Chapter 5:
Integer Compositions and Partitions and Set Partitions

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5.1. Compositions

- A *strict composition of n* is a tuple of *positive integers* that sum to n.
  The strict compositions of 4 are

  $$(4) \quad (3,1) \quad (1,3) \quad (2,2) \quad (2,1,1) \quad (1,2,1) \quad (1,1,2) \quad (1,1,1,1)$$

- It's a tuple, so $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ are all distinct.
  Later, we'll consider *integer partitions*, in which we regard those as equivalent and only use the one with decreasing entries, $(2,1,1)$.
- A *weak composition of n* is a tuple of *nonnegative integers* that sum to n.

  $(1,0,0,3)$ is a weak composition of 4.

- If strict or weak is not specified, a *composition* means a *strict composition*.

Notation and drawings of compositions

- *Tuple notation:* $3+1+1$ and $1+3+1$ both evaluate to 5. To properly distinguish between them, we represent them as tuples, $(3,1,1)$ and $(1,3,1)$, since tuples are distinguishable.
- Drawings:

Sum	**Tuple**	**Dots and bars**
$3+1+1$	$(3,1,1)$	$\cdot\cdot\cdot\mid\cdot\mid\cdot$
$1+3+1$	$(1,3,1)$	$\cdot\mid\cdot\cdot\cdot\mid\cdot$
$0+4+1$	$(0,4,1)$	$\mid\cdot\cdot\cdot\cdot\mid\cdot$
$4+1+0$	$(4,1,0)$	$\cdot\cdot\cdot\cdot\mid\cdot\mid$
$4+0+1$	$(4,0,1)$	$\cdot\cdot\cdot\cdot\mid\mid\cdot$
$4+0+0+1$	$(4,0,0,1)$	$\cdot\cdot\cdot\cdot\mid\mid\mid\cdot$

- If there is a bar at the beginning/end, the first/last part is 0. If there are any consecutive bars, some part(s) in the middle are 0.

How many strict compositions of n into k parts?

- A composition of n into k parts has n dots and $k-1$ bars.
  - Draw n dots: $\bullet \bullet \bullet \bullet \bullet$
  - There are $n-1$ spaces between the dots.
  - Choose $k-1$ of the spaces and put a bar in each of them.
  - For $n=5$, $k=3$: $\bullet | \bullet \bullet | \bullet \bullet$
- The bars split the dots into parts of sizes $\geqslant 1$, because there are no bars at the beginning or end, and no consecutive bars.
- Thus, there are $\binom{n-1}{k-1}$ strict compositions of n into k parts, for $n,k \geqslant 1$.
- For $n=5$ and $k=3$, we get $\binom{5-1}{3-1} = \binom{4}{2} = 6$.

Total # of strict compositions of $n \geqslant 1$ into any number of parts

- 2^{n-1} by placing bars in any subset (of any size) of the $n-1$ spaces.
- Or, $\displaystyle\sum_{k=1}^{n} \binom{n-1}{k-1}$, so the total is $\displaystyle 2^{n-1} = \sum_{k=1}^{n} \binom{n-1}{k-1}$.

How many weak compositions of n into k parts?

Review: We covered this when doing the Multinomial Theorem

- The diagram has n dots and $k-1$ bars in any order. No restriction on bars at the beginning/end/consecutively since parts=0 is OK.
- There are $n+k-1$ symbols.
  Choose n of them to be dots (or $k-1$ of them to be bars):

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

- For $n=5$ and $k=3$, we have

$$\binom{5+3-1}{5} = \binom{7}{5} = 21 \qquad \text{or} \qquad \binom{5+3-1}{3-1} = \binom{7}{2} = 21.$$

- The total number of weak compositions of n of all sizes is infinite, since we can insert any number of 0's into a strict composition of n.

Relation between weak and strict compositions

- Let $(a_1, \ldots, a_k)$ be a weak composition of n (parts $\geqslant 0$).
- Add 1 to each part to get a strict composition of $n+k$:

$$(a_1+1)+(a_2+1)+\cdots+(a_k+1)=(a_1+\cdots+a_k)+k=n+k$$

  The parts of $(a_1+1, \ldots, a_k+1)$ are $\geqslant 1$ and sum to $n+k$.
- $(2,0,3)$ is a weak composition of 5.
  $(3,1,4)$ is a strict composition of $5+3=8$.
- This is reversible and leads to a bijection between
  Weak compositions of n into k parts
  $\longleftrightarrow$ Strict compositions of $n+k$ into k parts
  (Forwards: add 1 to each part; reverse: subtract 1 from each part.)
- Thus, the number of weak compositions of n into k parts
  = The number of strict compositions of $n+k$ into k parts
  = $\binom{n+k-1}{k-1}$.

5.2. Set partitions

- A *partition of a set A* is a set of nonempty subsets of A called *blocks*, such that every element of A is in exactly one block.
- A set partition of $\{1, 2, 3, 4, 5, 6, 7\}$ into three blocks is
$$\{\{1, 3, 6\}, \{2, 7\}, \{4, 5\}\}.$$
- This is a set of sets. Since sets aren't ordered, the blocks can be put in another order, and the elements within each block can be written in a different order:
$$\{\{1, 3, 6\}, \{2, 7\}, \{4, 5\}\} = \{\{5, 4\}, \{6, 1, 3\}, \{7, 2\}\}.$$
- Define $S(n, k)$ as the number of partitions of an n-element set into k blocks. This is called the *Stirling Number of the Second Kind*. We will find a recursion and other formulas for $S(n, k)$.
- Must use capital 'S' in $S(n, k)$; later we'll define a separate function $s(n, k)$ with lowercase 's'.

How do partitions of $[n]$ relate to partitions of $[n-1]$?

- Define $[0] = \emptyset$ and $[n] = \{1, 2, \ldots, n\}$ for integers $n > 0$.
  It is convenient to use $[n]$ as an example of an n-element set.
- Examine what happens when we cross out n in a set partition of $[n]$, to obtain a set partition of $[n-1]$ (here, $n = 5$):

$$\begin{array}{rcl}
\{\{1,3\},\{2,4,\cancel{5}\}\} & \to & \{\{1,3\},\{2,4\}\} \\
\{\{1,3,\cancel{5}\},\{2,4\}\} & \to & \{\{1,3\},\{2,4\}\} \\
\{\{1,3\},\{2,4\},\cancel{\{5\}}\} & \to & \{\{1,3\},\{2,4\}\}
\end{array}$$

- For all three of the set partitions on the left, removing 5 yields the set partition $\{\{1,3\},\{2,4\}\}$.
- In the first two, 5 was in a block with other elements, and removing it yielded the same number of blocks.
- In the third, 5 was in its own block, so we also had to remove the block $\{5\}$ since only nonempty blocks are allowed.

How do partitions of $[n]$ relate to partitions of $[n-1]$?

- Reversing that, there are three ways to insert 5 into $\{\{1,3\},\{2,4\}\}$:

$$\{\{1,3\},\{2,4\}\} \to \begin{cases} \{\,\{1,3,\color{red}{5}\},\{2,4\}\} & \text{insert in 1}^{\text{st}}\text{ block;} \\ \{\,\{1,3\},\{2,4,\color{red}{5}\}\} & \text{insert in 2}^{\text{nd}}\text{ block;} \\ \{\,\{1,3\},\{2,4\},\color{red}{\{5\}}\} & \text{insert as new block.} \end{cases}$$

- Inserting n in an existing block keeps the same number of blocks.
- Inserting $\{n\}$ as a new block increases the number of blocks by 1.

Recursion for $S(n, k)$

Insert n into a partition of $[n-1]$ to obtain a partition of $[n]$ into k blocks:

- **Case: partitions of $[n]$ in which n is not in a block alone:**
  Choose a partition of $[n-1]$ into k blocks ($S(n-1,k)$ choices)
  Insert n into any of these blocks (k choices)

  Subtotal: $k \cdot S(n-1,k)$

- **Case: partitions of $[n]$ in which n is in a block alone:**
  Choose a partition of $[n-1]$ into $k-1$ blocks ($S(n-1,k-1)$ ways)
  and add a new block $\{n\}$ (one way)

  Subtotal: $S(n-1,k-1)$

- **Total:** $S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$

- This recursion requires $n-1 \geqslant 0$ and $k-1 \geqslant 0$, so $n, k \geqslant 1$.

Initial conditions for $S(n, k)$

When $n = 0$ or $k = 0$

$n = 0$: Partitions of $\emptyset$

- It is not valid to partition the null set as $\{\emptyset\}$, since that has an empty block.
- However, it *is* valid to partition it as $\{\} = \emptyset$. There are no blocks, so there are no empty blocks. The union of no blocks equals $\emptyset$.
- This is the only partition of $\emptyset$, so $S(0, 0)=1$ and $S(0, k)=0$ for $k>0$.

$k = 0$: partitions into 0 blocks

- $S(n, 0) = 0$ when $n > 0$
  since every partition of $[n]$ must have at least one block.

Not an initial condition, but related:

- $S(n, k) = 0$ for $k > n$
  since the partition of $[n]$ with the most blocks is $\{\{1\}, \ldots, \{n\}\}$.

Table of values of $S(n,k)$: Initial conditions

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$
$$S(n,0) = 0 \text{ if } n > 0$$
$$S(0,k) = 0 \text{ if } k > 0 \qquad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0				
$n=2$	0				
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$: Recursion

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1$$
$$S(n,0) = 0 \text{ if } n > 0$$
$$S(0,k) = 0 \text{ if } k > 0$$

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1) \quad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0				
$n=2$	0		$S(n{-}1,k{-}1)$	$S(n{-}1,k)$	
$n=3$	0			$S(n,k)$	
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$

$$S(n,0) = 0 \text{ if } n > 0 \qquad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$$S(0,k) = 0 \text{ if } k > 0$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
		$\downarrow \cdot 1$			
$n=1$	0	1			
$n=2$	0				
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1$$
$$S(n,0) = 0 \text{ if } n > 0$$
$$S(0,k) = 0 \text{ if } k > 0$$

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1) \quad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0		
$n=2$	0				
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1$$
$$S(n,0) = 0 \text{ if } n > 0$$
$$S(0,k) = 0 \text{ if } k > 0$$

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1) \quad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
		$\downarrow \cdot 1$	$\downarrow \cdot 2$	$\downarrow \cdot 3$	$\downarrow \cdot 4$
$n=1$	0	1	0	0	0
$n=2$	0				
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad\qquad S(n,k) = k \cdot S(n-1,k)$$
$$S(n,0) = 0 \text{ if } n > 0 \qquad\qquad + S(n-1,k-1)$$
$$S(0,k) = 0 \text{ if } k > 0 \qquad\qquad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1			
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,0) = 0 \text{ if } n > 0 \qquad S(0,k) = 0 \text{ if } k > 0$$

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1) \quad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1	1		
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$
$$S(n,0) = 0 \text{ if } n > 0 \qquad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$
$$S(0,k) = 0 \text{ if } k > 0$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1	1	0	0
$n=3$	0				
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$

$S(n,0) = 0$ if $n > 0$

$S(0,k) = 0$ if $k > 0$

if $n \geqslant 1$ and $k \geqslant 1$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1	1	0	0
$n=3$	0	1	3	1	0
$n=4$	0				

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$
$$S(n,0) = 0 \text{ if } n > 0 \qquad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$
$$S(0,k) = 0 \text{ if } k > 0$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1	1	0	0
$n=3$	0	1	3	1	0
$n=4$	0	1	7	6	1

Example and Bell numbers

- $S(n, k)$ is the number of set partitions of $[n]$ into k blocks. For $n = 4$:

$k = 1$	$k = 2$	$k = 3$	$k = 4$
	$\{\{1, 2, 3\}, \{4\}\}$		
	$\{\{1, 2, 4\}, \{3\}\}$	$\{\{1, 2\}, \{3\}, \{4\}\}$	
	$\{\{1, 3, 4\}, \{2\}\}$	$\{\{1, 3\}, \{2\}, \{4\}\}$	
$\{\{1, 2, 3, 4\}\}$	$\{\{2, 3, 4\}, \{1\}\}$	$\{\{1, 4\}, \{2\}, \{2\}\}$	$\{\{1\}, \{2\}, \{3\}, \{4\}\}$
	$\{\{1, 2\}, \{3, 4\}\}$	$\{\{2, 3\}, \{1\}, \{4\}\}$	
	$\{\{1, 3\}, \{2, 4\}\}$	$\{\{2, 4\}, \{1\}, \{3\}\}$	
	$\{\{1, 4\}, \{2, 3\}\}$	$\{\{3, 4\}, \{1\}, \{2\}\}$	
$S(4, 1) = 1$	$S(4, 2) = 7$	$S(4, 3) = 6$	$S(4, 4) = 1$

- The *Bell number* B_n is the total number of set partitions of $[n]$ into any number of blocks:

$$B_n = S(n, 0) + S(n, 1) + \cdots + S(n, n)$$

- Total: $B_4 = 1 + 7 + 6 + 1 = 15$

Table of Stirling numbers and Bell numbers

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$

$S(n,0) = 0$ if $n > 0$

$S(0,k) = 0$ if $k > 0$

if $n \geqslant 1$ and $k \geqslant 1$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	Row total B_n
$n=0$	1	0	0	0	0	0	1
$n=1$	0	1	0	0	0	0	1
$n=2$	0	1	1	0	0	0	2
$n=3$	0	1	3	1	0	0	5
$n=4$	0	1	7	6	1	0	15
$n=5$	0	1	15	25	10	1	52

Simplex locks

- Simplex brand locks were a popular combination lock with 5 buttons.
- The combination 13-25-4 means:
  - Push buttons 1 and 3 together.
  - Push buttons 2 and 5 together.
  - Push 4 alone.
  - Turn the knob to open.
- Buttons cannot be reused.
- We first consider the case that all buttons are used, and separately consider the case that some buttons aren't used.

Represent the combination 13-25-4 as an ordered set partition

- We may represent 13-25-4 as an *ordered set partition*

$$(\{1,3\},\{2,5\},\{4\})$$

  - Block $\{1,3\}$ is first, block $\{2,5\}$ is second, and block $\{4\}$ is third.
  - Blocks are sets, so can replace $\{1,3\}$ by $\{3,1\}$, or $\{2,5\}$ by $\{5,2\}$.
  - Parentheses on the outer level make it an ordered tuple:

$$\Big(\{1,3\},\{2,5\},\{4\}\Big)$$

- By contrast, a set partition is a set of blocks:

$$\Big\{\{1,3\},\{2,5\},\{4\}\Big\}$$

  - Braces on the outer level make it a set instead of an ordered tuple.
  - Reordering blocks just changes how we write it but doesn't give a new set partition: $\{\{1,3\},\{2,5\},\{4\}\} = \{\{5,2\},\{4\},\{1,3\}\}$

Number of combinations

- Let $n =$ # of buttons (which must all be used)
  $k =$ # groups of button pushes.
- There are $S(n, k)$ ways to split the buttons into k blocks
  $\times\, k!$ ways to order the blocks
  $= k! \cdot S(n, k)$ combinations.
- The # of combinations on $n = 5$ buttons and $k = 3$ groups of pushes is
$$3! \cdot S(5, 3) = 6 \cdot 25 = \boxed{\mathbf{150}}$$

Represent the combination 13-25-4 as a surjective (onto) function

- Define a function $f(i) = j$, where button i is in push number j:

$i =$ button number	$j =$ push number
1	1
2	2
3	1
4	3
5	2

- This gives a surjective (onto) function $f : [5] \to [3]$.
- The blocks of buttons pushed are
  1st: $f^{-1}(1) = \{1, 3\}$ 2nd: $f^{-1}(2) = \{2, 5\}$ 3rd: $f^{-1}(3) = \{4\}$

Theorem

The number of surjective (onto) functions $f : [n] \to [k]$ is $k! \cdot S(n, k)$.

Proof.

Split $[n]$ into k nonempty blocks in one of $S(n, k)$ ways.
Choose one of $k!$ orders for the blocks: $(f^{-1}(1), \ldots, f^{-1}(k))$. $\square$

How many combinations don't use all the buttons?

- The combination 3-25 does not use 1 and 4.
- Trick: write it as 3-25-(14), with *all* unused buttons in *one* "phantom" push at the end.
- There are three groups of buttons and we don't use the 3rd group.
- \# combinations with 2 pushes that don't use all buttons
  $=$ # combinations with 3 pushes that do use all buttons.
- For set partition $\{\{3\},\{2,5\},\{1,4\}\}$, the $3!$ orders of the blocks give:

Ordered 3-tuple	**Actual combination**	**+ phantom push**
$(\{3\},\{2,5\},\{1,4\})$	3-25	3-25-(14)
$(\{3\},\{1,4\},\{2,5\})$	3-14	3-14-(25)
$(\{2,5\},\{3\},\{1,4\})$	25-3	25-3-(14)
$(\{2,5\},\{1,4\},\{3\})$	25-14	25-14-(3)
$(\{1,4\},\{3\},\{2,5\})$	14-3	14-3-(25)
$(\{1,4\},\{2,5\},\{3\})$	14-25	14-25-(3)

How many combinations don't use all the buttons?

- Putting *all* unused buttons into *one* phantom push at the end gives a bijection between
  - Combinations with $k-1$ pushes that don't use all n buttons, and
  - Combinations with k pushes that do use all n buttons.

Lemma (General case)

For $n, k \geqslant 1$:

The # combinations with $k-1$ pushes that don't use all n buttons
= the # combinations with k pushes that do use all n buttons
$= k! \cdot S(n, k)$.

Counting the total number of functions $f : [n] \to [k]$

We will count the number of functions $f : [n] \to [k]$ in two ways.

First method

(k choices of $f(1)$) $\times$ (k choices of $f(2)$) $\times \cdots \times$ (k choices of $f(n)$) $= k^n$

Counting the total number of functions $f : [n] \to [k]$

Second method: Classify functions by their images and inverses

- Consider $f : [10] \to \{a, b, c, d, e\}$:

i =	1	2	3	4	5	6	7	8	9	10
f(i) =	a	c	c	a	c	d	c	a	c	d

- The *domain* is $[10]$.
- The *codomain* (or *target*) is $\{a, b, c, d, e\}$.
- The *image* is $\operatorname{image}(f) = \{f(1), \ldots, f(10)\} = \{a, c, d\}$.
  It's a subset of the codomain.
- The inverse blocks are

$$f^{-1}(a) = \{1, 4, 8\} \qquad f^{-1}(c) = \{2, 3, 5, 7, 9\}$$

$$f^{-1}(d) = \{6, 10\} \qquad f^{-1}(b) = f^{-1}(e) = \emptyset$$

- $f : [10] \to \{a, b, c, d, e\}$ is not onto, but $f : [10] \to \{a, c, d\}$ is onto.

Counting the total number of functions $f : [n] \to [k]$

Second method, continued

- Consider $f : [10] \to \{a, b, c, d, e\}$:

i =	1	2	3	4	5	6	7	8	9	10
f(i) =	a	c	c	a	c	d	c	a	c	d

- $f : [10] \to \{a, b, c, d, e\}$ is not onto, but $f : [10] \to \{a, c, d\}$ is onto.
- There are $S(10, 3) \cdot 3!$ surjective functions $f : [10] \to \{a, c, d\}$.
- Classify all $f : [10] \to \{a, b, c, d, e\}$ according to $T = \text{image}(f)$.
- There are $\binom{5}{3}$ subsets $T \subseteq \{a, b, c, d, e\}$ of size $|T| = 3$.
  Each T has $S(10, 3) \cdot 3!$ surjective functions $f : [10] \to T$.
  So $S(10, 3) \cdot 3! \cdot \binom{5}{3}$ functions $f : [10] \to \{a, \ldots, e\}$ have $|\text{image}(f)| = 3$.
- **Simplify:** $3! \cdot \binom{5}{3} = 3! \cdot \frac{5!}{3!\,2!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = (5)_3$
  So $S(10, 3) \cdot (5)_3$ functions $f : [10] \to [5]$ have $|\text{image}(f)| = 3$.

Counting the total number of functions $f : [n] \to [k]$

Second method, continued

- In general, $S(n,i) \cdot (k)_i$ functions $f : [n] \to [k]$ have $|\operatorname{image}(f)| = i$.
- Summing over all possible image sizes $i = 0, \ldots, n$ gives the total number of functions $f : [n] \to [k]$
$$\sum_{i=0}^{n} S(n,i) \cdot (k)_i$$
- Putting this together with the first method gives
$$k^n = \sum_{i=0}^{n} S(n,i) \cdot (k)_i \qquad \text{for all integers } n, k \geqslant 0$$

Counting the total number of functions $f : [n] \to [k]$

Second method, continued

$$k^n = \sum_{i=0}^{n} S(n,i) \cdot (k)_i \qquad \text{for all integers } n, k \geqslant 0$$

- $i = |\,\mathrm{image}(f)| = |\{f(1), \ldots, f(n)\}| \leqslant n$, so $i \leqslant n$.
- Also, $i \leqslant k$ since $\mathrm{image}(f) \subseteq [k]$.
- In the sum, upper bound $i = n$ may be replaced by k or $\min(n,k)$. Any terms added or removed in the sum by changing the upper bound don't affect the result since those terms equal 0:

$$\begin{aligned} S(n,i) &= 0 \qquad \text{for } i > n \\ (k)_i &= 0 \qquad \text{for } i > k. \end{aligned}$$

Identity for real numbers

The identity

$$k^n = \sum_{i=0}^{n} S(n,i) \cdot (k)_i \qquad \text{for all integers } n, k \geqslant 0$$

generalizes to

Theorem

$$x^n = \sum_{i=0}^{n} S(n,i) \cdot (x)_i \qquad \textit{for all real } x \textit{ and integer } n \geqslant 0.$$

Identity for real numbers

Theorem

$$x^n = \sum_{i=0}^{n} S(n,i) \cdot (x)_i \qquad \textit{for all real } x \textit{ and integer } n \geqslant 0.$$

Examples

For $n = 2$:

$$\begin{aligned} S(2,0)(x)_0 + S(2,1)(x)_1 + S(2,2)(x)_2 &= 0 \cdot 1 + 1 \cdot x + 1 \cdot x(x-1) \\ &= 0 + x + (x^2 - x) = x^2 \end{aligned}$$

For $n = 3$:

$$\begin{aligned} &S(3,0)(x)_0 + S(3,1)(x)_1 + S(3,2)(x)_2 + S(3,3)(x)_3 \\ &\quad = 0 \cdot 1 + 1 \cdot x + 3 \cdot x(x-1) + 1 \cdot x(x-1)(x-2) \\ &\quad = 0 + x + 3(x^2 - x) + (x^3 - 3x^2 + 2x) \\ &\quad = x^3 + (3-3)x^2 + (1-3+2)x \qquad = x^3 \end{aligned}$$

Lemma from Abstract Algebra

Lemma

If $f(x)$ and $g(x)$ are polynomials of degree $\leqslant n$ that agree on more than n distinct values of x, then $f(x) = g(x)$ as polynomials.

Proof.

- Let $h(x) = f(x) - g(x)$. This is a polynomial of degree $\leqslant n$.
- If $h(x) = 0$ identically, then $f(x) = g(x)$ as polynomials.
  Assume $h(x)$ is not identically 0.
- Let $x_1, \ldots, x_m$ (with $m > n$) be distinct values at which $f(x_i) = g(x_i)$.
  Then $h(x_i) = f(x_i) - g(x_i) = 0$ for $i = 1, \ldots, m$, so $h(x)$ factors as
  $$h(x) = p(x)(x - x_1)^{r_1}(x - x_2)^{r_2} \cdots (x - x_m)^{r_m} \cdots$$
  for some polynomial $p(x) \neq 0$ and some integers $r_1, \ldots, r_m \geqslant 1$.
- Then $h(x)$ has degree $\geqslant m > n$.
  But $h(x)$ has degree $\leqslant n$, a contradiction.
  Thus, $h(x) = 0$, so $f(x) = g(x)$. $\square$

Identity for real numbers

Theorem

$$x^n = \sum_{i=0}^{n} S(n,i) \cdot (x)_i \qquad \textit{for all real } x \textit{ and integer } n \geqslant 0.$$

Proof.

- Both sides of the equation are polynomials in x of degree n.
- They agree at an infinite number of values $x = 0, 1, \ldots$
- Since $\infty > n$, they're identical polynomials. $\square$

5.3. Integer partitions

- The compositions $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ are different.
  Sometimes the number of 1's, 2's, 3's, $\ldots$ matters but not the order.
- An *integer partition* of n is a tuple $(a_1, \ldots, a_k)$ of positive integers that sum to n, with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant 1$.
  The partitions of 4 are:
  $$(4) \qquad (3,1) \qquad (2,2) \qquad (2,1,1) \qquad (1,1,1,1)$$
- Define
  $$\begin{aligned} p(n) &= \text{\# integer partitions of } n \\ p_k(n) &= \text{\# integer partitions of } n \text{ into exactly } k \text{ parts} \end{aligned}$$
  $$\begin{aligned} p(4) &= 5 \\ p_1(4) &= 1 \quad p_2(4) = 2 \quad p_3(4) = 1 \quad p_4(4) = 1 \end{aligned}$$
- We will learn a method to compute these in Chapter 8.

Type of a set partition

- Consider this set partition of $[10]$:

$$\Big\{\{1,4\},\{7,6\},\{5\},\{8,2,3\},\{9\},\{10\}\Big\}$$

- The block lengths in the order it was written are $2, 2, 1, 3, 1, 1$.
- But the blocks of a set partition could be written in other orders. To make this unique, the *type* of a set partition is a tuple of the block lengths listed in decreasing order: $(3, 2, 2, 1, 1, 1)$.
- For a set of size n partitioned into k blocks, the type is an integer partition of n in k parts.

How many set partitions of $[10]$ have type $(3,2,2,1,1,1)$?

- Split $[10]$ into sets A,B,C,D,E,F of sizes $3,2,2,1,1,1$, respectively, in one of $\binom{10}{3,2,2,1,1,1} = \frac{10!}{3!\,2!^2\,1!^3} = 151200$ ways.
- But $\{A,\;\; B,C,\;\; D,E,F\} = \{A,\;\; C,B,\;\; F,E,D\}$, so we overcounted:
  - B,C could be reordered C,B: $2!$ ways.
  - D,E,F could be permuted in $3!$ ways.
  - If there are m_i blocks of size i, we overcounted by a factor of $m_i!$.
- Dividing by the overcounts gives

$$\frac{\binom{10}{3,2,2,1,1,1}}{1!\,2!\,3!} = \frac{151200}{1\cdot 2\cdot 6} = \boxed{\mathbf{12600}}$$

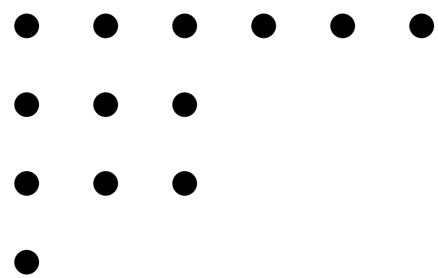
General formula

For an n element set, the number of set partitions of type $(a_1,a_2,\ldots,a_k)$ where $n = a_1 + a_2 + \cdots + a_k$ and m_i of the a's equal i, is

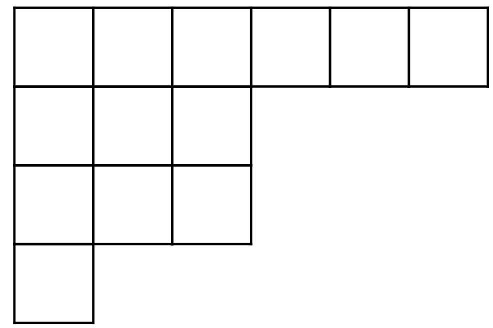
$$\frac{\binom{n}{a_1,a_2,\ldots,a_k}}{m_1!\,m_2!\,\cdots} = \frac{n!}{(1!^{m_1}\,m_1!)(2!^{m_2}\,m_2!)\cdots}$$

Ferrers diagrams and Young diagrams

Ferrers diagram of $(6, 3, 3, 1)$



Young diagram

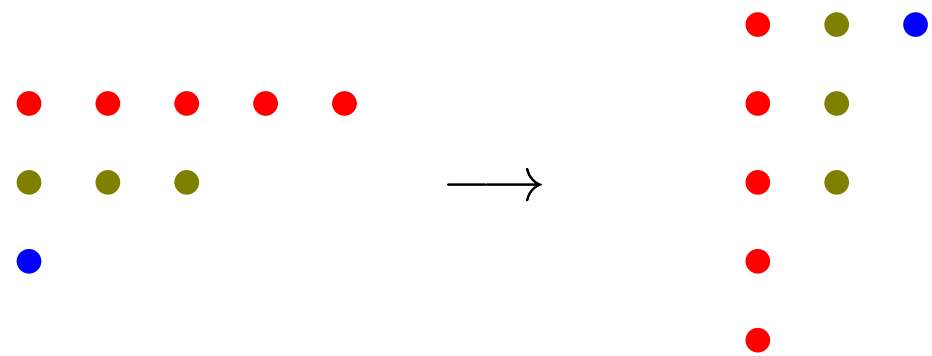


- Consider a partition $(a_1, \ldots, a_k)$ of n.
- *Ferrers diagram:* a_i dots in the ith row.
- *Young diagram:* squares instead of dots.
- The total number of dots or squares is n.

- Our book calls both of these *Ferrers diagrams*, but often they are given separate names.

Conjugate Partition

- Reflect a Ferrers diagram across its main diagonal:



$$\pi = (5,3,1) \qquad\qquad \pi' = (3,2,2,1,1)$$

- This transforms a partition π to its *conjugate partition*, denoted π'.
- The ith row of π turns into the ith column of π':
  the red, green, and blue rows of π turn into columns of π'.
  Also, the ith column of π turns into the ith row of π'.
- **Theorem:** $(\pi')' = \pi$
- **Theorem:** If π has k parts, then the largest part of π' is k.
  Here: π has 3 parts $\leftrightarrow$ the first column of π has length 3
  $\leftrightarrow$ the first row π' is 3
  $\leftrightarrow$ the largest part of π' is 3

Theorem

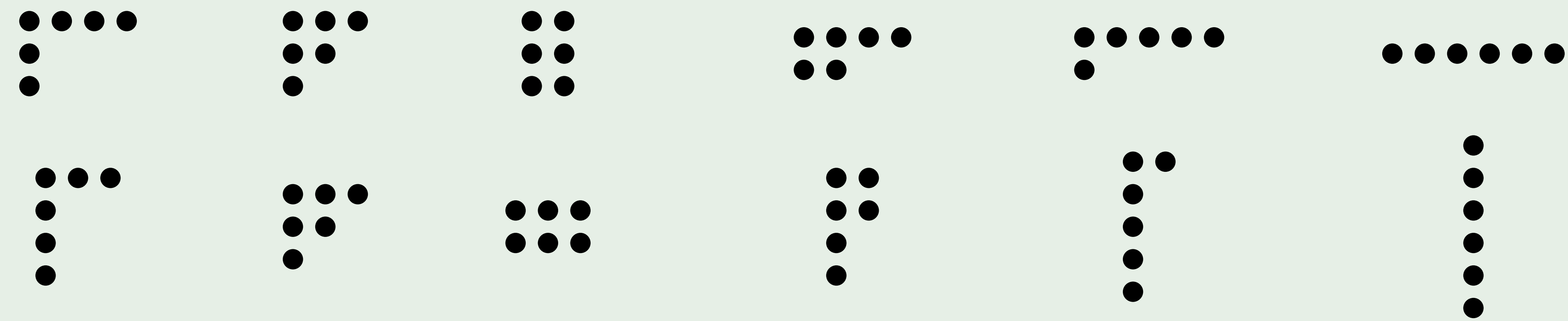
1. *The number of partitions of n into exactly k parts $(p_k(n))$*
   $=$ *the number of partitions of n where the largest part $= k$.*
2. *The number of partitions of n into $\leqslant k$ parts*
   $=$ *the number of partitions of n into parts that are each $\leqslant k$.*

Proof: Conjugation is a bijection between the two types of partitions.

Example: Partitions of 6 into 3 or $\leqslant 3$ parts

π with exactly 3 parts — π with < 3 parts

$(4, 1, 1)$ $(3, 2, 1)$ $(2, 2, 2)$ $(4, 2)$ $(5, 1)$ (6)



$(3, 1, 1, 1)$ $(3, 2, 1)$ $(3, 3)$ $(2, 2, 1, 1)$ $(2, 1, 1, 1, 1)$ $(1,1,1,1,1,1)$

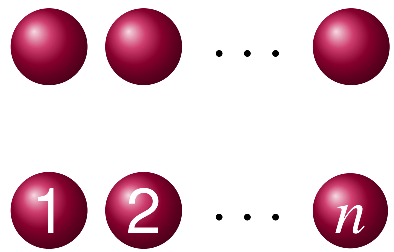
π' has largest part $= 3$ — π' has largest part < 3

Balls and boxes

Many combinatorial problems can be modeled as placing *balls* into *boxes*:

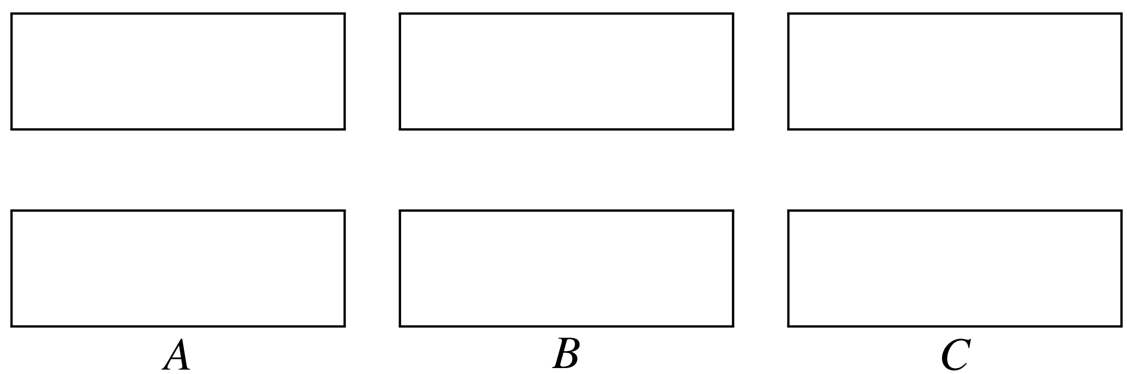
Indistinguishable balls:

Distinguishable balls:



Indistinguishable boxes:

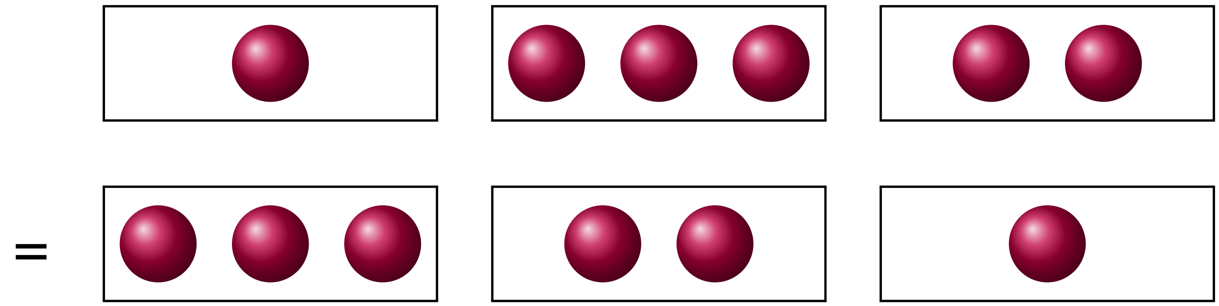
Distinguishable boxes:



Balls and boxes

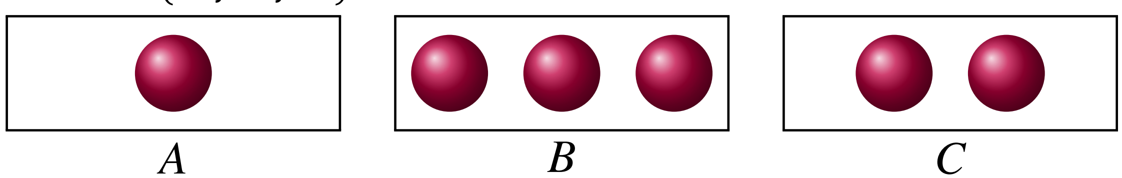
Indistinguishable balls

- **Integer partitions:** $(3, 2, 1)$



Indistinguishable balls.
Indistinguishable boxes.

- **Compositions:** $(1, 3, 2)$

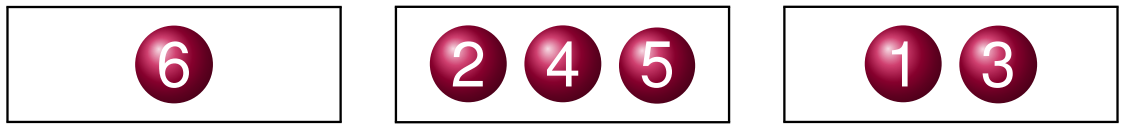


Indistinguishable balls.
Distinguishable boxes (which give the order).

Balls and boxes

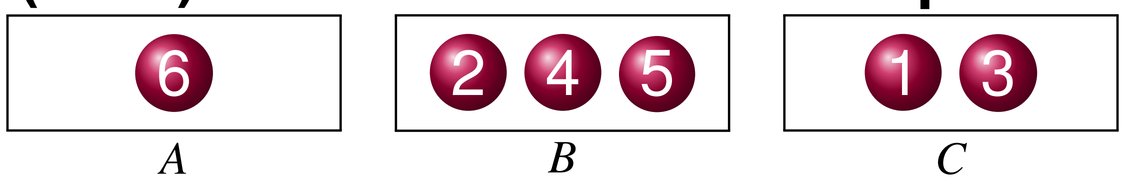
Distinguishable balls

- **Set partitions:** $\{\{6\}, \{2,4,5\}, \{1,3\}\}$



Distinguishable balls.
Indistinguishable boxes (so the blocks are not in any order).

- **Surjective (onto) functions / ordered set partitions:**



Distinguishable balls and distinguishable boxes.

Gives surjective function $f : [6] \to \{A, B, C\}$

$$f(6) = A \qquad f(2) = f(4) = f(5) = B \qquad f(1) = f(3) = C$$

or an ordered set partition $(\{6\}, \{2,4,5\}, \{1,3\})$