Chapter 3.3, 4.1, 4.3. Binomial Coefficient Identities

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Table of binomial coefficients

$\binom{n}{k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1	0	0	0	0	0	0
$n=1$	1	1	0	0	0	0	0
$n=2$	1	2	1	0	0	0	0
$n=3$	1	3	3	1	0	0	0
$n=4$	1	4	6	4	1	0	0
$n=5$	1	5	10	10	5	1	0
$n=6$	1	6	15	20	15	6	1

- Compute a table of binomial coefficients using $\displaystyle\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.
- We'll look at several patterns. First, the nonzero entries of each row are symmetric; e.g., row $n=4$ is

$$\tbinom{4}{0}, \tbinom{4}{1}, \tbinom{4}{2}, \tbinom{4}{3}, \tbinom{4}{4} \quad = \quad 1, 4, 6, 4, 1\ ,$$

which reads the same in reverse. **Conjecture:** $\binom{n}{k} = \binom{n}{n-k}$.

A binomial coefficient identity

Theorem

For nonegative integers $k \leqslant n$,

$$\binom{n}{k} = \binom{n}{n-k} \qquad \textit{including } \binom{n}{0} = \binom{n}{n} = 1$$

First proof: Expand using factorials:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad\qquad \binom{n}{n-k} = \frac{n!}{(n-k)!\,k!}$$

These are equal.

Theorem

For nonnegative integers $k \leqslant n$,

$$\binom{n}{k} = \binom{n}{n-k} \qquad \textit{including } \binom{n}{0} = \binom{n}{n} = 1$$

Second proof: A bijective proof.

- We'll give a bijection between two sets, one counted by the left side, $\binom{n}{k}$, and the other by the right side, $\binom{n}{n-k}$. Since there's a bijection, the sets have the same size, giving $\binom{n}{k} = \binom{n}{n-k}$.
- Let $\mathcal{P}$ be the set of k-element subsets of $[n]$. Note that $|\mathcal{P}| = \binom{n}{k}$.
- For example, with $n = 4$ and $k = 2$, we have

$$\mathcal{P} = \left\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\right\} \qquad |\mathcal{P}| = \binom{4}{2} = 6$$

- We'll use complements in $[n]$. For example, with subsets of $[4]$,

$$\{1,4\}^c = \{2,3\} \qquad \text{and} \qquad \{3\}^c = \{1,2,4\}.$$

Note that for any $A \subset [n]$, we have $|A^c| = n - |A|$ and $(A^c)^c = A$.

Theorem

For nonnegative integers $k \leqslant n$*,*

$$\binom{n}{k} = \binom{n}{n-k} \qquad \textit{including} \binom{n}{0} = \binom{n}{n} = 1$$

Second proof: A bijective proof.

- Let $\mathcal{P}$ be the set of k-element subsets of $[n]$. $|\mathcal{P}| = \binom{n}{k}$.
- Let $\mathcal{Q}$ be the set of $(n-k)$-element subsets of $[n]$. $|\mathcal{Q}| = \binom{n}{n-k}$.
- Define $f : \mathcal{P} \to \mathcal{Q}$ by $f(S) = S^c$ (complement of set S in $[n]$).
- Show that this is a bijection:
  - f *is onto:* Given $T \in \mathcal{Q}$, then $S = T^c$ satisfies $f(S) = (T^c)^c = T$.
    Note that $S \subset [n]$ and $|S| = n - |T| = n - (n-k) = k$, so $S \in \mathcal{P}$.
  - f *is one-to-one:* If $f(R) = f(S)$ then $R^c = S^c$.
    The complement of that is $(R^c)^c = (S^c)^c$, which simplifies to $R = S$.
- Thus, f is a bijection, so $|\mathcal{P}| = |\mathcal{Q}|$. Thus, $\binom{n}{k} = \binom{n}{n-k}$.

Sum of binomial coefficients

$\binom{n}{k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	Total
$n=0$	1	0	0	0	0	0	0	1
$n=1$	1	1	0	0	0	0	0	2
$n=2$	1	2	1	0	0	0	0	4
$n=3$	1	3	3	1	0	0	0	8
$n=4$	1	4	6	4	1	0	0	16
$n=5$	1	5	10	10	5	1	0	32
$n=6$	1	6	15	20	15	6	1	64

- Compute the total in each row.
- Any conjecture on the formula?
- The sum in row n seems to be $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Sum of binomial coefficients

Theorem

For integers $n \geqslant 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

First proof: Based on the Binomial Theorem.

- The Binomial Theorem gives $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
- Plug in $x = y = 1$:
  - $(1+1)^n = 2^n$
  - $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$.

Sum of binomial coefficients

Theorem

For integers $n \geqslant 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Second proof: Counting in two ways (also called "double counting")

- How many subsets are there of $[n]$?
- We'll compute this in two ways. The two ways give different formulas, but since they count the same thing, they must be equal.
- Right side (we already showed this method):
  - Choose whether or not to include 1 (2 choices).
  - Choose whether or not to include 2 (2 choices).
  - Continue that way up to n. In total, there are 2^n combinations of choices, leading to 2^n subsets.

Sum of binomial coefficients

Theorem

For integers $n \geqslant 0$,
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Second proof, continued: Left side:

- Subsets of $[n]$ have sizes between 0 and n.
- There are $\binom{n}{k}$ subsets of size k for each $k = 0, 1, \ldots, n$.
- The total number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$.
- Equating the two ways of counting gives $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Partition $\mathcal{P}([3])$ as $\{A_0, A_1, A_2, A_3\}$,
where A_k is the set of subsets of $[3]$ of size k:

$$A_0 = \{\emptyset\} \qquad A_2 = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$$
$$A_1 = \big\{\{1\},\{2\},\{3\}\big\} \qquad A_3 = \big\{\{1,2,3\}\big\}$$

Recall that parts of a partition should be nonempty.
Note A_0 is not equal to $\emptyset$, but rather has $\emptyset$ as an element.

Recursion for binomial coefficients

- A recursion involves solving a problem in terms of smaller instances of the same type of problem.
- **Example:** Consider 3-element subsets of $[5]$:

Subsets without 5	Subsets with 5
$\{1,2,3\}$	$\{1,2,5\}$
$\{1,2,4\}$	$\{1,3,5\}$
$\{1,3,4\}$	$\{1,4,5\}$
$\{2,3,4\}$	$\{2,3,5\}$
	$\{2,4,5\}$
	$\{3,4,5\}$

- **Subsets without 5:** These are actually 3-element subsets of $[4]$, so there are $\binom{4}{3} = 4$ of them.
- **Subsets with 5:** Take all 2-element subsets of $[4]$ and insert a 5 into them. So $\binom{4}{2} = 6$ of them.
- Thus, $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$.

Recursion for binomial coefficients

Theorem

For nonnegative integers n, k: $$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

- We will prove this by counting in two ways. It can also be done by expressing binomial coefficients in terms of factorials.
- How many $k+1$ element subsets are there of $[n+1]$?
- 1st way: There are $\binom{n+1}{k+1}$ subsets of $[n+1]$ of size $k+1$.
- 2nd way: Split the subsets into those that do / do not contain $n+1$:
  - Subsets without $n+1$ are actually $(k+1)$-element subsets of $[n]$, so there are $\binom{n}{k+1}$ of them.
  - Subsets with $n+1$ are obtained by taking k-element subsets of $[n]$ and inserting $n+1$ into them. There are $\binom{n}{k}$ of these.
  - In total, there are $\binom{n}{k+1} + \binom{n}{k}$ subsets of $[n+1]$ with $k+1$ elements.
- Equating the two counts gives the theorem.

Recursion for binomial coefficients

Theorem

For nonnegative integers n, k: $$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

- However, we can't compute $\binom{5}{0}$ from this (uses $k=-1$), nor $\binom{0}{5}$ (uses $n=-1$). We must handle those separately.
- The *initial conditions* are

$$\binom{n}{0} = 1 \text{ for } n \geqslant 0, \qquad \binom{0}{k} = 0 \text{ for } k \geqslant 1$$

- For $n \geqslant 0$, the only 0-element subset of $[n]$ is $\emptyset$, so $\binom{n}{0} = 1$.
- For $k \geqslant 1$, there are no k-element subsets of $[0] = \emptyset$, so $\binom{0}{k} = 0$.

Recursion for binomial coefficients

Initial conditions:

$\binom{n}{k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1	0	0	0	0	0	0
$n=1$	1						
$n=2$	1						
$n=3$	1						
$n=4$	1						
$n=5$	1						
$n=6$	1						

Recursion: $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$; these are positioned like this:

$$\begin{matrix} \binom{n}{k} & \binom{n}{k+1} \\ & \binom{n+1}{k+1} \end{matrix}$$

Use the recursion to fill in the table

$\binom{n}{k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1	0	0	0	0	0	0
$n=1$	1	1	0	0	0	0	0
$n=2$	1	2	1	0	0	0	0
$n=3$	1	3	3	1	0	0	0
$n=4$	1	4	6	4	1	0	0
$n=5$	1	5	10	10	5	1	0
$n=6$	1	6	15	20	15	6	1

Pascal's triangle

Alternate way to present the table of binomial coefficients $\binom{n}{k}$

$k=0$, $k=1$, $k=2$, $k=3$, $k=4$, $k=5$

$n=0$						1					
$n=1$					1		1				
$n=2$				1		2		1			
$n=3$			1		3		3		1		
$n=4$		1		4		6		4		1	
$n=5$	1		5		10		10		5		1

- **Initial conditions:** Each row starts with $\binom{n}{0}=1$ and ends with $\binom{n}{n}=1$.
- **Recursion:** For the rest, each entry is the sum of the two numbers it's in-between on the row above. E.g., $6+4=10$:

$$\begin{array}{ccc} \binom{n}{k} & & \binom{n}{k+1} \\ & \binom{n+1}{k+1} & \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} \binom{4}{2} & & \binom{4}{3} \\ & \binom{5}{3} & \end{array}$$

Diagonal sums

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

- Form a diagonal from a 1 on the right edge, in direction $\swarrow$ as shown, for any number of cells, and then turn $\searrow$ for one cell.
- Yellow: $1+2+3+4=10$ $\qquad \binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\binom{4}{1}=\binom{5}{2}$
- Pink: $1+4+10=15$ $\qquad \binom{3}{3}+\binom{4}{3}+\binom{5}{3}=\binom{6}{4}$
- Pattern: $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$

Diagonal sums

Theorem

For integers $0 \leqslant k \leqslant n$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

- Prove by counting $(k+1)$-element subsets of $[n+1]$ in two ways.
- **First way:** The number of such subsets is $\binom{n+1}{k+1}$.

Diagonal sums

Theorem

For integers $0 \leqslant k \leqslant n$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

- **Second way:** Categorize subsets by their largest element.
- For $k = 2$ and $n = 4$, the 3-element subsets of $[5]$ are

Largest element 3	**4**	**5**
$\{1,2,3\}$	$\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$	$\{1,2,5\}$, $\{2,3,5\}$, $\{1,3,5\}$, $\{2,4,5\}$, $\{1,4,5\}$, $\{3,4,5\}$
$\binom{2}{2}$ 2-elt subsets of $[2]$, plus a 3	$\binom{3}{2}$ 2-elt subsets of $[3]$, plus a 4	$\binom{4}{2}$ 2-elt subsets of $[4]$, plus a 5

- Thus, $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = \binom{5}{3}$.

Diagonal sums

Theorem

For integers $0 \leqslant k \leqslant n$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Partition the $(k+1)$-element subsets of $[n+1]$ by their largest element:

- The largest element ranges from $k+1$ to $n+1$.
- For largest element $j \in \{k+1, k+2, \ldots, n+1\}$, take any k-element subset of $[j-1]$ and insert j. There are $\binom{j-1}{k}$ subsets to choose.
- This gives the left side:

$$\sum_{j=k+1}^{n+1} \binom{j-1}{k} = \sum_{j=k}^{n} \binom{j}{k}.$$

Special cases of $(x+1)^n$

- By the Binomial Theorem,

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- We will give combinatorial interpretations of these special cases:
  - For $n \geqslant 0$, $\quad 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$: We already did this.
  - For $n > 0$, $\quad 0^n = (-1+1)^n$ gives $0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$
  - For $n \geqslant 0$, $\quad 3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$

Alternating sums

Alternating sum	
1	$= 1$
$1 - 1$	$= 0$
$1 - 2 + 1$	$= 0$
$1 - 3 + 3 - 1$	$= 0$
$1 - 4 + 6 - 4 + 1$	$= 0$
$1 - 5 + 10 - 10 + 5 - 1$	$= 0$

- Form an alternating sum in each row of Pascal's Triangle.
  It appears that
  $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1 & \text{for } n = 0; \\ 0 & \text{for } n > 0. \end{cases}$$
- This is the Binomial Theorem expansion of $(-1+1)^n$:
  - For $n > 0$: it's $0^n = 0$.
  - For $n = 0$: In general, 0^0 is not well-defined. Here, it arises from $(x+y)^0 = 1$, and then setting $x = -1$, $y = 1$.
- We'll also do a bijective proof.

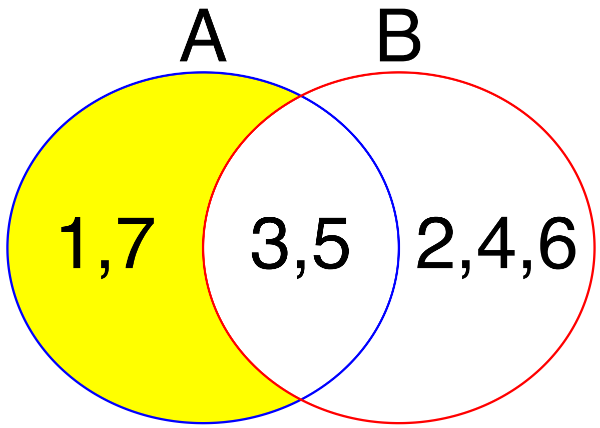
Notation for set differences

Let A and B be sets.

- The *set difference* $A \setminus B$ is the set of elements that are in A but not in B:

$$A \setminus B = A \cap B^c = \{ x \ : \ x \in A \ \text{ and } \ x \notin B \}$$

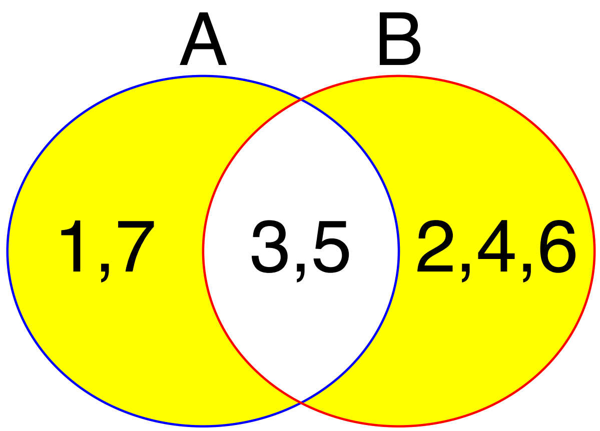
$$\{1,3,5,7\} \setminus \{2,3,4,5,6\} = \{1,7\}$$



- The *symmetric difference* $A \mathbin{\Delta} B$ is the set of elements that are in A or in B but not in both:

$$\begin{aligned} A \mathbin{\Delta} B &= \{ x \ : \ x \in A \cup B \ \text{ and } \ x \notin A \cap B \} \\ &= (A \cup B) \setminus (A \cap B) \\ &= (A \setminus B) \cup (B \setminus A) \end{aligned}$$

$$\{1,3,5,7\} \ \Delta \ \{2,3,4,5,6\} = \{1,2,4,6,7\}$$



Alternating sums (bijective proof)

Theorem

For $n \geqslant 1$,

$$\sum_{\substack{k=0\\ k \text{ even}}}^{n} \binom{n}{k} = \sum_{\substack{k=0\\ k \text{ odd}}}^{n} \binom{n}{k}$$

- Subtracting the right side from the left gives $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.
- Let $\mathcal{P}$ be the even-sized subsets of $[n]$.
- Let $\mathcal{Q}$ be the odd-sized subsets of $[n]$.
- This is a bijection $f : \mathcal{P} \to \mathcal{Q}$. For $A \in \mathcal{P}$,

$$f(A) = A \,\Delta\, \{1\} = \begin{cases} A \cup \{1\} & \text{if } 1 \notin A; \\ A \setminus \{1\} & \text{if } 1 \in A. \end{cases}$$

  The inverse $f^{-1} : \mathcal{Q} \to \mathcal{P}$ is $f^{-1}(B) = B \,\Delta\, \{1\}$.
- Thus, $|\mathcal{P}| = |\mathcal{Q}|$. Substitute $|\mathcal{P}| = \sum_{\substack{k=0\\ k \text{ even}}}^{n} \binom{n}{k}$ and $|\mathcal{Q}| = \sum_{\substack{k=0\\ k \text{ odd}}}^{n} \binom{n}{k}$.

Alternating sums (bijective proof)

Example: $n = 4$

Example: Subsets of $[4]$

$\mathcal{P}$ (even):	$\emptyset$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3,4\}$
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$
$\mathcal{Q}$ (odd):	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$

Example: $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$

Prove by counting in two ways

Let $n \geqslant 0$. How many pairs of sets (A, B) are there with $A \subset B \subset [n]$?

First count

- Choose the size of B: $k = 0, \ldots, n$.
- Choose $B \subset [n]$ of size k in one of $\binom{n}{k}$ ways.
- Choose $A \subset B$ in one of 2^k ways.
- Total: $\sum_{k=0}^{n} \binom{n}{k} 2^k$

Second count

- For each element $i = 1, \ldots, n$, choose one of:
  - i is in both A and B
  - i is in B only
  - i is in neither

  Cannot have i in A only, since then $A \not\subset B$.
- There are 3 choices for each i, giving a total 3^n.

Thus, $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$

Example: $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ (for $n \geqslant 1$)

- Example of the above identity at $n = 4$:

$$\begin{aligned} & 1\binom{4}{1} + 2\binom{4}{2} + 3\binom{4}{3} + 4\binom{4}{4} \\ &= 1 \cdot 4 + 2 \cdot 6 + 3 \cdot 4 + 4 \cdot 1 \\ &= 4 + 12 + 12 + 4 \\ &= 32 = 4 \cdot 2^{4-1} \end{aligned}$$

- We will prove the identity using "Counting in two ways" and also using Calculus.

Example: $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ (for $n \geqslant 1$)

First method: Counting in two ways

- **Scenario:** In a group of n people, we want to choose a subset to form a committee, and appoint one committee member as the chair of the committee. How may ways can we do this?
- Write this as follows:
  $$\mathcal{A} = \{\, (S,x) \;:\; S \subseteq [n] \text{ and } x \in S \,\}$$
  S represents the committee and x represents the chair.
- We will compute $|\mathcal{A}|$ by counting it in two ways:
  - Pick S first and then x.
  - Or, pick x first and then S.

Example: $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ (for $n \geqslant 1$)

First method: Counting in two ways

$$\mathcal{A} = \{\, (S, x) \;:\; S \subseteq [n] \text{ and } x \in S \,\}$$

Pick S first and x second

- Pick the size of the committee (S) first: $k = 1, \ldots, n$.
  It has to be at least 1, since the committee has a chair.
- Pick k committee members in one of $\binom{n}{k}$ ways.
- Pick one of the k committee members to be the chair, x.
- **Total:** $\sum_{k=1}^{n} \binom{n}{k} \cdot k$

Pick x first and S second

- Pick any element of $[n]$ to be the chair, x. There are n choices.
- Pick the remaining committee members by taking any subset S' of $[n] \setminus \{x\}$; there are 2^{n-1} ways to do this.
- Set $S = \{x\} \cup S'$. **Total:** $n \cdot 2^{n-1}$

Comparing the two totals gives: for $n \geqslant 1$, $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

Example: $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ (for $n \geqslant 0$)

Second method: Calculus

- By the Binomial Theorem, for any integer $n \geqslant 0$:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- Differentiate with respect to x:

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k\, x^{k-1}$$

- Set $x = 1$:

$$n(1+1)^{n-1} = n \cdot 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \qquad \text{for } n \geqslant 0.$$

- Note that this includes $k = 0$ and holds for $n \geqslant 0$, while the first method started at $k = 1$ and held for $n \geqslant 1$.
  - The $k = 0$ term is optional since $\binom{n}{0} \cdot 0 = 0$.
  - We could have done additional steps in either method to prove the other version of the identity.

Taylor series review

Recall the formula for the Taylor Series of $f(x)$ centered at $x = a$:

$$f(x) = \sum_{k=0}^{\infty} c_k\,(x-a)^k \qquad \text{where } c_k = \frac{f^{(k)}(a)}{k!}$$

We'll focus on $a = 0$ (also called the *Maclaurin series*).
Compute the k^{th} derivative as a function of x, and plug in $x = 0$:

$$\begin{array}{rcl|c}
f(x) & = & c_0 + c_1\,x + c_2\,x^2 + c_3\,x^3 + c_4\,x^4 + \cdots & \\
f(0) & = & c_0 + 0 + 0 + 0 + 0 + \cdots & c_0 = f(0) \\
 & & & \\
f'(x) & = & c_1 + 2c_2\,x + 3c_3\,x^2 + 4c_4\,x^3 + \cdots & \\
f'(0) & = & c_1 + 0 + 0 + 0 + \cdots & c_1 = f'(0) \\
 & & & \\
f''(x) & = & 2c_2 + 6c_3\,x + 12c_4\,x^2 + \cdots & \\
f''(0) & = & 2c_2 + 0 + 0 + \cdots & c_2 = f^{(2)}(0)/2 \\
 & & & \\
f'''(x) & = & 6c_3 + 24c_4\,x + \cdots & \\
f'''(0) & = & 6c_3 + 0 + \cdots & c_3 = f^{(3)}(0)/6
\end{array}$$

Binomial series $(1+x)^\alpha$ when α is a real number

We will compute the Taylor series of $(1+x)^\alpha$ for any real number α, not necessarily a positive integer:

$$f(x) = (1+x)^\alpha = \sum_{k=0}^{\infty} c_k x^k$$

$$f(0) = (1+0)^\alpha = 1$$
$$c_0 = f(0)/0! = 1$$

$$f'(x) = \alpha(1+x)^{\alpha-1}$$
$$f'(0) = \alpha(1+0)^{\alpha-1} = \alpha$$
$$c_1 = f'(0)/1! = \alpha$$

Binomial series $(1+x)^\alpha$ when α is a real number

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$
$$f''(0) = \alpha(\alpha-1)(1+0)^{\alpha-2} = \alpha(\alpha-1)$$
$$c_2 = f''(0)/2! = \alpha(\alpha-1)/2$$

$$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$
$$f'''(0) = \alpha(\alpha-1)(\alpha-2)(1+0)^{\alpha-3} = \alpha(\alpha-1)(\alpha-2)$$
$$c_3 = f'''(0)/3! = \alpha(\alpha-1)(\alpha-2)/6$$

In general:

$$c_k = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k$$

Binomial series $(1+x)^\alpha$ when α is a real number

- Let α be any real number, or a variable.
- For any integer $k > 0$, define the *falling factorial*
$$(\alpha)_k = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$$
and the *binomial coefficient*
$$\binom{\alpha}{k} = \frac{(\alpha)_k}{k!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$
- For $k=0$, set $(\alpha)_0 = \binom{\alpha}{0} = 1$.
- In this notation,
$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$
- When α is a nonnegative integer, $(\alpha)_k = \binom{\alpha}{k} = 0$ for $k > \alpha$.
E.g., for $\alpha = 3$ and $k \geqslant 4$: $(\alpha)_k = 3 \cdot 2 \cdot 1 \cdot 0 \cdot \cdots = 0$

  Thus, the series can be truncated at $k = \alpha$, giving the same result as the Binomial Theorem.

Binomial series $(1+x)^\alpha$ when α is a real number

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

For what values of x does this converge?

- If α is a nonnegative integer, the series terminates and gives a polynomial that converges for all x.
- Otherwise, the series doesn't terminate. Use the ratio test:

$$L = \lim_{k\to\infty} \left| \frac{\text{Term } k+1}{\text{Term } k} \right|$$

- Compute the ratio

$$\frac{\text{Term } k+1}{\text{Term } k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k)/(k+1)!}{\alpha(\alpha-1)\cdots(\alpha-(k-1))/k!} \cdot \frac{x^{k+1}}{x^k} = \frac{(\alpha-k)x}{k+1}$$

- As $k\to\infty$, this ratio approaches $-x$, with absolute value $L = |x|$.

Binomial series $(1+x)^\alpha$ when α is a real number

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

- Ratio test:
  - $L = \lim_{k\to\infty} \left| \frac{\text{Term } k+1}{\text{Term } k} \right| = |x|$
  - If $L < 1$ ($|x| < 1$), it converges.
    If $L > 1$ ($|x| > 1$), it diverges.
    If $L = 1$ ($|x| = 1$), the test is inconclusive.
- It turns out:
  - When α is a nonnegative integer, it converges for all x.
  - For $\alpha \geqslant 0$ that isn't an integer, it converges in $-1 \leqslant x \leqslant 1$.
  - For $\alpha < 0$, it converges in $-1 < x < 1$.
- We will use this series later in Catalan numbers (Chapter 8).

Summary

We already used these proof methods:

- **Induction**
- **Proof by contradiction**

Here we used several additional methods to prove identities:

- **Counting in two ways:** Find two formulas for the size of a set, and equate them.
- **Bijections:** Show that there is a bijection between sets P and Q. Then equate formulas for their sizes, $|P| = |Q|$.
- **Calculus:** Manipulate functions such as polynomials or power series using derivatives and other methods from algebra and calculus.