Chapter 5:
Integer Compositions and Partitions
and Set Partitions

Prof. Tesler

Math 184A
Fall 2017
5.1. Compositions

- A strict composition of \( n \) is a tuple of positive integers that sum to \( n \). The strict compositions of \( 4 \) are

\[
(4) \ (3, 1) \ (1, 3) \ (2, 2) \ (2, 1, 1) \ (1, 2, 1) \ (1, 1, 2) \ (1, 1, 1, 1)
\]

- It’s a tuple, so \((2, 1, 1), (1, 2, 1), (1, 1, 2)\) are all distinct. Later, we’ll consider integer partitions, in which we regard those as equivalent and only use the one with decreasing entries, \((2, 1, 1)\).

- A weak composition of \( n \) is a tuple of nonnegative integers that sum to \( n \).

\[
(1, 0, 0, 3)
\]

is a weak composition of \( 4 \).

- If strict or weak is not specified, a composition means a strict composition.
**Tuple notation:** \(3 + 1 + 1\) and \(1 + 3 + 1\) both evaluate to 5. To properly distinguish between them, we represent them as tuples, \((3, 1, 1)\) and \((1, 3, 1)\), since tuples are distinguishable.

**Drawings:**

<table>
<thead>
<tr>
<th>Sum</th>
<th>Tuple</th>
<th>Dots and bars</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3 + 1 + 1)</td>
<td>((3, 1, 1))</td>
<td>· · ·</td>
</tr>
<tr>
<td>(1 + 3 + 1)</td>
<td>((1, 3, 1))</td>
<td>·</td>
</tr>
<tr>
<td>(0 + 4 + 1)</td>
<td>((0, 4, 1))</td>
<td></td>
</tr>
<tr>
<td>(4 + 1 + 0)</td>
<td>((4, 1, 0))</td>
<td>· · ·</td>
</tr>
<tr>
<td>(4 + 0 + 1)</td>
<td>((4, 0, 1))</td>
<td>· · ·</td>
</tr>
<tr>
<td>(4 + 0 + 0 + 1)</td>
<td>((4, 0, 0, 1))</td>
<td>· · ·</td>
</tr>
</tbody>
</table>

If there is a bar at the beginning/end, the first/last part is 0. If there are any consecutive bars, some part(s) in the middle are 0.
How many strict compositions of $n$ into $k$ parts?

- A composition of $n$ into $k$ parts has $n$ dots and $k - 1$ bars.
  - Draw $n$ dots: •••••
  - There are $n - 1$ spaces between the dots.
  - Choose $k - 1$ of the spaces and put a bar in each of them.
  - For $n = 5$, $k = 3$: • | • • | • •

- The bars split the dots into parts of sizes $\geq 1$, because there are no bars at the beginning or end, and no consecutive bars.

- Thus, there are $\binom{n-1}{k-1}$ strict compositions of $n$ into $k$ parts, for $n,k \geq 1$.

- For $n = 5$ and $k = 3$, we get $\binom{5-1}{3-1} = \binom{4}{2} = 6$.

**Total # of strict compositions of $n \geq 1$ into any number of parts**

- $2^{n-1}$ by placing bars in any subset (of any size) of the $n - 1$ spaces.

- Or, $\sum_{k=1}^{n} \binom{n-1}{k-1}$, so the total is $2^{n-1} = \sum_{k=1}^{n} \binom{n-1}{k-1}$. 
How many weak compositions of $n$ into $k$ parts?

Review: We covered this when doing the Multinomial Theorem

- The diagram has $n$ dots and $k - 1$ bars in any order. No restriction on bars at the beginning/end/consecutively since parts=0 is OK.

- There are $n + k - 1$ symbols. Choose $n$ of them to be dots (or $k - 1$ of them to be bars):

$$\binom{n + k - 1}{n} = \binom{n + k - 1}{k - 1}$$

- For $n = 5$ and $k = 3$, we have

$$\binom{5 + 3 - 1}{5} = \binom{7}{5} = 21 \quad \text{or} \quad \binom{5 + 3 - 1}{3 - 1} = \binom{7}{2} = 21.$$  

- The total number of weak compositions of $n$ of all sizes is infinite, since we can insert any number of 0’s into a strict composition of $n.$
Relation between weak and strict compositions

- Let \((a_1, \ldots, a_k)\) be a weak composition of \(n\) (parts \(\geq 0\)).
- Add 1 to each part to get a strict composition of \(n + k\):
  \[
  (a_1 + 1) + (a_2 + 1) + \cdots + (a_k + 1) = (a_1 + \cdots + a_k) + k = n + k
  \]
  The parts of \((a_1 + 1, \ldots, a_k + 1)\) are \(\geq 1\) and sum to \(n + k\).
- \((2, 0, 3)\) is a weak composition of 5.
- \((3, 1, 4)\) is a strict composition of \(5 + 3 = 8\).
- This is reversible and leads to a bijection between
  Weak compositions of \(n\) into \(k\) parts
  \[\longleftrightarrow\]
  Strict compositions of \(n + k\) into \(k\) parts
  (Forwards: add 1 to each part; reverse: subtract 1 from each part.)
- Thus, the number of weak compositions of \(n\) into \(k\) parts
  \(=\) The number of strict compositions of \(n + k\) into \(k\) parts
  \(=\) \(\binom{n+k-1}{k-1}\).
5.2. Set partitions

- A *partition of a set* $A$ is a set of nonempty subsets of $A$ called *blocks*, such that every element of $A$ is in exactly one block.

- A set partition of $\{1, 2, 3, 4, 5, 6, 7\}$ into three blocks is

$$\left\{ \{1, 3, 6\}, \{2, 7\}, \{4, 5\} \right\}.$$ 

- This is a set of sets. Since sets aren’t ordered, the blocks can be put in another order, and the elements within each block can be written in a different order:

$$\left\{ \{1, 3, 6\}, \{2, 7\}, \{4, 5\} \right\} = \left\{ \{5, 4\}, \{6, 1, 3\}, \{7, 2\} \right\}.$$ 

- Define $S(n, k)$ as the number of partitions of an $n$-element set into $k$ blocks. This is called the *Stirling Number of the Second Kind*. We will find a recursion and other formulas for $S(n, k)$.

- Must use capital ‘S’ in $S(n, k)$; later we’ll define a separate function $s(n, k)$ with lowercase ‘s’.
How do partitions of \([n]\) relate to partitions of \([n - 1]\)?

- Define \([0] = \emptyset\) and \([n] = \{1, 2, \ldots, n\}\) for integers \(n > 0\).
  It is convenient to use \([n]\) as an example of an \(n\)-element set.

- Examine what happens when we cross out \(n\) in a set partition of \([n]\), to obtain a set partition of \([n - 1]\) (here, \(n = 5\)): 
  
  \[
  \begin{align*}
  \{\{1, 3\}, \{2, 4, 5\}\} & \rightarrow \{\{1, 3\}, \{2, 4\}\} \\
  \{\{1, 3, 5\}, \{2, 4\}\} & \rightarrow \{\{1, 3\}, \{2, 4\}\} \\
  \{\{1, 3\}, \{2, 4\}, \{5\}\} & \rightarrow \{\{1, 3\}, \{2, 4\}\}
  \end{align*}
  \]

- For all three of the set partitions on the left, removing 5 yields the set partition \(\{\{1, 3\}, \{2, 4\}\}\).

- In the first two, 5 was in a block with other elements, and removing it yielded the same number of blocks.

- In the third, 5 was in its own block, so we also had to remove the block \(\{5\}\) since only nonempty blocks are allowed.
How do partitions of \([n]\) relate to partitions of \([n-1]\)?

- Reversing that, there are three ways to insert 5 into \({\{1, 3\}, \{2, 4\}}\):

  \[
  \{\{1, 3\}, \{2, 4\}\} \rightarrow \begin{cases}
  \{\{1, 3, 5\}, \{2, 4\}\} & \text{insert in 1}\text{st block;} \\
  \{\{1, 3\}, \{2, 4, 5\}\} & \text{insert in 2}\text{nd block;} \\
  \{\{1, 3\}, \{2, 4\}, \{5\}\} & \text{insert as new block.}
  \end{cases}
  \]

- Inserting \(n\) in an existing block keeps the same number of blocks.
- Inserting \(\{n\}\) as a new block increases the number of blocks by 1.
Recursion for $S(n, k)$

Insert $n$ into a partition of $[n-1]$ to obtain a partition of $[n]$ into $k$ blocks:

- **Case: partitions of $[n]$ in which $n$ is not in a block alone:**  
  Choose a partition of $[n-1]$ into $k$ blocks ($S(n-1, k)$ choices)  
  Insert $n$ into any of these blocks ($k$ choices)  
  **Subtotal:** $k \cdot S(n-1, k)$

- **Case: partitions of $[n]$ in which $n$ is in a block alone:**  
  Choose a partition of $[n-1]$ into $k-1$ blocks ($S(n-1, k-1)$ ways)  
  and add a new block $\{n\}$  
  **Subtotal:** $S(n-1, k-1)$

**Total:** $S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1)$

- This recursion requires $n-1 \geq 0$ and $k-1 \geq 0$, so $n, k \geq 1$. 

Prof. Tesler  
Ch. 5: Compositions and Partitions  
Math 184A / Fall 2017
Initial conditions for $S(n, k)$

When $n = 0$ or $k = 0$

$n = 0$: Partitions of $\emptyset$

- It is not valid to partition the null set as $\{\emptyset\}$, since that has an empty block.
- However, it is valid to partition it as $\emptyset = \emptyset$. There are no blocks, so there are no empty blocks. The union of no blocks equals $\emptyset$.
- This is the only partition of $\emptyset$, so $S(0, 0) = 1$ and $S(0, k) = 0$ for $k > 0$.

$k = 0$: partitions into 0 blocks

- $S(n, 0) = 0$ when $n > 0$ since every partition of $[n]$ must have at least one block.

Not an initial condition, but related:

- $S(n, k) = 0$ for $k > n$ since the partition of $[n]$ with the most blocks is $\{\{1\}, \ldots, \{n\}\}$. 
Table of values of $S(n, k)$: Initial conditions

Compute $S(n, k)$ from the recursion and initial conditions:

\[ S(0, 0) = 1 \quad \quad S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1) \]
\[ S(n, 0) = 0 \text{ if } n > 0 \]
\[ S(0, k) = 0 \text{ if } k > 0 \]
\[ S(0, k) = 0 \text{ if } k > 0 \]

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table of values of $S(n, k)$: Recursion

Compute $S(n, k)$ from the recursion and initial conditions:

- $S(0, 0) = 1$
- $S(n, 0) = 0$ if $n > 0$
- $S(0, k) = 0$ if $k > 0$
- $S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1)$ if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td></td>
<td>$S(n-1, k-1)$</td>
<td>$S(n-1, k)$</td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>$S(n, k)$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table of values of $S(n, k)$

Compute $S(n, k)$ from the recursion and initial conditions:

- $S(0, 0) = 1$
- $S(n, 0) = 0$ if $n > 0$
- $S(0, k) = 0$ if $k > 0$
- $S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1)$ if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Table of values of $S(n, k)$**

Compute $S(n, k)$ from the recursion and initial conditions:

\[
S(0, 0) = 1 
\]

\[
S(n, 0) = 0 \text{ if } n > 0 
\]

\[
S(0, k) = 0 \text{ if } k > 0 
\]

\[
S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1) \text{ if } n \geq 1 \text{ and } k \geq 1 
\]

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Compute $S(n, k)$ from the recursion and initial conditions:

$$S(0, 0) = 1 \quad \quad \quad S(n, k) = k \cdot S(n - 1, k)$$

$$S(n, 0) = 0 \text{ if } n > 0 \quad \quad \quad S(n - 1, k - 1)$$

$$S(0, k) = 0 \text{ if } k > 0 \quad \quad \quad \text{if } n \geq 1 \text{ and } k \geq 1$$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
Table of values of $S(n, k)$

Compute $S(n, k)$ from the recursion and initial conditions:

- $S(0, 0) = 1$
- $S(n, 0) = 0$ if $n > 0$
- $S(0, k) = 0$ if $k > 0$
- $S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1)$ if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table of values of $S(n, k)$

Compute $S(n, k)$ from the recursion and initial conditions:

\[
S(0, 0) = 1 \quad \quad \quad S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1)
\]

$S(n, 0) = 0$ if $n > 0$

$S(0, k) = 0$ if $k > 0$

$S(n, k) = 0$ if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table of values of $S(n, k)$

Compute $S(n, k)$ from the recursion and initial conditions:

$S(0, 0) = 1$

$S(n, 0) = 0$ if $n > 0$

$S(0, k) = 0$ if $k > 0$

$S(n, k) = k \cdot S(n - 1, k)
+ S(n - 1, k - 1)$

if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table of values of $S(n, k)$

Compute $S(n, k)$ from the recursion and initial conditions:

- $S(0, 0) = 1$
- $S(n, 0) = 0$ if $n > 0$
- $S(0, k) = 0$ if $k > 0$
- $S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1)$ if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table of values of $S(n, k)$

Compute $S(n, k)$ from the recursion and initial conditions:

- $S(0, 0) = 1$
- $S(n, 0) = 0$ if $n > 0$
- $S(0, k) = 0$ if $k > 0$
- $S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1)$ if $n \geq 1$ and $k \geq 1$

<table>
<thead>
<tr>
<th>$S(n, k)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
Example and Bell numbers

- \( S(n, k) \) is the number of set partitions of \([n]\) into \(k\) blocks. For \(n = 4\):

<table>
<thead>
<tr>
<th></th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 3)</th>
<th>(k = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>{{1, 2, 3}, {4}}</td>
<td>{{1, 2, 4}, {3}}</td>
<td>{{1, 3, 4}, {2}}</td>
<td>{{1, 3, 4}, {2}}</td>
</tr>
<tr>
<td>(2)</td>
<td>{{1, 2, 3}, {4}}</td>
<td>{{1, 2, 4}, {3}}</td>
<td>{{1, 3, 4}, {2}}</td>
<td>{{1, 3, 4}, {2}}</td>
</tr>
<tr>
<td>(3)</td>
<td>{{1, 2, 3}, {4}}</td>
<td>{{1, 2, 4}, {3}}</td>
<td>{{1, 3, 4}, {2}}</td>
<td>{{1, 3, 4}, {2}}</td>
</tr>
<tr>
<td>(4)</td>
<td>{{1, 2, 3}, {4}}</td>
<td>{{1, 2, 4}, {3}}</td>
<td>{{1, 3, 4}, {2}}</td>
<td>{{1, 3, 4}, {2}}</td>
</tr>
</tbody>
</table>

- \( S(4, 1) = 1 \quad S(4, 2) = 7 \quad S(4, 3) = 6 \quad S(4, 4) = 1 \)

- The **Bell number** \(B_n\) is the total number of set partitions of \([n]\) into any number of blocks:

\[
B_n = S(n, 0) + S(n, 1) + \cdots + S(n, n)
\]

- Total: \(B_4 = 1 + 7 + 6 + 1 = 15\)
Compute \( S(n, k) \) from the recursion and initial conditions:

\[
S(0, 0) = 1 \\
S(n, 0) = 0 \text{ if } n > 0 \\
S(0, k) = 0 \text{ if } k > 0 \\
S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1) \text{ if } n \geq 1 \text{ and } k \geq 1
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( S(n, k) )</th>
<th>( k = 0 )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>Row total ( B_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
<td>0</td>
<td>52</td>
</tr>
</tbody>
</table>
Simplex locks

- Simplex brand locks were a popular combination lock with 5 buttons.

- The combination 13-25-4 means:
  - Push buttons 1 and 3 together.
  - Push buttons 2 and 5 together.
  - Push 4 alone.
  - Turn the knob to open.

- Buttons cannot be reused.

- We first consider the case that all buttons are used, and separately consider the case that some buttons aren’t used.
Represent the combination 13-25-4 as an ordered set partition

- We may represent 13-25-4 as an ordered set partition
  \((\{1, 3\}, \{2, 5\}, \{4\})\)
  Block \(\{1, 3\}\) is first, block \(\{2, 5\}\) is second, and block \(\{4\}\) is third.
  Blocks are sets, so can replace \(\{1, 3\}\) by \(\{3, 1\}\), or \(\{2, 5\}\) by \(\{5, 2\}\).
- Note that if we don’t say it’s ordered, then a set partition is a set of blocks, not a tuple of blocks, and the blocks can be reordered:
  \(\{\{1, 3\}, \{2, 5\}, \{4\}\} = \{\{5, 2\}, \{4\}, \{1, 3\}\}\)

Number of combinations

- Let \(n = \#\) of buttons (which must all be used)
  \(k = \#\) groups of button pushes.
- There are \(S(n, k)\) ways to split the buttons into \(k\) blocks
  \(\times k!\) ways to order the blocks
  \(= k! \cdot S(n, k)\) combinations.
- The \# of combinations on \(n = 5\) buttons and \(k = 3\) groups of pushes is
  \(3! \cdot S(5, 3) = 6 \cdot 25 = 150\)
Represent the combination 13-25-4 as a surjective (onto) function

- Define a function \( f(i) = j \), where button \( i \) is in push number \( j \):
  
  \[
  \begin{array}{c|c}
  i = \text{button number} & j = \text{push number} \\
  \hline
  1 & 1 \\
  2 & 2 \\
  3 & 1 \\
  4 & 3 \\
  5 & 2 \\
  \end{array}
  \]

- This gives a surjective (onto) function \( f : [5] \to [3] \).

- The blocks of buttons pushed are
  
  1\textsuperscript{st}: \( f^{-1}(1) = \{1, 3\} \)  
  2\textsuperscript{nd}: \( f^{-1}(2) = \{2, 5\} \)  
  3\textsuperscript{rd}: \( f^{-1}(3) = \{4\} \)

**Theorem**

The number of surjective (onto) functions \( f : [n] \to [k] \) is \( k! \cdot S(n, k) \).

**Proof.**

Split \([n]\) into \( k \) nonempty blocks in one of \( S(n, k) \) ways.
Choose one of \( k! \) orders for the blocks: \( (f^{-1}(1), \ldots, f^{-1}(k)) \).
How many combinations don’t use all the buttons?

- The combination 3-25 does not use 1 and 4.
- Trick: write it as 3-25-(14), with all unused buttons in one “phantom” push at the end.
- There are three groups of buttons and we don’t use the 3rd group.
- # combinations with 2 pushes that don’t use all buttons = # combinations with 3 pushes that do use all buttons.
- For set partition \{{{3}\}, {{2, 5}\}, {{1, 4}\}}, the 3! orders of the blocks give:

<table>
<thead>
<tr>
<th>Ordered 3-tuple</th>
<th>Actual combination</th>
<th>+ phantom push</th>
</tr>
</thead>
<tbody>
<tr>
<td>{{{3}}, {{2, 5}}, {{1, 4}}}</td>
<td>3-25</td>
<td>3-25-(14)</td>
</tr>
<tr>
<td>{{{3}}, {{1, 4}}, {{2, 5}}}</td>
<td>3-14</td>
<td>3-14-(25)</td>
</tr>
<tr>
<td>{{{2, 5}}, {{3}}, {{1, 4}}}</td>
<td>25-3</td>
<td>25-3-(14)</td>
</tr>
<tr>
<td>{{{2, 5}}, {{1, 4}}, {{3}}}</td>
<td>25-14</td>
<td>25-14-(3)</td>
</tr>
<tr>
<td>{{{1, 4}}, {{3}}, {{2, 5}}}</td>
<td>14-3</td>
<td>14-3-(25)</td>
</tr>
<tr>
<td>{{{1, 4}}, {{2, 5}}, {{3}}}</td>
<td>14-25</td>
<td>14-25-(3)</td>
</tr>
</tbody>
</table>
How many combinations don’t use all the buttons?

Putting *all* unused buttons into *one* phantom push at the end gives a bijection between
- Combinations with $k - 1$ pushes that *don’t* use all $n$ buttons, and
- Combinations with $k$ pushes that *do* use all $n$ buttons.

Lemma (General case)

For $n, k \geq 1$:

The # combinations with $k - 1$ pushes that *don’t* use all $n$ buttons
= the # combinations with $k$ pushes that *do* use all $n$ buttons
= $k! \cdot S(n, k)$. 
We will count the number of functions $f : [n] \rightarrow [k]$ in two ways.

**First method**

$$(k \text{ choices of } f(1)) \times (k \text{ choices of } f(2)) \times \cdots \times (k \text{ choices of } f(n)) = k^n$$
Counting the total number of functions $f : [n] \rightarrow [k]$

Second method: Classify functions by their images and inverses

- Consider $f : [10] \rightarrow \{a, b, c, d, e\}$:
  
  $i =$
  
  $\begin{array}{ccccccccc}
  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  f(i) = & a & c & c & a & c & d & c & a & c & d \\
  \end{array}$

- The **domain** is $[10]$.

- The **codomain** (or **target**) is $\{a, b, c, d, e\}$.

- The **image** is $\text{image}(f) = \{f(1), \ldots, f(10)\} = \{a, c, d\}$. It’s a subset of the codomain.

- The inverse blocks are
  
  $f^{-1}(a) = \{1, 4, 8\}$
  
  $f^{-1}(c) = \{2, 3, 5, 7, 9\}$
  
  $f^{-1}(d) = \{6, 10\}$
  
  $f^{-1}(b) = f^{-1}(e) = \emptyset$

- $f : [10] \rightarrow \{a, b, c, d, e\}$ is not onto, but $f : [10] \rightarrow \{a, c, d\}$ is onto.
Counting the total number of functions \( f : [n] \to [k] \)

Second method, continued

- Consider \( f : [10] \to \{a, b, c, d, e\} : \)
  
  \[
  \begin{array}{cccccccccc}
  i = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  f(i) = & a & c & c & a & c & d & c & a & c & d \\
  \end{array}
  \]
  
  \( f : [10] \to \{a, b, c, d, e\} \) is not onto, but \( f : [10] \to \{a, c, d\} \) is onto.

- There are \( S(10, 3) \cdot 3! \) surjective functions \( f : [10] \to \{a, c, d\} \).

- Classify all \( f : [10] \to \{a, b, c, d, e\} \) according to \( T = \text{image}(f) \).

- There are \( \binom{5}{3} \) subsets \( T \subseteq \{a, b, c, d, e\} \) of size \( |T| = 3 \).
  
  Each \( T \) has \( S(10, 3) \cdot 3! \) surjective functions \( f : [10] \to T \).

  So \( S(10, 3) \cdot 3! \cdot \binom{5}{3} \) functions \( f : [10] \to \{a, \ldots, e\} \) have \( |\text{image}(f)| = 3 \).

- **Simplify:** \( 3! \cdot \binom{5}{3} = 3! \cdot \binom{5}{3} = 5 \cdot 4 \cdot 3 = (5)_3 \)

  So \( S(10, 3) \cdot (5)_3 \) functions \( f : [10] \to [5] \) have \( |\text{image}(f)| = 3 \).
In general, $S(n, i) \cdot (k)_i$ functions $f : [n] \to [k]$ have $|\text{image}(f)| = i$.

Summing over all possible image sizes $i = 0, \ldots, n$ gives the total number of functions $f : [n] \to [k]$

$$\sum_{i=0}^{n} S(n, i) \cdot (k)_i$$

Putting this together with the first method gives

$$k^n = \sum_{i=0}^{n} S(n, i) \cdot (k)_i$$

for all integers $n, k \geq 0$.
Counting the total number of functions $f : [n] \rightarrow [k]$

Second method, continued

\[ k^n = \sum_{i=0}^{n} S(n, i) \cdot (k)_i \quad \text{for all integers } n, k \geq 0 \]

- $i = |\text{image}(f)| = |\{f(1), \ldots, f(n)\}| \leq n$, so $i \leq n$.

- Also, $i \leq k$ since $\text{image}(f) \subseteq [k]$.

- In the sum, upper bound $i = n$ may be replaced by $k$ or $\min(n, k)$. Any terms added or removed in the sum by changing the upper bound don’t affect the result since those terms equal $0$:

\[ S(n, i) = 0 \quad \text{for } i > n \]
\[ (k)_i = 0 \quad \text{for } i > k. \]
The identity

\[ k^n = \sum_{i=0}^{n} S(n, i) \cdot (k)_i \quad \text{for all integers } n, k \geq 0 \]

generalizes to

**Theorem**

\[ x^n = \sum_{i=0}^{n} S(n, i) \cdot (x)_i \quad \text{for all real } x \text{ and integer } n \geq 0. \]
Identity for real numbers

**Theorem**

\[ x^n = \sum_{i=0}^{n} S(n, i) \cdot (x)_i \quad \text{for all real } x \text{ and integer } n \geq 0. \]

**Examples**

For \( n = 2 \):

\[
S(2, 0)(x)_0 + S(2, 1)(x)_1 + S(2, 2)(x)_2 = 0 \cdot 1 + 1 \cdot x + 1 \cdot x(x - 1) = 0 + x + (x^2 - x) = x^2
\]

For \( n = 3 \):

\[
S(3, 0)(x)_0 + S(3, 1)(x)_1 + S(3, 2)(x)_2 + S(3, 3)(x)_3 = 0 \cdot 1 + 1 \cdot x + 3 \cdot x(x - 1) + 1 \cdot x(x - 1)(x - 2) = 0 + x + 3(x^2 - x) + (x^3 - 3x^2 + 2x) = x^3 + (3 - 3)x^2 + (1 - 3 + 2)x = x^3
\]
Lemma

If \( f(x) \) and \( g(x) \) are polynomials of degree \( \leq n \) that agree on more than \( n \) distinct values of \( x \), then \( f(x) = g(x) \) as polynomials.

Proof.

- Let \( h(x) = f(x) - g(x) \). This is a polynomial of degree \( \leq n \).
- If \( h(x) = 0 \) identically, then \( f(x) = g(x) \) as polynomials. Assume \( h(x) \) is not identically 0.
- Let \( x_1, \ldots, x_m \) (with \( m > n \)) be distinct values at which \( f(x_i) = g(x_i) \). Then \( h(x_i) = f(x_i) - g(x_i) = 0 \) for \( i = 1, \ldots, m \), so \( h(x) \) factors as
  \[
  h(x) = p(x)(x - x_1)^{r_1}(x - x_2)^{r_2} \cdots (x - x_m)^{r_m} \cdots
  \]
  for some polynomial \( p(x) \neq 0 \) and some integers \( r_1, \ldots, r_m \geq 1 \).
- Then \( h(x) \) has degree \( \geq m > n \).
  But \( h(x) \) has degree \( \leq n \), a contradiction.
  Thus, \( h(x) = 0 \), so \( f(x) = g(x) \).
Identity for real numbers

Theorem

\[ x^n = \sum_{i=0}^{n} S(n, i) \cdot (x)_i \quad \text{for all real } x \text{ and integer } n \geq 0. \]

Proof.

- Both sides of the equation are polynomials in \( x \) of degree \( n \).
- They agree at an infinite number of values \( x = 0, 1, \ldots \).
- Since \( \infty > n \), they’re identical polynomials. 

\[ \square \]
5.3. Integer partitions

- The compositions \((2, 1, 1), (1, 2, 1), (1, 1, 2)\) are different. Sometimes the number of 1’s, 2’s, 3’s, \ldots matters but not the order.

- An \textit{integer partition} of \(n\) is a tuple \((a_1, \ldots, a_k)\) of positive integers that sum to \(n\), with \(a_1 \geq a_2 \geq \cdots \geq a_k \geq 1\).

  The partitions of 4 are:
  \[
  (4) \quad (3, 1) \quad (2, 2) \quad (2, 1, 1) \quad (1, 1, 1, 1)
  \]

- Define \(p(n) = \#\) integer partitions of \(n\)

  \[
  p_k(n) = \#\text{ integer partitions of } n \text{ into exactly } k \text{ parts}
  \]

  \[
  p(4) = 5 \quad p_1(4) = 1 \quad p_2(4) = 2 \quad p_3(4) = 1 \quad p_4(4) = 1
  \]

- We will learn a method to compute these in Chapter 8.
Type of a set partition

- Consider this set partition of [10]:

  \[ \left\{ \{1, 4\}, \{7, 6\}, \{5\}, \{8, 2, 3\}, \{9\}, \{10\} \right\} \]

- The block lengths in the order it was written are 2, 2, 1, 3, 1, 1.

- But the blocks of a set partition could be written in other orders. To make this unique, the type of a set partition is a tuple of the block lengths listed in decreasing order: (3, 2, 2, 1, 1, 1).

- For a set of size \( n \) partitioned into \( k \) blocks, the type is an integer partition of \( n \) in \( k \) parts.
How many set partitions of \([10]\) have type \((3, 2, 2, 1, 1, 1)\)?

- Split \([10]\) into sets \(A, B, C, D, E, F\) of sizes 3, 2, 2, 1, 1, 1, respectively, in one of \(\binom{10}{3,2,2,1,1,1} = \frac{10!}{3!2!2!1!} = 151200\) ways.

- But \(\{A, B, C, D, E, F\} = \{A, C, B, F, E, D\}\), so we overcounted:  
  - \(B, C\) could be reordered \(C, B\): \(2!\) ways.  
  - \(D, E, F\) could be permuted in \(3!\) ways.  
  - If there are \(m_i\) blocks of size \(i\), we overcounted by a factor of \(m_i!\).

- Dividing by the overcounts gives  
  \[
  \frac{\binom{10}{3,2,2,1,1,1}}{1!\ 2!\ 3!} = \frac{151200}{1 \cdot 2 \cdot 6} = \boxed{12600}
  \]

**General formula**  
For an \(n\) element set, the number of set partitions of type \((a_1, a_2, \ldots, a_k)\) where \(n = a_1 + a_2 + \cdots + a_k\) and \(m_i\) of the \(a\)'s equal \(i\), is  
\[
\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{m_1!\ m_2! \cdots} \left(\frac{1}{1!^{m_1}}\ m_1!\right)\left(\frac{1}{2!^{m_2}}\ m_2!\right) \cdots
\]
Ferrers diagrams and Young diagrams

Consider a partition \((a_1, \ldots, a_k)\) of \(n\).

- **Ferrers diagram:** \(a_i\) dots in the \(i\)th row.
- **Young diagram:** squares instead of dots.
- The total number of dots or squares is \(n\).

Our book calls both of these *Ferrers diagrams*, but often they are given separate names.
Conjugate Partition

- Reflect a Ferrers diagram across its main diagonal:

  $\pi = (5, 3, 1)$  \hspace{1cm}  $\pi' = (3, 2, 2, 1, 1)$

- This transforms a partition $\pi$ to its *conjugate partition*, denoted $\pi'$.

- The $i$th row of $\pi$ turns into the $i$th column of $\pi'$: the red, green, and blue rows of $\pi$ turn into columns of $\pi'$.

- Also, the $i$th column of $\pi$ turns into the $i$th row of $\pi'$.

**Theorem:** $(\pi')' = \pi$

**Theorem:** If $\pi$ has $k$ parts, then the largest part of $\pi'$ is $k$.

Here: $\pi$ has 3 parts $\iff$ the first column of $\pi$ has length 3 $\iff$ the first row $\pi'$ is 3 $\iff$ the largest part of $\pi'$ is 3
Theorem

1. The number of partitions of \( n \) into exactly \( k \) parts \( (p_k(n)) \)
   \( = \) the number of partitions of \( n \) where the largest part \( = k \).

2. The number of partitions of \( n \) into \( \leq k \) parts
   \( = \) the number of partitions of \( n \) into parts that are each \( \leq k \).

Proof: Conjugation is a bijection between the two types of partitions.

Example: Partitions of 6 into 3 or \( \leq 3 \) parts

\[
\begin{array}{ccc}
\pi \text{ with exactly 3 parts} & \pi \text{ with } < 3 \text{ parts} \\
(4, 1, 1) & (4, 2) & (5, 1) \\
(3, 2, 1) & (3, 2) & (6) \\
(2, 2, 2) & & \\

\pi' \text{ has largest part } = 3 & & \pi' \text{ has largest part } < 3 \\
(3, 1, 1, 1) & (2, 2, 1, 1) & (1,1,1,1,1,1) \\
(3, 2, 1) & (2, 1, 1, 1, 1) & \\
(3, 3) & & \\
\end{array}
\]
Many combinatorial problems can be modeled as placing \textit{balls} into \textit{boxes}:

\begin{itemize}
  \item \textbf{Indistinguishable balls:} \ball \ball \cdots \ball
  \item \textbf{Distinguishable balls:} \ball \ball \cdots \ball
  \item \textbf{Indistinguishable boxes:} \hspace{2cm} \hspace{2cm}
  \item \textbf{Distinguishable boxes:} \hspace{2cm} \hspace{2cm}
\end{itemize}
Balls and boxes

Indistinguishable balls

- **Integer partitions:** \((3, 2, 1)\)

  \[
  \begin{array}{ccc}
  & \bullet & \bullet \\
  \bullet & & \bullet \bullet \bullet \\
  & & \bullet \bullet \\
  \end{array}
  = \begin{array}{ccc}
  & \bullet & \bullet \bullet \\
  \bullet & & \bullet \bullet \\
  & & \bullet \\
  \end{array}
  \]

  Indistinguishable balls.
  Indistinguishable boxes.

- **Compositions:** \((1, 3, 2)\)

  \[
  \begin{array}{ccc}
  A & & \\
  \bullet & & \bullet \bullet \bullet \\
  & & \bullet \bullet \\
  \end{array}
  \]

  \[
  \begin{array}{ccc}
  B & & \\
  \bullet & & \bullet \bullet \bullet \\
  & & \bullet \bullet \\
  \end{array}
  \]

  \[
  \begin{array}{ccc}
  C & & \\
  \bullet & & \bullet \bullet \bullet \\
  & & \bullet \bullet \\
  \end{array}
  \]

  Indistinguishable balls.
  Distinguishable boxes (which give the order).
Distinguishable balls

Set partitions: \{\{6\}, \{2, 4, 5\}, \{1, 3\}\}

Distinguishable balls.
Indistinguishable boxes (so the blocks are not in any order).

Surjective (onto) functions / ordered set partitions:

Distinguishable balls and distinguishable boxes.

Gives surjective function \(f : [6] \rightarrow \{A, B, C\}\)

\[f(6) = A \quad f(2) = f(4) = f(5) = B \quad f(1) = f(3) = C\]

or an ordered set partition \(\left(\{6\}, \{2, 4, 5\}, \{1, 3\}\right)\)