calculus.
Example:  $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$  \hspace{1cm} (for $n \geq 1$)

First method: Counting in two ways

- **Scenario:** In a group of $n$ people, we want to choose a subset to form a committee, and appoint one committee member as the chair of the committee. How may ways can we do this?

- **Write this as follows:**
  \[ \mathcal{A} = \{ (S, x) : S \subseteq [n] \text{ and } x \in S \} \]
  $S$ represents the committee and $x$ represents the chair.

- **We will compute $|\mathcal{A}|$ by counting it in two ways:**
  - Pick $S$ first and then $x$.
  - Or, pick $x$ first and then $S$. 
Example: $\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$ \hspace{1cm} (for $n \geq 1$)

First method: Counting in two ways

Let $\mathcal{A} = \{(S, x) : S \subseteq [n] \text{ and } x \in S\}$

**Pick $S$ first and $x$ second**

- Pick the size of the committee ($S$) first: $k = 1, \ldots, n$.
  It has to be at least 1, since the committee has a chair.
- Pick $k$ committee members in one of $\binom{n}{k}$ ways.
- Pick one of the $k$ committee members to be the chair, $x$.
- **Total:** $\sum_{k=1}^{n} \binom{n}{k} \cdot k$

**Pick $x$ first and $S$ second**

- Pick any element of $[n]$ to be the chair, $x$. There are $n$ choices.
- Pick the remaining committee members by taking any subset $S'$ of $[n] \setminus \{x\}$; there are $2^{n-1}$ ways to do this.
- Set $S = \{x\} \cup S'$. **Total:** $n \cdot 2^{n-1}$

Comparing the two totals gives: for $n \geq 1$, $\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$. 
Example: \( \sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1} \) (for \( n \geq 0 \))

Second method: Calculus

- By the Binomial Theorem, for any integer \( n \geq 0 \):
  \[
  (x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
  \]

- Differentiate with respect to \( x \):
  \[
  n(x + 1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}
  \]

- Set \( x = 1 \):
  \[
  n(1 + 1)^{n-1} = n \cdot 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \quad \text{for} \ n \geq 0.
  \]

  Note that this includes \( k = 0 \) and holds for \( n \geq 0 \), while the first method started at \( k = 1 \) and held for \( n \geq 1 \).

  - The \( k = 0 \) term is optional since \( \binom{n}{0} \cdot 0 = 0 \).
  - We could have done additional steps in either method to prove the other version of the identity.
Taylor series review

Recall the formula for the Taylor Series of $f(x)$ centered at $x = a$:

$$f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \quad \text{where} \quad c_k = \frac{f^{(k)}(a)}{k!}$$

We’ll focus on $a = 0$.

Compute the $k^{th}$ derivative as a function of $x$, and plug in $x = 0$:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$
$$f(0) = c_0 + 0 + 0 + 0 + 0 + \cdots$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$
$$f'(0) = c_1 + 0 + 0 + 0 + \cdots$$

$$f''(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$$
$$f''(0) = 2c_2 + 0 + 0 + \cdots$$

$$f'''(x) = 6c_3 + 24c_4 x + \cdots$$
$$f'''(0) = 6c_3 + 0 + \cdots$$

$\quad c_0 = f(0)$

$\quad c_1 = f'(0)$

$\quad c_2 = f''(0)/2$  

$\quad c_3 = f'''(0)/6$
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

We will compute the Taylor series of \((1 + x)^\alpha\) for any real number \(\alpha\), not necessarily a positive integer:

\[
f(x) = (1 + x)^\alpha = \sum_{k=0}^{\infty} c_k x^k
\]

\[
f(0) = (1 + 0)^\alpha = 1
\]

\[
c_0 = f(0)/0! = 1
\]

\[
f'(x) = \alpha(1 + x)^{\alpha - 1}
\]

\[
f'(0) = \alpha(1 + 0)^{\alpha - 1} = \alpha
\]

\[
c_1 = f'(0)/1! = \alpha
\]
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

\[
f''(x) = \alpha(\alpha - 1)(1 + x)^{\alpha - 2}
\]
\[
f''(0) = \alpha(\alpha - 1)(1 + 0)^{\alpha - 2} = \alpha(\alpha - 1)
\]
\[
c_2 = f''(0)/2! = \alpha(\alpha - 1)/2
\]
\[
f'''(x) = \alpha(\alpha - 1)(\alpha - 2)(1 + x)^{\alpha - 3}
\]
\[
f'''(0) = \alpha(\alpha - 1)(\alpha - 2)(1 + 0)^{\alpha - 3} = \alpha(\alpha - 1)(\alpha - 2)
\]
\[
c_3 = f'''(0)/3! = \alpha(\alpha - 1)(\alpha - 2)/6
\]

In general:

\[
c_k = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!}
\]

\[
(1 + x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} x^k
\]
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

- Let \(\alpha\) be any real number, or a variable.
- For any integer \(k > 0\), define the **falling factorial**
  \[
  (\alpha)_k = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)
  \]
  and the **binomial coefficient**
  \[
  \binom{\alpha}{k} = \frac{(\alpha)_k}{k!} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!}
  \]

- For \(n = 0\), set \((\alpha)_0 = \binom{\alpha}{0} = 1\).
- In this notation,
  \[
  (1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
  \]

- When \(\alpha\) is a nonnegative integer, \((\alpha)_k = \binom{\alpha}{k} = 0\) for \(k > \alpha\).
  E.g., for \(\alpha = 3\), note \((\alpha)_k = 3 \cdot 2 \cdot 1 \cdot 0 \cdot \ldots = 0\) when \(k \geq 4\).
  Thus, the series can be truncated at \(k = \alpha\), giving the same result as the Binomial Theorem.
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

\[(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k\]

For what values of \(x\) does this converge?

- If \(\alpha\) is a nonnegative integer, the series terminates and gives a polynomial that converges for all \(x\).
- Otherwise, the series doesn’t terminate. Use the ratio test:

\[L = \lim_{k \to \infty} \left| \frac{\text{Term}_{k+1}}{\text{Term}_k} \right|\]

Compute the ratio

\[
\frac{\text{Term}_{k+1}}{\text{Term}_k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k)/(k + 1)! \cdot x^{k+1}}{\alpha(\alpha - 1) \cdots (\alpha - (k-1))/k! \cdot x^k} = \frac{(\alpha - k)x}{k + 1}
\]

As \(k \to \infty\), this ratio approaches \(-x\), with absolute value \(L = |x|\).
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

\[
(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
\]

- **Ratio test:**
  - \(L = \lim_{k \to \infty} \left| \frac{\text{Term}_{k+1}}{\text{Term}_k} \right| = |x|\)
  - If \(L < 1\) (\(|x| < 1\)), it converges.
  - If \(L > 1\) (\(|x| > 1\)), it diverges.
  - If \(L = 1\) (\(|x| = 1\)), the test is inconclusive.

- It turns out:
  - When \(\alpha\) is a nonnegative integer, it converges for all \(x\).
  - For \(\alpha \geq 0\) that isn’t an integer, it converges in \(-1 \leq x \leq 1\).
  - For \(\alpha < 0\), it converges in \(-1 < x < 1\).

We will use this series later in Catalan numbers (Chapter 8).
Summary

We already used these proof methods:

- **Induction**
- **Proof by contradiction**

Here we used several additional methods to prove identities:

- **Counting in two ways**: Find two formulas for the size of a set, and equate them.

- **Bijections**: Show that there is a bijection between sets $P$ and $Q$. Then equate formulas for their sizes, $|P| = |Q|$.

- **Calculus**: Manipulate functions such as polynomials or power series using derivatives and other methods from algebra and calculus.