Chapter 6.1. Cycles in Permutations

Prof. Tesler

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Notations for permutations

- Consider a permutation in *1-line form*:

$$f \quad = \quad 6 \quad 5 \quad 2 \quad 7 \quad 1 \quad 3 \quad 4 \quad 8$$

- This represents a function $f : [8] \to [8]$

$$\begin{aligned} f(1) &= 6 & \qquad f(5) &= 1 \\ f(2) &= 5 & f(6) &= 3 \\ f(3) &= 2 & f(7) &= 4 \\ f(4) &= 7 & f(8) &= 8 \end{aligned}$$

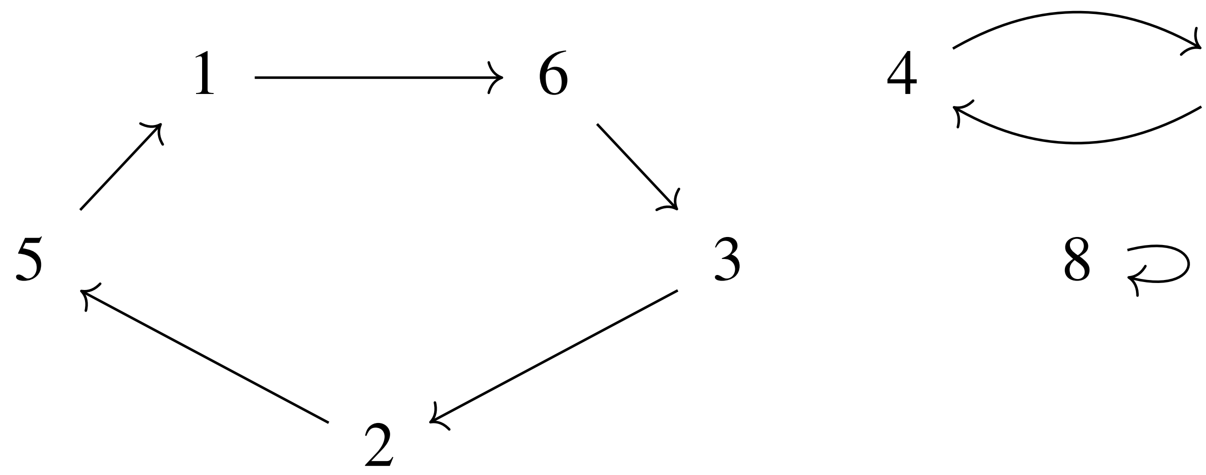
- The *2-line form* is

$$f = \begin{pmatrix} i_1 & i_2 & \cdots \\ f(i_1) & f(i_2) & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 2 & 7 & 1 & 3 & 4 & 8 \end{pmatrix}$$

Cycles in permutations

$f = 6\ 5\ 2\ 7\ 1\ 3\ 4\ 8$

- Draw a picture with points numbered $1, \ldots, n$ and arrows $i \to f(i)$.



- Each number has one arrow in and one out: $f^{-1}(i) \to i \to f(i)$
  Each chain closes upon itself, splitting the permutation into cycles.

- The *cycle decomposition* is $f = (1,6,3,2,5)(4,7)(8)$
  If all numbers are 1 digit, we may abbreviate: $f = (16325)(47)(8)$

- The cycles can be written in any order.
  Within each cycle, we can start at any number.
  $$f = (1,6,3,2,5)(4,7)(8) = (8)(7,4)(3,2,5,1,6) = \cdots$$

Multiplying permutations

$$f = (1,2,4)(3,6)(5) = 246153$$
$$g = (1,3)(2,5)(4,6) = 351624$$

- There are two conventions for multiplying permutations, corresponding to two conventions for composing functions.

Left-to-right composition (our book and often in Abstract Algebra)

$$(f\,g)(i) = g(f(i)) \qquad (f\,g)(1) = g(f(1)) = g(2) = 5$$

Right-to-left composition (usual convention in Calculus)

$$(f\,g)(i) = f(g(i)) \qquad (f\,g)(1) = f(g(1)) = f(3) = 6$$

- Note that multiplication of permutations is *not* commutative. E.g., with the left-to-right convention,

$(f\,g)(1) = g(f(1)) = g(2) = 5$ while
$(gf)(1) = f(g(1)) = f(3) = 6,$

so $(f\,g)(1) \neq (gf)(1)$, so $f\,g \neq gf$.

Multiplying permutations: left-to-right composition

$$f = (1,2,4)(3,6)(5) = 246153$$
$$g = (1,3)(2,5)(4,6) = 351624$$

i	$(1,2,4)$	$(3,6)$	(5)	$(1,3)$	$(2,5)$	$(4,6)$	$(f\,g)(i)$
1	2				5		$(f\,g)(1) = 5$
2	4					6	$(f\,g)(2) = 6$
3		6				4	$(f\,g)(3) = 4$
4	1			3			$(f\,g)(4) = 3$
5			5		2		$(f\,g)(5) = 2$
6		3		1			$(f\,g)(6) = 1$

So $f\,g = (1,2,4)(3,6)(5)(1,3)(2,5)(4,6) = 564321 = (1,5,2,6)(3,4)$.

Multiplying permutations: right-to-left composition

$$f = (1,2,4)(3,6)(5) = 246153$$
$$g = (1,3)(2,5)(4,6) = 351624$$

$(f\,g)(i)$	$(1,2,4)$	$(3,6)$	(5)	$(1,3)$	$(2,5)$	$(4,6)$	i
$(f\,g)(1) = 6$		6		3			1
$(f\,g)(2) = 5$			5		5		2
$(f\,g)(3) = 2$	2			1			3
$(f\,g)(4) = 3$		3				6	4
$(f\,g)(5) = 4$	4				2		5
$(f\,g)(6) = 1$	1					4	6

So $f\,g = (1,2,4)(3,6)(5)(1,3)(2,5)(4,6) = 652341 = (1,6)(2,5,4,3)$.

Inverse permutation

- The *identity permutation* on $[n]$ is $f(i) = i$ for all i. Call it
$$\mathrm{id}_n = 12 \cdots n = (1)(2) \cdots (n)$$
It satisfies $f \cdot \mathrm{id}_n = \mathrm{id}_n \cdot f = f$.

- The *inverse* of a permutation f is the inverse function f^{-1}.
$$f = 246153 \qquad\qquad f^{-1} = 416253$$
It satisfies $f(f^{-1}(i)) = i$ and $f^{-1}(f(i)) = i$ for all i.
Equivalently, $f \cdot f^{-1} = f^{-1} \cdot f = \mathrm{id}_n$.

- In cycle form, just reverse the direction of each cycle:
$$f = (1,2,4)(3,6)(5) \qquad\qquad f^{-1} = (4,2,1)(6,3)(5)$$

- The inverse of a product is $(f\,g)^{-1} = g^{-1} f^{-1}$
since $g^{-1} \cdot f^{-1} \cdot f \cdot g = g^{-1} \cdot \mathrm{id}_n \cdot g = g^{-1} \cdot g = \mathrm{id}_n$.

Type of a permutation

- The *type* of a permutation is the integer partition formed from putting the cycle lengths into decreasing order:
  $f \;=\; 6\;5\;2\;7\;1\;3\;4\;8 \;=\; (1,6,3,2,5)(4,7)(8) \qquad \text{type}(f) = (5,2,1)$

How many permutations of size 8 have type $(5,2,1)$?

- Draw a pattern with blanks for cycles of lengths $5, 2, 1$:
  $$(\_,\_,\_,\_,\_)(\_,\_)(\_)$$
- Fill in the blanks in one of $8! = 40320$ ways.
- Each cycle can be restarted anywhere:
  $(1,6,3,2,5) = (6,3,2,5,1) = (3,2,5,1,6) = (2,5,1,6,3) = (5,1,6,3,2)$
- We overcounted each cycle of length ℓ a total of ℓ times, so divide by the product of the cycle lengths:
  $$\frac{8!}{5\cdot 2\cdot 1} = \frac{40320}{10} = \boxed{\mathbf{4032}}$$

How many permutations of size 15 have 5 cycles of length 3?

- Draw a pattern with blanks for 5 cycles of length 3:
$$(\_,\_,\_)(\_,\_,\_)(\_,\_,\_)(\_,\_,\_)(\_,\_,\_)$$
These comprise $5 \cdot 3 = 15$ entries.

- Fill in the blanks in one of $15!$ ways.

- Each cycle has 3 representations matching this format (by restarting at any of 3 places), so divide by 3^5.

- The order of the whole cycles can be changed while keeping the pattern, e.g., $(1,2,3)(4,5,6) = (4,5,6)(1,2,3)$.
Divide by $5!$ ways to reorder the cycles.

- Total:
$$\frac{15!}{3^5 \cdot 5!} = \boxed{\mathbf{44844800}}$$

General formula for the number of permutations of each type

- Given these parameters:

Number of cycles of length i:	m_i
Permutation size:	$n = \sum_i m_i \cdot i$
Number of cycles:	$\sum_i m_i$

- The number of permutations of this type is

$$\frac{n!}{1^{m_1}\, 2^{m_2}\, 3^{m_3}\, \cdots m_1!\, m_2!\, m_3! \cdots} = \frac{n!}{1^{m_1}\, m_1!\, 2^{m_2}\, m_2!\, 3^{m_3} m_3!\, \cdots}$$

Example: 10 cycles of length 3 and 5 cycles of length 4

- type $= (4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$
- $n = 10 \cdot 3 + 5 \cdot 4 = 30 + 20 = 50$
- $10 + 5 = 15$ cycles
- Number of permutations $= \dfrac{50!}{3^{10} \cdot 4^5 \cdot 10! \cdot 5!}$

Stirling Numbers of the First Kind

- Let $c(n,k) =$ # of permutations of n elements with exactly k cycles. This is called the *Signless Stirling Number of the First Kind*.
- We will work out the values of $c(4,k)$, so $n=4$ and k varies.

$k=4$	$(1)(2)(3)(4)$	$c(4,4)=1$
$k=3$	$(\_,\_)(\_)(\_)$	$c(4,3)=\frac{4!}{2^1 \cdot 1^2 \cdot 1! \cdot 2!}=\frac{24}{4}=6$
$k=2$	$(\_,\_)(\_,\_)$	$\frac{4!}{2^2 \cdot 2!}=\frac{24}{4 \cdot 2}=3$
	$(\_,\_,\_)(\_)$	$\frac{4!}{3 \cdot 1 \cdot 1! \cdot 1!}=\frac{24}{3}=8$
		$c(4,2)=3+8=11$
$k=1$	$(\_,\_,\_,\_)$	$c(4,1)=\frac{4!}{4^1 \cdot 1!}=\frac{24}{4}=6$
$k \neq 1,2,3,4$		$c(4,k)=0$
		Total $=1+6+11+6=24=4!$

- For $c(n,k)$: the possible permutation types are integer partitions of n into k parts. Compute the number of permutations of each type. Add them up to get $c(n,k)$.

Recursive formula for $c(n, k)$

What permutations can be formed by inserting $n = 6$ into $(1, 4, 2)(3, 5)$ (a permutation of size $n - 1$)?

- **Case:** Insert 6 into an existing cycle in one of $n - 1 = 5$ ways:

$$\begin{aligned}
&(1, 6, 4, 2)(3, 5) \\
&(1, 4, 6, 2)(3, 5) \\
&(1, 4, 2, 6)(3, 5) = (6, 1, 4, 2)(3, 5) \\
&(1, 4, 2)(3, 6, 5) \\
&(1, 4, 2)(3, 5, 6) = (1, 4, 2)(6, 3, 5)
\end{aligned}$$

- ***Note:*** inserting a number at the start or end of a cycle is the same, so don't double-count it.

- **Case:** Insert (6) as a new cycle; there is only one way to do this:

$$(1, 4, 2)(3, 5)(6)$$

- To obtain k cycles, insert 6 into a permutation of $[5]$ with k cycles (if added to an existing cycle) or $k - 1$ cycles (if added as a new cycle).

Recursive formula for $c(n,k)$

Insert n into a permutation of $[n{-}1]$
to obtain a permutation of $[n]$ with k cycles:

- **Case: permutations of $[n]$ in which n is not in a cycle alone:**
  Choose a permutation of $[n{-}1]$ into k cycles ($c(n{-}1,k)$ ways)
  Insert n into an existing cycle after any of $1,\ldots,n{-}1$ ($n{-}1$ ways)

  Subtotal: $(n-1)\cdot c(n-1,k)$

- **Case: permutations of $[n]$ in which n is in a cycle alone:**
  Choose a permutation of $[n{-}1]$ into $k{-}1$ cycles ($c(n{-}1,k{-}1)$ ways)
  and add a new cycle (n) with one element (one way)

  Subtotal: $c(n-1,k-1)$

- **Total:** $c(n,k) = (n-1)\cdot c(n-1,k) + c(n-1,k-1)$

- This recursion requires using $n-1 \geqslant 0$ and $k-1 \geqslant 0$, so $n,k \geqslant 1$.

Initial conditions for $c(n,k)$

When $n = 0$ or $k = 0$

$n = 0$: Permutations of $\emptyset$

- There is only one "empty function" $f : \emptyset \to \emptyset$.
- It is vacuously one-to-one, onto, and a bijection.
- As a permutation, it has no cycles.
- $c(0,0) = 1$ and $c(0,k) = 0$ for $k > 0$.

$k = 0$: Permutations into 0 cycles

- $c(n,0) = 0$ when $n > 0$
  since every permutation of $[n]$ must have at least one cycle.

Not an initial condition, but related:

- $c(n,k) = 0$ for $k > n$
  since the permutation of $[n]$ with the most cycles is $(1)(2)\cdots(n)$.

Table of values of $c(n,k)$

Compute $c(n,k)$ from the recursion and initial conditions:

$$c(0,0) = 1 \qquad c(n,k) = (n-1)\cdot c(n-1,k) + c(n-1,k-1) \text{ if } n \geqslant 1 \text{ and } k \geqslant 1$$

$$c(n,0) = 0 \text{ if } n > 0$$

$$c(0,k) = 0 \text{ if } k > 0$$

$c(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0				
$n=2$	0				
$n=3$	0				
$n=4$	0				

Compute $c(n,k)$ from the recursion and initial conditions:

$$c(0,0) = 1$$
$$c(n,0) = 0 \text{ if } n > 0$$
$$c(0,k) = 0 \text{ if } k > 0$$

$$\begin{aligned} c(n,k) &= \color{blue}{(n-1)\cdot c(n-1,k)} \\ &\quad + \color{red}{c(n-1,k-1)} \\ &\quad \text{if } n \geqslant 1 \text{ and } k \geqslant 1 \end{aligned}$$

$c(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0				
$n=2$	0		$c(n-1,k-1)$	$c(n-1,k) \;\big\downarrow \cdot(n-1)$	
$n=3$	0			$c(n,k)$	
$n=4$	0				

Table of values of $c(n,k)$

Compute $c(n,k)$ from the recursion and initial conditions:

$$c(0,0) = 1 \qquad c(n,k) = {\color{blue}(n-1)\cdot c(n-1,k)}$$
$$c(n,0) = 0 \text{ if } n > 0 \qquad \quad + {\color{red}c(n-1,k-1)}$$
$$c(0,k) = 0 \text{ if } k > 0 \qquad \quad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$c(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	**Total: $n!$**
$n=0$	1	0	0	0	0	**1**
		$\downarrow\cdot 0$	$\downarrow\cdot 0$	$\downarrow\cdot 0$	$\downarrow\cdot 0$	
$n=1$	0	1	0	0	0	**1**
		$\downarrow\cdot 1$	$\downarrow\cdot 1$	$\downarrow\cdot 1$	$\downarrow\cdot 1$	
$n=2$	0	1	1	0	0	**2**
		$\downarrow\cdot 2$	$\downarrow\cdot 2$	$\downarrow\cdot 2$	$\downarrow\cdot 2$	
$n=3$	0	2	3	1	0	**6**
		$\downarrow\cdot 3$	$\downarrow\cdot 3$	$\downarrow\cdot 3$	$\downarrow\cdot 3$	
$n=4$	0	6	11	6	1	**24**

Generating function for $c(n,k)$

Theorem

Let n be a positive integer. Then

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

Example

- For $n=3$: $$\begin{aligned} x(x+1)(x+2) &= 2x+3x^2+x^3 \\ &= 0x^0+2x^1+3x^2+1x^3 \end{aligned}$$
  Compare with row $n=3$ of the $c(n,k)$ table: $0\ \ 2\ \ 3\ \ 1$
- For $n=4$: $$\begin{aligned} x(x+1)(x+2)(x+3) &= 6x+11x^2+6x^3+x^4 \\ &= 0x^0+6x^1+11x^2+6x^3+1x^4 \end{aligned}$$
  Compare with row $n=4$ of the $c(n,k)$ table: $0\ \ 6\ \ 11\ \ 6\ \ 1$
- So this theorem gives another way (besides the recurrence) to compute $c(n,k)$.

Generating function for $c(n,k)$

Example of going from $n=3$ to $n=4$

With values of $c(n,k)$ plugged in

$$
\begin{aligned}
\overbrace{\underbrace{x(x+1)(x+2)}_{\text{case } n=3}(x+3)}^{\text{case } n=4} &= (2x+3x^2+x^3)\cdot(x+3) \\
&= (2x+3x^2+x^3)\cdot x + (2x+3x^2+x^3)\cdot 3 \\
&= \qquad\qquad 2x^2 + 3x^3 + x^4 \\
&\quad + 6x + 9x^2 + 3x^3 \\
\hline
&= 6x + 11x^2 + 6x^3 + x^4
\end{aligned}
$$

With $c(n,k)$ unevaluated

$$
\begin{aligned}
&(c(3,0)\,x^0 + c(3,1)x^1 + c(3,2)x^2 + c(3,3)x^3)\cdot(x+3) \\
&= \qquad\qquad c(3,0)x^1 + c(3,1)x^2 + c(3,2)x^3 + c(3,3)x^4 \\
&\quad + 3c(3,0)x^0 + 3c(3,1)x^1 + 3c(3,2)x^2 + 3c(3,3)x^3 \\
\hline
&= c(4,0)x^0 + c(4,1)x^1 + c(4,2)x^2 + c(4,3)x^3 + c(4,4)x^4
\end{aligned}
$$

Here, $n=4$, and for $0<k<n$, the coefficient of x^k is

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$

Generating function for $c(n,k)$

Theorem

Let n be a positive integer. Then $\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$

Proof:

- *Base case $n=1$:* $\qquad c(1,0)+c(1,1)x = 0+1x = x$
- *Induction:* For $n \geqslant 2$, assume it holds for $n-1$:

$$x(x+1)\cdots(x+n-2) = \sum_{k=0}^{n-1} c(n-1,k)x^k$$

- Multiply by $x+n-1$ to get $x(x+1)\cdots(x+n-1)$ on one side:

$$x(x+1)\cdots(x+n-1) = \left(\sum_{k=0}^{n-1} c(n-1,k)x^k\right)(x+n-1)$$

We'll show that the other side equals $\sum_{k=0}^{n} c(n,k)\,x^k$.

Generating function for $c(n,k)$

Proof continued (induction step)

$$x(x+1)\cdots(x+n-1) = \left(\sum_{k=0}^{n-1} c(n-1,k)x^k\right)(x+n-1)$$

- Expand the product on the right side:

$$= \underbrace{\sum_{k=0}^{n-1} c(n-1,k)x^{k+1}}_{=\sum_{k=1}^{n} c(n-1,k-1)x^k} \quad + \quad \sum_{k=0}^{n-1} (n-1)c(n-1,k)x^k$$

- Combine terms with the same power of x:

$$= \underbrace{(n-1)c(n-1,0)}_{=0=c(n,0)}x^0 + \left(\sum_{k=1}^{n-1} \underbrace{\left(c(n-1,k-1)+(n-1)c(n-1,k)\right)}_{=c(n,k)}x^k\right) + \underbrace{c(n-1,n-1)}_{=1=c(n,n)}x^n$$

- This equals $\sum_{k=0}^{n} c(n,k)x^k$, so the induction step is complete.

Signs in the Stirling Number of the First Kind

- We showed that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

- Substitute $x \to -x$:

$$\begin{aligned}\sum_{k=0}^{n} c(n,k)(-1)^k x^k &= (-x)(-x+1)\cdots(-x+n-1)\\ &= (-1)^n\, x(x-1)\cdots(x-n+1) = (-1)^n\,(x)_n\end{aligned}$$

- Multiply by $(-1)^n$: $\displaystyle\sum_{k=0}^{n}(-1)^{n-k}c(n,k)x^k = (x)_n$

- Set $s(n,k) = (-1)^{n-k}c(n,k)$:

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

Signs in the Stirling Number of the First Kind

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

- This also holds for $n=0$: left $= s(0,0)x^0 = (-1)^{0-0}1x^0 = 1$
  $\sum_{k=0}^{0} s(0,k)x^k = (x)_0$ right $= (x)_0 = 1$
- $s(n,k) = (-1)^{n-k}c(n,k)$ is the *Stirling Number of the First Kind*.
  Recall $c(n,k)$ is the *Signless Stirling Number of the First Kind*.

Duality between Stirling numbers of the first and second kind

- For all nonnegative integers n, we can convert between powers of x and falling factorials in x in both directions:

$$x^n = \sum_{k=0}^{n} S(n,k) \cdot (x)_k \qquad\qquad (x)_n = \sum_{k=0}^{n} s(n,k) x^k$$

Linear algebra interpretation

- A basis of the space of polynomials is $x^0, x^1, x^2, \ldots$ Any polynomial can be expressed as a unique linear combination of these.
- $(x)_0$, $(x)_1$, $(x)_2$, ... is also a basis!
- $(x)_n$ has leading term $1x^n$. E.g., $(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x$.

Express $f(x) = 4x^3 - 5x + 6$ in the basis $(x)_0$, $(x)_1$, ...

- Start with $4(x)_3$ to get the leading term correct:
$$4(x)_3 = 4x^3 - 12x^2 + 8x$$
- Add $12(x)_2 = 12x(x-1)$ to get the x^2 term correct:
$$4(x)_3 + 12(x)_2 = 4x^3 - 12x^2 + 8x + 12x(x-1) = 4x^3 - 4x$$
- Subtract $(x)_1 = x$ to get the x^1 term correct:
$$4(x)_3 + 12(x)_2 - (x)_1 = 4x^3 - 5x$$
- Add $6(x)_0 = 6$ to get the x^0 term correct:
$$4(x)_3 + 12(x)_2 - (x)_1 + 6(x)_0 = 4x^3 - 5x + 6$$
- So $f(x) = 4(x)_3 + 12(x)_2 - (x)_1 + 6(x)_0$

Linear algebra interpretation

Coefficient vectors of $f(x)$ in each basis:

$f(x)$	**Basis**	**Coefficient vector**
$4x^3 - 5x + 6$	$x^0, \ldots, x^3$	$[6, -5, 0, 4]$
$4(x)_3 + 12(x)_2 - (x)_1 + 6(x)_0$	$(x)_0, \ldots, (x)_3$	$[6, -1, 12, 4]$

Lin. alg. interp. of $x^n = \sum_{k=0}^{n} S(n,k)\,(x)_k$ and $(x)_n = \sum_{k=0}^{n} s(n,k)x^k$

- Form matrices $[S(n,k)]$ and $[s(n,k)]$ for $0 \leqslant n,k \leqslant 3$:

$$\mathbf{S} = [S(n,k)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \qquad \mathbf{s} = [s(n,k)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

-

$f(x)$	**Basis**	**Coefficient vector**
$4x^3 - 5x + 6$	$x^0, \ldots, x^3$	$[6, -5, 0, 4]$
$4(x)_3 + 12(x)_2 - (x)_1 + 6(x)_0$	$(x)_0, \ldots, (x)_3$	$[6, -1, 12, 4]$

- $\mathbf{S}$ and $\mathbf{s}$ are the transition matrices between the two bases:
  $[6,-5,0,4]\mathbf{S} = [6,-1,12,4]$ and $[6,-1,12,4]\mathbf{s} = [6,-5,0,4]$

- The matrices are inverses: $\mathbf{Ss} = \mathbf{sS} =$ identity matrix.

- For polynomials of degree $\leqslant N$, form $(N+1) \times (N+1)$ matrices where the indices are $0 \leqslant k,n \leqslant N$.

Proof of $\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$ using weights

Optional material to read after we cover Chapter 8

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

- In addition to how we already proved this formula, there is another method based on material coming up in Chapter 8. The following is optional material that may be read after we cover Chapter 8.
- Define the *weight* of a permutation as the number of cycles it has. E.g., $\sigma = (1,3,5,4)(2)(6)$ has weight $w(\sigma) = 3$.
- Consider summing $x^{w(\sigma)}$ over all permutations σ of $[n]$.
  - There are $c(n,k)$ permutations of weight k, which will combine to give a term $c(n,k)x^k$. Thus, the sum is $\sum_{k=0}^{n} c(n,k)x^k$.
  - The following construction will show it also equals the right side.

Proof using weights

Optional material to read after we cover Chapter 8

- Let $n \geqslant 1$ and set

$$\mathcal{A} = \left\{ [i_1, \ldots, i_n] \ : \ 1 \leqslant i_j \leqslant j \qquad \text{for } j = 1, \ldots, n \right\}$$

- In elements of $\mathcal{A}$:
  - 1st number is 1
  - 2nd number is 1 or 2
  - 3rd number is 1, 2, or 3
  - Etc.
  - So $|\mathcal{A}| = n!$.
- **Example:** $[1, 2, 1, 3, 3, 6] \in \mathcal{A}$, but $[1, 3, 1, 2, 3, 6] \notin \mathcal{A}$.
- We'll give a bijection between $\mathcal{A}$ and permutations of $[n]$. It works similarly to the recursion for $c(n, k)$ from earlier in these slides, so review that if you need to.

Proof using weights

Optional material to read after we cover Chapter 8

- Given $[i_1, \ldots, i_n] \in \mathcal{A}$, construct a permutation as follows:
  - Start with an empty permutation. (weight 0)
  - Loop over $j = 1, \ldots, n$:
    If $i_j = j$, insert a new cycle (j). (increases weight by 1)
    Otherwise, insert j after i_j in i_j's cycle. (weight unchanged)

Example: input $[1, 2, 1, 3, 3, 6] \in \mathcal{A}$

- Start with empty permutation
- $i_1 = 1$ isn't in the permutation. Insert new cycle (1): (1)
- $i_2 = 2$ isn't in the permutation. Insert new cycle (2): $(1)(2)$
- $i_3 = 1$ is in the permutation. Insert 3 after 1: $(1, 3)(2)$
- $i_4 = 3$ is in the permutation. Insert 4 after 3: $(1, 3, 4)(2)$
- $i_5 = 3$ is in the permutation. Insert 5 after 3: $(1, 3, 5, 4)(2)$
- $i_6 = 6$ isn't in the permutation. Insert new cycle (6): $(1, 3, 5, 4)(2)(6)$
- This permutation has 3 cycles, so its weight is 3.

Proof using weights

Optional material to read after we cover Chapter 8

- Given $[i_1, \ldots, i_n] \in \mathcal{A}$, construct a permutation as follows:
  - Start with an empty permutation. (weight 0)
  - Loop over $j = 1, \ldots, n$:
    If $i_j = j$, insert a new cycle (j). (increases weight by 1)
    Otherwise, insert j after i_j in i_j's cycle. (weight unchanged)
- At step j,
  - 1 choice adds weight 1;
  - $j-1$ choices add weight 0,

  so step j contributes a factor $1x^1 + (j-1)x^0 = x + j - 1$.
- The total weight over $j = 1, \ldots, n$ is $\prod_{j=1}^{n}(x+j-1)$.
- This construction gives every permutation exactly once, weighted by its number of cycles, so the total weight is also $\sum_{k=0}^{n} c(n,k)x^k$.