Chapter 3.3, 4.1, 4.3. Binomial Coefficient Identities

Prof. Tesler

Math 184A
Winter 2019
## Table of binomial coefficients

<table>
<thead>
<tr>
<th>( (n \atop k) )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>( k = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

- Compute a table of binomial coefficients using \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

- We’ll look at several patterns. First, the nonzero entries of each row are symmetric; e.g., row \( n = 4 \) is

  \[
  \binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4} = 1, 4, 6, 4, 1,
  \]

  which reads the same in reverse.  **Conjecture:** \( \binom{n}{k} = \binom{n}{n-k} \).  


A binomial coefficient identity

Theorem

For nonegative integers \( k \leq n \),

\[
\binom{n}{k} = \binom{n}{n-k} \quad \text{including} \quad \binom{n}{0} = \binom{n}{n} = 1
\]

First proof: Expand using factorials:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \binom{n}{n-k} = \frac{n!}{(n-k)!k!}
\]

These are equal.
Theorem

For nonnegative integers $k \leq n$,

$${n \choose k} = {n \choose n-k} \quad \text{including} \quad {n \choose 0} = {n \choose n} = 1$$

Second proof: A bijective proof.

- We’ll give a bijection between two sets, one counted by the left side, $${n \choose k}$$, and the other by the right side, $${n \choose n-k}$$. Since there’s a bijection, the sets have the same size, giving $${n \choose k} = {n \choose n-k}$$.
- Let $\mathcal{P}$ be the set of $k$-element subsets of $[n]$. Note that $|\mathcal{P}| = {n \choose k}$.
- For example, with $n = 4$ and $k = 2$, we have

$$\mathcal{P} = \left\{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \right\} \quad |\mathcal{P}| = \binom{4}{2} = 6$$

- We’ll use complements in $[n]$. For example, with subsets of $[4]$,

$$\{1, 4\}^c = \{2, 3\} \quad \text{and} \quad \{3\}^c = \{1, 2, 4\}.$$ 

Note that for any $A \subset [n]$, we have $|A^c| = n - |A|$ and $(A^c)^c = A$. 

Theorem

For nonegative integers \( k \leq n \),

\[
\binom{n}{k} = \binom{n}{n-k} \quad \text{including} \quad \binom{n}{0} = \binom{n}{n} = 1
\]

Second proof: A bijective proof.

- Let \( \mathcal{P} \) be the set of \( k \)-element subsets of \([n]\). \( |\mathcal{P}| = \binom{n}{k} \).
- Let \( \mathcal{Q} \) be the set of \( (n-k) \)-element subsets of \([n]\). \( |\mathcal{Q}| = \binom{n}{n-k} \).
- Define \( f : \mathcal{P} \to \mathcal{Q} \) by \( f(S) = S^c \) (complement of set \( S \) in \([n]\)).
- Show that this is a bijection:
  - \( f \) is onto: Given \( T \in \mathcal{Q} \), then \( S = T^c \) satisfies \( f(S) = (T^c)^c = T \).
    Note that \( S \subset [n] \) and \( |S| = n - |T| = n - (n-k) = k \), so \( S \in \mathcal{P} \).
  - \( f \) is one-to-one: If \( f(R) = f(S) \) then \( R^c = S^c \).
    The complement of that is \( (R^c)^c = (S^c)^c \), which simplifies to \( R = S \).
- Thus, \( f \) is a bijection, so \( |\mathcal{P}| = |\mathcal{Q}| \). Thus, \( \binom{n}{k} = \binom{n}{n-k} \).
### Sum of binomial coefficients

<table>
<thead>
<tr>
<th>( \binom{n}{k} )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>( k = 6 )</th>
<th><strong>Total</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>32</td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
<td>64</td>
</tr>
</tbody>
</table>

- Compute the total in each row.
- Any conjecture on the formula?
- The sum in row \( n \) seems to be \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).
Sum of binomial coefficients

**Theorem**

For integers $n \geq 0$,

$$
\sum_{k=0}^{n} \binom{n}{k} = 2^n
$$

**First proof:** Based on the Binomial Theorem.

- The Binomial Theorem gives $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
- Plug in $x = y = 1$:
  - $(1 + 1)^n = 2^n$
  - $(1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$. 
Theorem

For integers \( n \geq 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Second proof: Counting in two ways (also called “double counting”)

- How many subsets are there of \([n]\)?
- We’ll compute this in two ways. The two ways give different formulas, but since they count the same thing, they must be equal.
- Right side (we already showed this method):
  - Choose whether or not to include 1 (2 choices).
  - Choose whether or not to include 2 (2 choices).
  - Continue that way up to \( n \). In total, there are \( 2^n \) combinations of choices, leading to \( 2^n \) subsets.
Theorem

For integers $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Second proof, continued: Left side:

- Subsets of $[n]$ have sizes between 0 and $n$.
- There are $\binom{n}{k}$ subsets of size $k$ for each $k = 0, 1, \ldots, n$.
- The total number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$.
- Equating the two ways of counting gives $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Partition $\mathcal{P}([3])$ as $\{A_0, A_1, A_2, A_3\}$, where $A_k$ is the set of subsets of $[3]$ of size $k$:

- $A_0 = \{\emptyset\}$
- $A_2 = \\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$
- $A_1 = \\{\{1\}, \{2\}, \{3\}\}$
- $A_3 = \\{\{1, 2, 3\}\}$

Recall that parts of a partition should be nonempty. Note $A_0$ is not equal to $\emptyset$, but rather has $\emptyset$ as an element.
A recursion involves solving a problem in terms of smaller instances of the same type of problem.

**Example:** Consider 3-element subsets of \([5]\):

<table>
<thead>
<tr>
<th>Subsets without 5</th>
<th>Subsets with 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3}</td>
<td>{1, 2, 5}</td>
</tr>
<tr>
<td>{1, 2, 4}</td>
<td>{1, 3, 5}</td>
</tr>
<tr>
<td>{1, 3, 4}</td>
<td>{1, 4, 5}</td>
</tr>
<tr>
<td>{2, 3, 4}</td>
<td>{2, 3, 5}</td>
</tr>
<tr>
<td></td>
<td>{2, 4, 5}</td>
</tr>
<tr>
<td></td>
<td>{3, 4, 5}</td>
</tr>
</tbody>
</table>

**Subsets without 5:** These are actually 3-element subsets of \([4]\), so there are \(\binom{4}{3} = 4\) of them.

**Subsets with 5:** Take all 2-element subsets of \([4]\) and insert a 5 into them. So \(\binom{4}{2} = 6\) of them.

Thus, \(\binom{5}{3} = \binom{4}{3} + \binom{4}{2}\).
Theorem

For nonnegative integers $n, k$:

\[
\binom{n + 1}{k + 1} = \binom{n}{k} + \binom{n}{k + 1}
\]

- We will prove this by counting in two ways. It can also be done by expressing binomial coefficients in terms of factorials.

- How many $k + 1$ element subsets are there of $[n + 1]$?

  1\textsuperscript{st} way: There are $\binom{n + 1}{k + 1}$ subsets of $[n + 1]$ of size $k + 1$.

  2\textsuperscript{nd} way: Split the subsets into those that do / do not contain $n + 1$:
  - Subsets without $n + 1$ are actually $(k + 1)$-element subsets of $[n]$, so there are $\binom{n}{k+1}$ of them.
  - Subsets with $n + 1$ are obtained by taking $k$-element subsets of $[n]$ and inserting $n + 1$ into them. There are $\binom{n}{k}$ of these.

- In total, there are $\binom{n}{k+1} + \binom{n}{k}$ subsets of $[n + 1]$ with $k + 1$ elements.

- Equating the two counts gives the theorem.
Recursion for binomial coefficients

**Theorem**

For nonnegative integers $n, k$:

\[
\binom{n + 1}{k + 1} = \binom{n}{k} + \binom{n}{k + 1}
\]

- However, we can’t compute $\binom{5}{0}$ from this (uses $k = -1$), nor $\binom{0}{5}$ (uses $n = -1$). We must handle those separately.
- The initial conditions are

\[
\binom{n}{0} = 1 \text{ for } n \geq 0, \quad \binom{0}{k} = 0 \text{ for } k \geq 1
\]

- For $n \geq 0$, the only 0-element subset of $[n]$ is $\emptyset$, so $\binom{n}{0} = 1$.
- For $k \geq 1$, there are no $k$-element subsets of $[0] = \emptyset$, so $\binom{0}{k} = 0$. 
Recursion for binomial coefficients

Initial conditions:

\[
\begin{array}{|c|cccccccc|}
\hline
n & k = 0 & k = 1 & k = 2 & k = 3 & k = 4 & k = 5 & k = 6 \\
\hline
n = 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
n = 1 & 1 & & & & & & \\
n = 2 & 1 & & & & & & \\
n = 3 & 1 & & & & & & \\
n = 4 & 1 & & & & & & \\
n = 5 & 1 & & & & & & \\
n = 6 & 1 & & & & & & \\
\hline
\end{array}
\]

Recursion: \( \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \); these are positioned like this:

\[
\begin{align*}
\binom{n}{k} & \quad \binom{n}{k+1} \\
\binom{n+1}{k+1} & \quad \binom{n+1}{k+1}
\end{align*}
\]
Use the recursion to fill in the table

<table>
<thead>
<tr>
<th>( \binom{n}{k} )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>( k = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
Pascal’s triangle

Alternate way to present the table of binomial coefficients \( \binom{n}{k} \)

\[
\begin{array}{cccccc}
\text{n = 0} & & & & & \\
& 1 & & & & \\
\text{n = 1} & & & & & \\
& 1 & 1 & & & \\
\text{n = 2} & & & & & \\
& 1 & 2 & 1 & & \\
\text{n = 3} & & & & & \\
& 1 & 3 & 3 & 1 & \\
\text{n = 4} & & & & & \\
& 1 & 4 & 6 & 4 & 1 \\
\text{n = 5} & & & & & \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

- **Initial conditions:** Each row starts with \( \binom{n}{0} = 1 \) and ends with \( \binom{n}{n} = 1 \).
- **Recursion:** For the rest, each entry is the sum of the two numbers it’s in-between on the row above. E.g., \( 6 + 4 = 10 \):
  \[
  \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}
  \]
  \[
  \binom{4}{2} + \binom{4}{3} = \binom{5}{3}
  \]
Form a diagonal from a 1 on the right edge, in direction ↙ as shown, for any number of cells, and then turn ↘ for one cell.

- **Yellow:** \(1 + 2 + 3 + 4 = 10\)
  \[\frac{1}{1} + \frac{2}{1} + \frac{3}{1} + \frac{4}{1} = \frac{5}{2}\]

- **Pink:** \(1 + 4 + 10 = 15\)
  \[\frac{3}{3} + \frac{4}{3} + \frac{5}{3} = \frac{6}{4}\]

- **Pattern:**
  \[\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}\]
For integers $0 \leq k \leq n$,

\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}
\]

Prove by counting $(k+1)$-element subsets of $[n+1]$ in two ways.

- **First way:** The number of such subsets is $\binom{n+1}{k+1}$. 
Diagonal sums

Theorem

For integers $0 \leq k \leq n$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

- **Second way:** Categorize subsets by their largest element.

For $k = 2$ and $n = 4$, the 3-element subsets of $[5]$ are

<table>
<thead>
<tr>
<th>Largest element</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>{1, 2, 3}</td>
<td>{1, 2, 4}</td>
<td>{1, 2, 5}, {2, 3, 5}, {1, 3, 5}, {2, 4, 5}, {1, 4, 5}, {3, 4, 5}</td>
</tr>
<tr>
<td>$\binom{2}{2}$ 2-elt subsets</td>
<td>(\binom{3}{2}) 2-elt subsets</td>
<td>$\binom{4}{2}$ 2-elt subsets</td>
<td></td>
</tr>
<tr>
<td>of [2], plus a 3</td>
<td>of [3], plus a 4</td>
<td>of [4], plus a 5</td>
<td></td>
</tr>
</tbody>
</table>

Thus, $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = \binom{5}{3}$.
### Theorem

For integers $0 \leq k \leq n$,

\[
\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}
\]

Partition the $(k+1)$-element subsets of $[n+1]$ by their largest element:

- The largest element ranges from $k+1$ to $n+1$.
- For largest element $j \in \{k+1, k+2, \ldots, n+1\}$, take any $k$-element subset of $[j-1]$ and insert $j$. There are $\binom{j-1}{k}$ subsets to choose.
- This gives the left side:

\[
\sum_{j=k+1}^{n+1} \binom{j-1}{k} = \sum_{j=k}^{n} \binom{j}{k}.
\]
Special cases of \((x + 1)^n\)

- By the Binomial Theorem,

\[
(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

- We will give combinatorial interpretations of these special cases:
  - For \(n \geq 0\), \(2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k}\): We already did this.
  - For \(n > 0\), \(0^n = (-1 + 1)^n\) gives \(0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\)
  - For \(n \geq 0\), \(3^n = (2 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k\)
Alternating sums

**Alternating sum**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$= 1$</td>
</tr>
<tr>
<td>$1 - 1$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>$1 - 2 + 1$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>$1 - 3 + 3 - 1$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>$1 - 4 + 6 - 4 + 1$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>$1 - 5 + 10 - 10 + 5 - 1$</td>
<td>$= 0$</td>
</tr>
</tbody>
</table>

Form an alternating sum in each row of Pascal’s Triangle. It appears that

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1 & \text{for } n = 0; \\ 0 & \text{for } n > 0. \end{cases} \]

This is the Binomial Theorem expansion of $(-1 + 1)^n$:

- For $n > 0$: it’s $0^n = 0$.
- For $n = 0$: In general, $0^0$ is not well-defined. Here, it arises from $(x + y)^0 = 1$, and then setting $x = -1$, $y = 1$.

We’ll also do a bijective proof.
Notation for set differences

Let $A$ and $B$ be sets.

- The **set difference** $A \setminus B$ is the set of elements that are in $A$ but not in $B$:
  \[ A \setminus B = A \cap B^c = \{ x : x \in A \text{ and } x \notin B \} \]
  \[ \{1, 3, 5, 7\} \setminus \{2, 3, 4, 5, 6\} = \{1, 7\} \]

- The **symmetric difference** $A \Delta B$ is the set of elements that are in $A$ or in $B$ but not in both:
  \[ A \Delta B = \{ x : x \in A \cup B \text{ and } x \notin A \cap B \} \]
  \[ = (A \cup B) \setminus (A \cap B) \]
  \[ = (A \setminus B) \cup (B \setminus A) \]
  \[ \{1, 3, 5, 7\} \Delta \{2, 3, 4, 5, 6\} = \{1, 2, 4, 6, 7\} \]
Alternating sums (bijective proof)

**Theorem**

For $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}$$

$k$ even

$k$ odd

Subtracting the right side from the left gives $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Let $\mathcal{P}$ be the even-sized subsets of $[n]$.

Let $\mathcal{Q}$ be the odd-sized subsets of $[n]$.

This is a bijection $f : \mathcal{P} \rightarrow \mathcal{Q}$. For $A \in \mathcal{P}$,

$$f(A) = A \Delta \{1\} = \begin{cases} A \cup \{1\} & \text{if } 1 \notin A; \\ A \setminus \{1\} & \text{if } 1 \in A. \end{cases}$$

The inverse $f^{-1} : \mathcal{Q} \rightarrow \mathcal{P}$ is $f^{-1}(B) = B \Delta \{1\}$.

Thus, $|\mathcal{P}| = |\mathcal{Q}|$. Substitute $|\mathcal{P}| = \sum_{k=0}^{n} \binom{n}{k}$ and $|\mathcal{Q}| = \sum_{k=0}^{n} \binom{n}{k}$.
Example: Subsets of $[4]$

\begin{align*}
\mathcal{P} \text{ (even)}: & \quad \emptyset \quad \{1,2\} \quad \{1,3\} \quad \{1,4\} \quad \{2,3\} \quad \{2,4\} \quad \{3,4\} \quad \{1,2,3,4\} \\
\mathcal{Q} \text{ (odd)}: & \quad \{1\} \quad \{2\} \quad \{3\} \quad \{4\} \quad \{1,2,3\} \quad \{1,2,4\} \quad \{1,3,4\} \quad \{2,3,4\}
\end{align*}
Example: \[ \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n \]

Prove by counting in two ways

Let \( n \geq 0 \). How many pairs of sets \((A, B)\) are there with \( A \subset B \subset [n] \)?

First count

- Choose the size of \( B \): \( k = 0, \ldots, n \).
- Choose \( B \subset [n] \) of size \( k \) in one of \( \binom{n}{k} \) ways.
- Choose \( A \subset B \) in one of \( 2^k \) ways.
- Total: \[ \sum_{k=0}^{n} \binom{n}{k} 2^k \]

Second count

- For each element \( i = 1, \ldots, n \), choose one of:
  - \( i \) is in both \( A \) and \( B \)
  - \( i \) is in \( B \) only
  - \( i \) is in neither
  
  Cannot have \( i \) in \( A \) only, since then \( A \not\subset B \).
- There are 3 choices for each \( i \), giving a total \( 3^n \).

Thus, \[ \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n \]
Example: \( \sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \) (for \( n \geq 1 \))

- Example of the above identity at \( n = 4 \):

\[
\begin{align*}
1 \binom{4}{1} + 2 \binom{4}{2} + 3 \binom{4}{3} + 4 \binom{4}{4} &= 1 \cdot 4 + 2 \cdot 6 + 3 \cdot 4 + 4 \cdot 1 \\
&= 4 + 12 + 12 + 4 \\
&= 32 = 4 \cdot 2^{4-1}
\end{align*}
\]

- We will prove the identity using “Counting in two ways” and also using Calculus.
Example: \( \sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \) (for \( n \geq 1 \))

First method: Counting in two ways

- **Scenario:** In a group of \( n \) people, we want to choose a subset to form a committee, and appoint one committee member as the chair of the committee. How may ways can we do this?

- Write this as follows:
  \[
  \mathcal{A} = \{ (S, x) : S \subseteq [n] \text{ and } x \in S \}
  \]
  \( S \) represents the committee and \( x \) represents the chair.

- We will compute \( |\mathcal{A}| \) by counting it in two ways:
  - Pick \( S \) first and then \( x \).
  - Or, pick \( x \) first and then \( S \).
Example: \[ \sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \quad \text{(for } n \geq 1) \]

First method: Counting in two ways

\[ A = \{(S, x) : S \subseteq [n] \text{ and } x \in S\} \]

**Pick S first and x second**

- Pick the size of the committee (S) first: \( k = 1, \ldots, n \).
  It has to be at least 1, since the committee has a chair.
- Pick \( k \) committee members in one of \( \binom{n}{k} \) ways.
- Pick one of the \( k \) committee members to be the chair, \( x \).
- **Total:** \[ \sum_{k=1}^{n} \binom{n}{k} \cdot k \]

**Pick x first and S second**

- Pick any element of \([n]\) to be the chair, \( x \). There are \( n \) choices.
- Pick the remaining committee members by taking any subset \( S' \) of \([n] \setminus \{x\}\); there are \( 2^{n-1} \) ways to do this.
- **Set** \( S = \{x\} \cup S' \). **Total:** \( n \cdot 2^{n-1} \)

Comparing the two totals gives: for \( n \geq 1 \), \[ \sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \].
Example: \( \sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1} \) \hspace{1em} (for \( n \geq 0 \))

**Second method: Calculus**

- By the Binomial Theorem, for any integer \( n \geq 0 \):
  \[
  (x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
  \]
- Differentiate with respect to \( x \):
  \[
  n(x + 1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}
  \]
- Set \( x = 1 \):
  \[
  n(1 + 1)^{n-1} = n \cdot 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \hspace{1em} \text{for} \hspace{1em} n \geq 0.
  \]
- Note that this includes \( k = 0 \) and holds for \( n \geq 0 \), while the first method started at \( k = 1 \) and held for \( n \geq 1 \).
  - The \( k = 0 \) term is optional since \( \binom{n}{0} \cdot 0 = 0 \).
  - We could have done additional steps in either method to prove the other version of the identity.
Taylor series review

Recall the formula for the Taylor Series of \( f(x) \) centered at \( x = a \):

\[
f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \quad \text{where} \quad c_k = \frac{f^{(k)}(a)}{k!}
\]

We’ll focus on \( a = 0 \) (also called the Maclaurin series). Compute the \( k \)th derivative as a function of \( x \), and plug in \( x = 0 \):

\[
\begin{align*}
f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\
f(0) &= c_0 + 0 + 0 + 0 + 0 + \cdots \\
f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots \\
f'(0) &= c_1 + 0 + 0 + 0 + \cdots \\
f''(x) &= 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots \\
f''(0) &= 2c_2 + 0 + 0 + \cdots \\
f'''(x) &= 6c_3 + 24c_4 x + \cdots \\
f'''(0) &= 6c_3 + 0 + \cdots
\end{align*}
\]

\( c_0 = f(0) \)

\( c_1 = f'(0) \)

\( c_2 = f''(0)/2 \)

\( c_3 = f'''(0)/6 \)
Binomial series $(1 + x)^\alpha$ when $\alpha$ is a real number

We will compute the Taylor series of $(1 + x)^\alpha$ for any real number $\alpha$, not necessarily a positive integer:

$$f(x) = (1 + x)^\alpha = \sum_{k=0}^{\infty} c_k x^k$$

$$f(0) = (1 + 0)^\alpha = 1$$
$$c_0 = f(0)/0! = 1$$

$$f'(x) = \alpha(1 + x)^{\alpha - 1}$$
$$f'(0) = \alpha(1 + 0)^{\alpha - 1} = \alpha$$
$$c_1 = f'(0)/1! = \alpha$$
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

\[
f''(x) = \alpha(\alpha - 1)(1 + x)^{\alpha - 2}
\]
\[
f''(0) = \alpha(\alpha - 1)(1 + 0)^{\alpha - 2} = \alpha(\alpha - 1)
\]
\[
c_2 = f''(0)/2! = \alpha(\alpha - 1)/2
\]

\[
f'''(x) = \alpha(\alpha - 1)(\alpha - 2)(1 + x)^{\alpha - 3}
\]
\[
f'''(0) = \alpha(\alpha - 1)(\alpha - 2)(1 + 0)^{\alpha - 3} = \alpha(\alpha - 1)(\alpha - 2)
\]
\[
c_3 = f'''(0)/3! = \alpha(\alpha - 1)(\alpha - 2)/6
\]

In general:

\[
c_k = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!}
\]

\[
(1 + x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} x^k
\]
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

- Let \(\alpha\) be any real number, or a variable.
- For any integer \(k > 0\), define the **falling factorial**
  \[
  (\alpha)_k = \alpha (\alpha - 1) (\alpha - 2) \cdots (\alpha - k + 1)
  \]
  and the **binomial coefficient**
  \[
  \binom{\alpha}{k} = \frac{(\alpha)_k}{k!} = \frac{\alpha (\alpha - 1) (\alpha - 2) \cdots (\alpha - k + 1)}{k!}
  \]
- For \(k = 0\), set \((\alpha)_0 = \binom{\alpha}{0} = 1\).
- In this notation,
  \[
  (1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
  \]
- When \(\alpha\) is a nonnegative integer, \((\alpha)_k = \binom{\alpha}{k} = 0\) for \(k > \alpha\).
  
  E.g., for \(\alpha = 3\) and \(k \geq 4\):
  \[
  (\alpha)_k = 3 \cdot 2 \cdot 1 \cdot 0 \cdots = 0
  \]
  Thus, the series can be truncated at \(k = \alpha\), giving the same result as the Binomial Theorem.
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

\[
(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
\]

For what values of \(x\) does this converge?

- If \(\alpha\) is a nonnegative integer, the series terminates and gives a polynomial that converges for all \(x\).
- Otherwise, the series doesn’t terminate. Use the ratio test:

\[
L = \lim_{k \to \infty} \left| \frac{\text{Term}_{k+1}}{\text{Term}_k} \right|
\]

Compute the ratio

\[
\frac{\text{Term}_{k+1}}{\text{Term}_k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k)/(k + 1)!}{\alpha(\alpha - 1) \cdots (\alpha - (k - 1))/k!} \cdot \frac{x^{k+1}}{x^k} = \frac{(\alpha - k)x}{k + 1}
\]

As \(k \to \infty\), this ratio approaches \(-x\), with absolute value \(L = |x|\).
Binomial series \((1 + x)^\alpha\) when \(\alpha\) is a real number

\[
(1 + x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
\]

- **Ratio test:**
  \[L = \lim_{k \to \infty} \left| \frac{\text{Term}_{k+1}}{\text{Term}_k} \right| = |x|\]
  - If \(L < 1\) (\(|x| < 1\)), it converges.
  - If \(L > 1\) (\(|x| > 1\)), it diverges.
  - If \(L = 1\) (\(|x| = 1\)), the test is inconclusive.

- **It turns out:**
  - When \(\alpha\) is a nonnegative integer, it converges for all \(x\).
  - For \(\alpha \geq 0\) that isn’t an integer, it converges in \(-1 \leq x \leq 1\).
  - For \(\alpha < 0\), it converges in \(-1 < x < 1\).

- **We will use this series later in Catalan numbers (Chapter 8).**
Summary

We already used these proof methods:

- **Induction**
- **Proof by contradiction**

Here we used several additional methods to prove identities:

- **Counting in two ways:** Find two formulas for the size of a set, and equate them.

- **Bijections:** Show that there is a bijection between sets $P$ and $Q$. Then equate formulas for their sizes, $|P| = |Q|$.

- **Calculus:** Manipulate functions such as polynomials or power series using derivatives and other methods from algebra and calculus.