Chapter 8.1.1-8.1.2. Generating Functions

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Ordinary Generating Functions (OGF)

- Let a_n $(n = 0, 1, \ldots)$ be a sequence.
- The *ordinary generating function* (OGF) of this sequence is

$$G(x) = \sum_{n=0}^{\infty} a_n \, x^n \ .$$

  *N*ote: If the range of n is different, sum over the range of n instead.

- This is the *Maclaurin series* of $G(x)$, which is the Taylor series of $G(x)$ centered at $x = 0$.
- Chapter 8.2 has another kind of generating function called an *exponential generating function*. But "generating function" without specifying which type usually refers to the "ordinary generating function" defined above.

Example: 3^n

- Let $a_n = 3^n$ for $n \geqslant 0$: $\qquad 1, 3, 9, 27, 81, 243, \ldots$

$$G(x) = \sum_{n=0}^{\infty} 3^n\, x^n = \frac{1}{1-3x}$$

- This is a geometric series:

First term $(\boldsymbol{n=0})$	$3^0 x^0 = 1$
Ratio	$\frac{\text{Term } n+1}{\text{Term } n} = \frac{3^{n+1}x^{n+1}}{3^n x^n} = 3x$
Sum	$\frac{\text{First term}}{1-\text{ratio}} = \frac{1}{1-3x}$

- This series converges for $|3x| < 1$; that is, for $|x| < 1/3$.

Example: n doesn't have to start at 0

- Let $a_n = 3$ for $n \geqslant 1$.

$$G(x) = \sum_{n=1}^{\infty} 3\,x^n = \frac{3x}{1-x}$$

- This is a geometric series:

First term $(\boldsymbol{n=1})$	$3x$
Ratio	$\frac{\text{Term } n+1}{\text{Term } n} = \frac{3x^{n+1}}{3x^n} = x$
Sum	$\frac{\text{First term}}{1-\text{ratio}} = \frac{3x}{1-x}$

- This series converges for $|x| < 1$.

- $b_n = 10$ for $n \geqslant -2$ has generating function

$$B(x) = \sum_{n=-2}^{\infty} 10\,x^n = \frac{10/x^2}{1-x} = \frac{10}{x^2(1-x)},$$

which converges for $0 < |x| < 1$.

Example: $1/n!$

- Let $a_n = \frac{1}{n!}$ for $n \geqslant 0$.

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \text{(using Taylor series)}$$

- This series converges for all x.
- We're focusing on nonnegative integer sequences. However, generating functions are defined for any sequence. The series above arises in Exponential Generating Functions (Ch. 8.2).
- In Probability (Math 180 series), generating functions are used for sequences a_n of real numbers in the range $0 \leqslant a_n \leqslant 1$.

Example: $\binom{10}{n}$

- Let $a_n = \binom{10}{n}$ for $n \geqslant 0$.
- Since $\binom{10}{n} = 0$ for $n > 10$, we can restrict the sum to $n = 0, \ldots, 10$.
- By the Binomial Theorem,

$$G(x) = \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} \binom{10}{n} x^n = \sum_{n=0}^{10} \binom{10}{n} x^n = (x+1)^{10}$$

- This is a polynomial. It converges for all x.

- Let $b_n = \binom{10}{n}$ for $n \geqslant 2$.

$$B(x) = \sum_{n=2}^{\infty} \binom{10}{n} x^n = \left(\sum_{n=0}^{\infty} \binom{10}{n} x^n \right) - \left(\sum_{n=0}^{1} \binom{10}{n} x^n \right)$$
$$= (x+1)^{10} - (1 + 10x)$$

Example: $n!$

- Let $a_n = n!$ for $n \geqslant 0$. This is the # permutations of n elements.

$$G(x) = \sum_{n=0}^{\infty} n!\, x^n$$

- This diverges at all $x \neq 0$.

Divergence is not necessarily a problem

- We are using Taylor series to encode infinite sequences a_n.
- We do various operations ($G(x) + H(x)$, $G(x) \cdot H(x)$, $G'(x)$, ...) on Taylor series to obtain new Taylor series. Then we determine the sequences they represent.
- These operations usually don't involve plugging in values of x, so the radius of convergence doesn't matter.
- Convergence only matters if we have to plug in a value of x. There's an example at the end of these slides.

Finding a coefficient

- Find the coefficient of x^3 in the Maclaurin series of this function:

$$f(x) = \frac{1+x}{1-3x} + (x+10)^8 = \sum_{n=0}^{\infty} a_n\, x^n$$

- Compute just enough terms:

$$\frac{1+x}{1-3x} = (1+x)\cdot(1-3x)^{-1} = (1+x)(1+3x+9x^2+27x^3+\cdots)$$

$$(x+10)^8 = \cdots + \binom{8}{3} x^3 \cdot 10^5 + \cdots$$

- In $f(x)$, the term x^3 arises from

$$1\cdot 27x^3 + x\cdot 9x^2 + \binom{8}{3} x^3\cdot 10^5 = 27x^3 + 9x^3 + (56x^3)(100000)$$

$$= 5600036x^3\ .$$

- The coefficient is $\boxed{\mathbf{5600036}}$.

Using Calculus to find generating function of $a_n = n$

- Let $a_n = n$ for $n \geqslant 0$.

$$G(x) = \sum_{n=0}^{\infty} n\,x^n = ?$$

- Use Calculus:

$$\begin{array}{lrl} & \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \text{Differentiate } (d/dx): & \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} n\,x^{n-1} \\ \text{Times } x: & \frac{x}{(1-x)^2} &= \sum_{n=0}^{\infty} n\,x^n \end{array}$$

So $G(x) = \frac{x}{(1-x)^2}$.

Using Calculus to find generating function of $a_n = n$

- Let $a_n = n$ for $n \geqslant 0$.

$$G(x) = \sum_{n=0}^{\infty} n\,x^n = ?$$

- Note that $\frac{d}{dx}x^n = n\,x^{n-1}$ for all values of n, including $n = 0$.

  Sometimes the constant term $x^0 = 1$ is separated out for derivatives. Here, that's valid but adds more work:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

  Differentiate (d/dx): $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^{n-1}$

  Times x: $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^n$

  $G(x)$ should include term $n = 0$, but this sum starts at $n = 1$.
  So, add the $n = 0^{\text{th}}$ term $n\,x^n$ as $0\,x^0$ on the right and 0 on the left:

$$0 + \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n\,x^n$$

- So again, $G(x) = x/(1-x)^2$.

Using Calculus to find generating function of $a_n = n$

- Let $a_n = n$ for $n \geqslant 0$.

$$G(x) = \sum_{n=0}^{\infty} n\,x^n = \frac{x}{(1-x)^2}$$

- By the Ratio Test, this converges for $|x| < 1$.
- Example:

$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \cdots = \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}$$

Solving a recursion

- Consider the recursion

$$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= 2a_n + 1 \quad \text{for } n \geqslant 0 \end{aligned}$$

- The first few terms are

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 2a_0 + 1 = 2(1) + 1 = 3 \\ a_2 &= 2a_1 + 1 = 2(3) + 1 = 7 \\ a_3 &= 2a_2 + 1 = 2(7) + 1 = 15 \end{aligned}$$

- Any conjectures on the formula?
- We will study this using induction and using generating functions.

Solving a recursion

- Consider the recursion

$$a_0 = 1 \qquad\qquad a_{n+1} = 2a_n + 1 \quad \text{for } n \geqslant 0$$

- The first few terms are

$$a_0 = 1 \qquad a_1 = 3 \qquad a_2 = 7 \qquad a_3 = 15$$

- Conjecture: $a_n = 2^{n+1} - 1$ for $n \geqslant 0$.
- We will prove this using induction.

Solving a recursion

Theorem

The recursion

$$a_0 = 1\,, \qquad a_{n+1} = 2a_n + 1 \quad \textit{for } n \geqslant 0$$

has solution

$$a_n = 2^{n+1} - 1 \quad \textit{for } n \geqslant 0$$

Base case:

- The base case is $n = 0$.
- $a_0 = 1$ is given.
- The formula gives $2^{0+1} - 1 = 2^1 - 1 = 1$. $\checkmark$
- They agree, so the base case holds.

Solving a recursion

Theorem

The recursion $a_0 = 1\,, \qquad a_{n+1} = 2a_n + 1$ *for* $n \geqslant 0$
has solution $(*)\ \ a_n = 2^{n+1} - 1$ *for* $n \geqslant 0.$

Induction step:

- Assume $(*)$ holds at $n = k$: $a_k = 2^{k+1} - 1$.
  We will prove it also holds for $n = k + 1$.
- Apply the recursion:

$$\begin{aligned} a_{k+1} &= 2a_k + 1 \\ &= 2(2^{k+1} - 1) + 1 \quad \text{by the induction hypothesis} \\ &= 2^{k+2} - 2 + 1 = 2^{k+2} - 1 = 2^{(k+1)+1} - 1 \end{aligned}$$

- Thus, $(*)$ holds for $n = k + 1$ as well.
- Thus, $(*)$ holds for all integers $n \geqslant 0$.

Solving a recursion using generating functions

$a_0 = 1$, $a_{n+1} = 2a_n + 1$ for $n \geqslant 0$

- That method required us to conjecture the solution. Now we'll use generating functions, which will solve it without guessing.
- Define $A(x) = \sum_{n=0}^{\infty} a_n\, x^n$ and substitute the recursion into it.
- Separate a_0 vs. a_n for $n \geqslant 1$, since they're defined differently:

$$A(x) = a_0\, x^0 + \sum_{n=1}^{\infty} a_n\, x^n$$

- Rewrite $a_{n+1} = 2a_n + 1$ for $n \geqslant 0$
  as $a_n = 2a_{n-1} + 1$ for $n \geqslant 1$, and plug in:

$$= 1 \cdot 1 + \sum_{n=1}^{\infty} (2\, a_{n-1} + 1)\, x^n$$

$$= 1 + 2 \sum_{n=1}^{\infty} a_{n-1}\, x^n + \sum_{n=1}^{\infty} x^n$$

Solving a recursion using generating functions

$a_0 = 1$, $a_{n+1} = 2a_n + 1$ for $n \geqslant 0$

$$A(x) = 1 + 2\sum_{n=1}^{\infty} a_{n-1}\,x^n + \sum_{n=1}^{\infty} x^n$$

Evaluate this in terms of $A(x)$ and known Taylor series.

- $1 + \sum_{n=1}^{\infty} x^n = 1 + \frac{x}{1-x} = \frac{1}{1-x}$
- $\displaystyle\sum_{n=1}^{\infty} a_{n-1}\,x^n = a_0\,x^1 + a_1\,x^2 + a_2\,x^3 + \cdots = x\,A(x)$
- Or, substitute $m = n - 1$, so $n = m + 1$:
  - Lower limit $n = 1$ becomes $m = 1 - 1 = 0$.
  - Upper limit $n = \infty$ becomes $m = \infty - 1 = \infty$.
  - Term $a_{n-1}\,x^n$ becomes $a_m\,x^{m+1}$.

$$\sum_{n=1}^{\infty} a_{n-1}\,x^n = \sum_{m=0}^{\infty} a_m\,x^{m+1} = x\sum_{m=0}^{\infty} a_m\,x^m = x\,A(x)$$

Solving a recursion using generating functions

$a_0 = 1$, $a_{n+1} = 2a_n + 1$ for $n \geqslant 0$

$$A(x) = 1 + 2\sum_{n=1}^{\infty} a_{n-1}\,x^n + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} + 2\,x\,A(x)$$

- Solve for $A(x)$:

$$A(x)\cdot(1-2\,x) = \frac{1}{1-x} \qquad\qquad A(x) = \frac{1}{(1-x)(1-2\,x)}$$

Solving a recursion using generating functions

$a_0 = 1$, $a_{n+1} = 2a_n + 1$ for $n \geqslant 0$

$$A(x) = \frac{1}{(1-x)(1-2x)}$$

- Use Calculus to find the Taylor series of $A(x)$.
- This is a rational function. We will use partial fractions to show that

$$A(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

- Expand these with the geometric series:

$$A(x) = 2\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(2(2^n)-1)x^n = \sum_{n=0}^{\infty}(2^{n+1}-1)x^n$$

- So $a_n = 2^{n+1} - 1$ for $n \geqslant 0$.

Comments

- We did not need to conjecture the formula; this method found the formula for us.
- We performed operations on Taylor series representing sequences, in order to get other Taylor series representing other sequences. We did not need to plug in values for x, so convergence at certain x's is not an issue.

Partial fractions

- We seek to rewrite
$$\frac{1}{(1-x)(1-2x)} = \frac{B}{1-x} + \frac{C}{1-2x}$$
with B, C constants. This should hold for all values of x.

- Clear denominators by multiplying by $(1-x)(1-2x)$, and collect:
$$1 = B(1-2x) + C(1-x) = (B+C) + (-2B-C)x$$

- Since this is true for *all* x, we need $B+C=1$ and $-2B-C=0$.

- Solve: $C=-2B$ so $1=B+C=B+(-2B)=-B$.
Then $B=-1$ and $C=-2B=-2(-1)=2$, giving
$$\frac{1}{(1-x)(1-2x)} = -\frac{1}{1-x} + \frac{2}{1-2x}$$

- Double-check that (next slide) and expand using geometric series to get a_n (already shown).

Partial fractions

Double-check the partial fraction expansion:

$$
\begin{aligned}
-\frac{1}{1-x}+\frac{2}{1-2x} &= \frac{-(1-2x)+2(1-x)}{(1-x)(1-2x)} \\
&= \frac{-1+2x+2-2x}{(1-x)(1-2x)} \\
&= \frac{1}{(1-x)(1-2x)} \qquad \checkmark
\end{aligned}
$$

Multiplying generating functions

- What's the coefficient of x^2 in this product?

$$(1 + 2x + 3x^2)(4 + 5x + 6x^2)$$

- Work out all the ways x^2 arises in the product:

$$\begin{aligned} & 1 \cdot 6x^2 + 2x \cdot 5x + 3x^2 \cdot 4 \\ = {} & 6x^2 + 10x^2 + 12x^2 \\ = {} & 28x^2 \end{aligned}$$

so the coefficient is 28.

Multiplication rule

- Let $$A(x) = \sum_{n=0}^{\infty} a_n\, x^n \qquad\qquad B(x) = \sum_{n=0}^{\infty} b_n\, x^n$$
  and define $$C(x) = A(x)B(x) = \sum_{n=0}^{\infty} c_n\, x^n$$
- Then for all $n \geqslant 0$,
  $$c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$$
- This is the sum of contributions to the coefficient of x^n from
  $$(a_k\, x^k \text{ in } A(x)) \cdot (b_{n-k}\, x^{n-k} \text{ in } B(x)) = a_k\, b_{n-k}\, x^n$$
- For any specific n, computing c_n only requires a finite # of arithmetic operations, even though the series may be infinite. So c_n is well-defined regardless of whether the series converge.

Coefficient of x^{13} in $(1+x)^{10}\cdot(1+x)^5=(1+x)^{15}$

$$\begin{array}{ccccc}
(1+x)^{10} & \cdot & (1+x)^5 & = & (1+x)^{15} \\[2ex]
\displaystyle\sum_{i=0}^{10}\binom{10}{i}x^i & \cdot & \displaystyle\sum_{j=0}^{5}\binom{5}{j}x^j & = & \displaystyle\sum_{k=0}^{15}\binom{15}{k}x^k
\end{array}$$

- **Left side:** x^{13} arises from $x^{10}\cdot x^3$, $x^9\cdot x^4$, $x^8\cdot x^5$, with coefficients:

$$\begin{array}{cccccccc}
 & x^{10}\cdot x^3 & & x^9\cdot x^4 & & x^8\cdot x^5 & & \\ \hline
 & \binom{10}{10}\binom{5}{3} & + & \binom{10}{9}\binom{5}{4} & + & \binom{10}{8}\binom{5}{5} & & \\
= & 1\cdot 10 & + & 10\cdot 5 & + & 45\cdot 1 & & \\
= & 10 & + & 50 & + & 45 & = & \boxed{\mathbf{105}}
\end{array}$$

- **Right side:** $\binom{15}{13}=\frac{15\cdot 14}{2}=\boxed{\mathbf{105}}$ $\checkmark$

Coefficient of x^k in $(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$

First method: Multiplying as polynomials

$$\begin{aligned}(1+x)^m \quad\cdot\quad (1+x)^n \quad &= \quad (1+x)^{m+n} \\ \sum_{i=0}^{m} \tbinom{m}{i} x^i \cdot \sum_{j=0}^{n} \tbinom{n}{j} x^j \quad &= \quad \sum_{k=0}^{m+n} \tbinom{m+n}{k} x^k\end{aligned}$$

Coefficient of x^k on the left:

- Need $\underbrace{i+j=k}_{j=k-i}$ with $0 \leqslant i \leqslant m$ and $\underbrace{0 \leqslant j \leqslant n.}_{\substack{0\leqslant k-i\leqslant n \\ \\ -k\leqslant -i\leqslant n-k, \text{ so } k-n\leqslant i\leqslant k}}$
- **Limits:** $i \geqslant 0$ and $i \geqslant k-n$ give $i \geqslant \max\{0, k-n\}$.
  $i \leqslant m$ and $i \leqslant k$ give $i \leqslant \min\{m, k\}$.
- Coefficient of x^k on the left vs. on the right:

$$\sum_{i=\max\{0,k-n\}}^{\min\{k,m\}} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$$

Coefficient of x^k in $\quad (1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$

First method: Multiplying as polynomials

$$\sum_{i=\max\{0,k-n\}}^{\min\{k,m\}} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$$

For the coefficient of x^{13} in $(1+x)^{10}(1+x)^5 = (1+x)^{15}$:

- $m=10$, $n=5$, $k=13$
- Lower limit $\max\{0, 13-5\} = \max\{0, 8\} = 8$
- Upper limit $\min\{13, 10\} = 10$

$$\sum_{i=8}^{10} \binom{10}{i}\binom{5}{13-i} = \binom{15}{13}$$

- This matches the equation from a few slides ago,

$$\binom{10}{10}\binom{5}{3} + \binom{10}{9}\binom{5}{4} + \binom{10}{8}\binom{5}{5} = \binom{15}{13}$$

Coefficient of x^k in $(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$

Second method: Multiplying as infinite series

$$\begin{array}{ccccc} (1+x)^m & \cdot & (1+x)^n & = & (1+x)^{m+n} \\ \displaystyle\sum_{i=0}^{\infty} \tbinom{m}{i} x^i & \cdot & \displaystyle\sum_{j=0}^{\infty} \tbinom{n}{j} x^j & = & \displaystyle\sum_{k=0}^{\infty} \tbinom{m+n}{k} x^k \end{array}$$

- We changed the limits to 0 to ∞.
- For integers $m, n, k \geqslant 0$, the multiplication rule gives the *Chu-Vandermonde Identity* (Chu 1303; Vandermonde 1772):

$$\boxed{\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}}$$

- Reconcile this with multiplying as polynomials:
  - Any terms with $i > m$ have $\binom{m}{i} = 0$.
  - Any terms with $k - i > n$ have $\binom{n}{k-i} = 0$.
  - All additional terms in this version $= 0$, so the sums are equal.
- This method is easier.

Chu-Vandermonde Identity works for complex numbers!

$$(1+x)^{\alpha} \cdot (1+x)^{\beta} = (1+x)^{\alpha+\beta}$$

$$\sum_{i=0}^{\infty} \binom{\alpha}{i} x^i \cdot \sum_{j=0}^{\infty} \binom{\beta}{j} x^j = \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} x^k$$

- Let α and β be any complex numbers, and $k \geqslant 0$ be an integer.
- Apply the multiplication rule to get:

$$\boxed{\sum_{i=0}^{k} \binom{\alpha}{i}\binom{\beta}{k-i} = \binom{\alpha+\beta}{k}}$$

Example for $k = 2$

Evaluate both sides as polynomials in α, β, and then use algebra to show they're the same.

Left: $\binom{\alpha}{0}\binom{\beta}{2} + \binom{\alpha}{1}\binom{\beta}{1} + \binom{\alpha}{2}\binom{\beta}{0} = 1 \cdot \frac{\beta(\beta-1)}{2} + \alpha\beta + \frac{\alpha(\alpha-1)}{2} \cdot 1$

Right: $\binom{\alpha+\beta}{2} = \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}$

Counting in two ways / bijective proof

Third method to prove the Chu-Vandermonde identity (for integers $m, n, k \geqslant 0$)

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

How many k element subsets are there of $[m+n]$?

- It's $\binom{m+n}{k}$.
- Or, pick $i = 0, \ldots, k$ and then

  pick i elements from $\{1, \ldots, m\}$ in $\binom{m}{i}$ ways

  and $k-i$ elements from $\{m+1, \ldots, m+n\}$ in $\binom{n}{k-i}$ ways.
- Total: $\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$
- E.g., for $m = 10$, $n = 5$, and $i = 3$:

$$\begin{array}{rrl}
\text{picking} & \{1,3,10\} \subseteq \{1,\ldots,10\} & \text{of size } 3 \\
\text{and} & \{11,13\} \subseteq \{11,\ldots,15\} & \text{of size } 2 \\
\text{gives} & \{1,3,10,11,13\} \subseteq \{1,\ldots,15\} & \text{of size } 3+2=5.
\end{array}$$

Formalizing the bijection

- We'll formalize the bijection just used, and turn it into a generating function proof.
- Let
$$\begin{aligned} \mathcal{A} &= \mathcal{P}(\{1,\ldots,m\}) \\ \mathcal{B} &= \mathcal{P}(\{m+1,\ldots,m+n\}) \\ \mathcal{C} &= \mathcal{P}(\{1,\ldots,m+n\}) \end{aligned}$$
- $\mathcal{A} \times \mathcal{B}$ is isomorphic to $\mathcal{C}$: There is a bijection respecting set size.
  - Given $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, then $\gamma = \alpha \cup \beta \in \mathcal{C}$.
  - Given $\gamma \in \mathcal{C}$, form $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ by
$$\begin{aligned} \alpha &= \gamma \cap \{1,\ldots,m\} \\ \beta &= \gamma \cap \{m+1,\ldots,m+n\}. \end{aligned}$$
  - Sizes satisfy $|\alpha| + |\beta| = |\gamma|$.
- E.g., for $m = 10$ and $n = 5$,
$$(\alpha, \beta) = (\{1,3,10\}, \{11,13\}) \quad \leftrightarrow \quad \gamma = \{1,3,10,11,13\}$$
sizes $3 + 2 = 5$.

Combining the bijection with generating functions

$\mathcal{A} = \mathcal{P}(\{1,\ldots,m\}), \quad \mathcal{B} = \mathcal{P}(\{m+1,\ldots,m+n\}), \quad \mathcal{C} = \mathcal{P}(\{1,\ldots,m+n\})$.

- Let **Gen. fn.**

$a_i =$	# elements in $\mathcal{A}$ of size i	$= \binom{m}{i}$	$A(x) = \sum_{i \geqslant 0} a_i\, x^i$
$b_j =$	$\ldots$ $\mathcal{B}$ $\ldots$ j	$= \binom{n}{j}$	$B(x) = \sum_{j \geqslant 0} b_j\, x^j$
$c_k =$	$\ldots$ $\mathcal{C}$ $\ldots$ k	$= \binom{m+n}{k}$	$C(x) = \sum_{k \geqslant 0} c_k\, x^k$

- The number of elements in $\mathcal{C}$ of size k is

$$c_k = \sum_{i=0}^{k} (\text{\# in } \mathcal{A} \text{ of size } i)(\text{\# in } \mathcal{B} \text{ of size } k-i) = \sum_{i=0}^{k} a_i\, b_{k-i}$$

- These match the coefficient of x^k on both sides of $C(x) = A(x)B(x)$.

Generating function for # of subsets of a set, by size

- Let $\mathcal{A} = \mathcal{P}([3]) = \{\, \alpha \;:\; \alpha \subseteq [3] \,\}$.
- There are $a_i = \binom{3}{i}$ elements in $\mathcal{A}$ of size i.
- Three equivalent formulas for the generating function of this:

$$A(x) = \sum_{i=0}^{3} a_i\, x^i = 1 + 3x + 3x^2 + x^3 = (1+x)^3$$

$$\begin{array}{ll}
A(x) = \displaystyle\sum_{\alpha \in \mathcal{A}} x^{|\alpha|} = x^0 & \underline{\alpha} \\
 & \emptyset, \\
\qquad + x^1 + x^1 + x^1 & \{1\},\{2\},\{3\}, \\
\qquad + x^2 + x^2 + x^2 & \{1,2\},\{1,3\},\{2,3\}, \\
\qquad + x^3 & \{1,2,3\} \\
\qquad = 1 + 3x + 3x^2 + x^3 = (1+x)^3 &
\end{array}$$

$$A(x) = \sum_{\alpha \subseteq [3]} x^{|\alpha|} \quad \text{since } \alpha \in \mathcal{A} \text{ iff } \alpha \subseteq [3]$$

Generating function for # of subsets of a set, by size

- Let S be a set of size m, and $\mathcal{A} = \mathcal{P}(S)$ (the power set of S).
- Let a_i be the number of elements of $\mathcal{A}$ with size i:

$$\begin{aligned} a_i &= \left|\{\, \alpha \in \mathcal{A} \;:\; |\alpha| = i \,\}\right| \\ &= \left|\{\, \alpha \subseteq S \;:\; |\alpha| = i \,\}\right| \qquad \text{using } \mathcal{A} = \mathcal{P}(S) = \{\, \alpha \;:\; \alpha \subseteq S \,\}. \end{aligned}$$

- The generating function for the # of sets in $\mathcal{A}$ having each size is

$$G(x) = \sum_{i \geqslant 0} a_i\, x^i = \sum_{i=0}^{m} \binom{m}{i} x^i = \boxed{(\mathbf{1} + \boldsymbol{x})^{\boldsymbol{m}}}$$

It can also be written

$$= \sum_{i \geqslant 0} (\#\ \alpha \in \mathcal{A} \text{ with } |\alpha| = i)\ x^i = \boxed{\sum_{\alpha \in \mathcal{A}} \boldsymbol{x}^{|\alpha|}}$$

$$= \sum_{i \geqslant 0} (\#\ \alpha \subseteq S \text{ with } |\alpha| = i)\ x^i = \boxed{\sum_{\alpha \subseteq S} \boldsymbol{x}^{|\alpha|}}$$

Structures

We generalize this method:

- Let $\mathcal{A}$ be a set, whose elements we call *structures*.
- Define a *weight* function $w : \mathcal{A} \to \mathbb{N}$,
  where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers.
- The generating function for the # of structures in $\mathcal{A}$ by weight is

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{w(\alpha)} = \sum_{i=0}^{\infty} a_i x^i$$

where a_i is the # of structures in $\mathcal{A}$ with weight i:

$$a_i = \Big| \{ \alpha \in \mathcal{A} \ : \ w(\alpha) = i \} \Big|$$

Example

$\mathcal{A}$ is a set of sets, and the weight of $\alpha \in \mathcal{A}$ is $w(\alpha) = |\alpha|$.

$$\begin{aligned} \mathcal{A} &= \{\{1\} \ , \ \{2\} \ , \ \{1,3\} \ , \ \{1,3,5\}\} \\ A(x) &= x^1 + x^1 + x^2 + x^3 = 2x + x^2 + x^3 \end{aligned}$$

Multiplication rule for generating functions of structures

Earlier example with subsets of $[m+n]$

$$\mathcal{A} = \mathcal{P}(\{1,\ldots,m\}) \qquad A(x) = (1+x)^m$$
$$\mathcal{B} = \mathcal{P}(\{m+1,\ldots,m+n\}) \qquad B(x) = (1+x)^n$$
$$\mathcal{C} = \mathcal{P}(\{1,\ldots,m+n\}) \qquad C(x) = (1+x)^{m+n} = A(x)B(x)$$

Isomorphism $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$

$(\alpha, \beta) \mapsto \gamma = \alpha \cup \beta$ has weights $|\alpha| + |\beta| = |\gamma|$

Generalization

- Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be sets of structures,
  with weight functions $w_\mathcal{A}, w_\mathcal{B}, w_\mathcal{C}$
  and generating functions $A(x)$, $B(x)$, $C(x)$.
- Suppose there's an isomorphism $\mathcal{A} \times \mathcal{B} \cong \mathcal{C}$,
  and when $(\alpha, \beta) \leftrightarrow \gamma$ under this isomorphism,
  $$w_\mathcal{A}(\alpha) + w_\mathcal{B}(\beta) = w_\mathcal{C}(\gamma).$$
  Then $A(x)B(x) = C(x)$.

Proof of multiplication rule

To simplify notation, assume all weight functions are the same, w

$$C(x) = \sum_{\gamma \in \mathcal{C}} x^{w(\gamma)}$$ Definition of generating function

$$= \sum_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} x^{w(\alpha)+w(\beta)}$$ Using isomorphism

$$= \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} x^{w(\alpha)} \cdot x^{w(\beta)}$$ Unravelling Cartesian product

$$= \left(\sum_{\alpha \in \mathcal{A}} x^{w(\alpha)} \right) \left(\sum_{\beta \in \mathcal{B}} x^{w(\beta)} \right)$$ Separate variables

$$= A(x)\,B(x)$$ Definition of generating function

Addition rule for generating functions of structures

- Let $\mathcal{A}, \mathcal{B}$ be disjoint sets of structures: $\mathcal{A} \cap \mathcal{B} = \emptyset$
- Let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$.
- Then their generating functions $A(x), B(x), C(x)$ satisfy
$$C(x) = A(x) + B(x).$$

Example

Consider these collections of sets, with weight $w(S) = |S|$.

$$\mathcal{A} = \Big\{\{1\},\{2\},\{1,3\},\{1,3,5\}\Big\} \qquad A(x) = 2x + x^2 + x^3$$

$$\mathcal{B} = \Big\{\{4\},\{5\},\{6\},\{4,5\}\Big\} \qquad B(x) = 3x + x^2$$

$$\mathcal{A} \cap \mathcal{B} = \emptyset$$

$$\mathcal{C} = \mathcal{A} \cup \mathcal{B} \qquad C(x) = 5x + 2x^2 + x^3$$

How many ways can we make change for n cents?

- Let c_n be the number of ways to make change for n¢ from 5 pennies and 2 nickels.

n		c_n
$0, \ldots, 4$	n pennies	1
5	5 pennies, or 1 nickel	2
$6, 7, 8, 9$	1 nickel and $n-5$ pennies	1
10	1 nickel and 5 pennies, or 2 nickels	2
$11, \ldots, 15$	2 nickels and $n-10$ pennies	1
otherwise		0

- Coins of the same type are considered indistinguishable for this count. For 1¢, we just use one penny, and count $c_1 = 1$; we don't count the 5 ways to choose which penny.

How many ways can we make change for n cents?

- We have 5 pennies and 2 nickels.
- Make a choice from each column, and combine them.

Column A	**Column B**
0 pennies	0 nickels
1 penny	1 nickel
2 pennies	2 nickels
3 pennies	
4 pennies	
5 pennies	

- The *weight* of a combination of coins is the total value in cents.
- The weight of r pennies and s nickels is $(r + 5s)$¢.

How many ways can we make change for n cents?

	Column A Pennies	**Column B** Nickels
# coins	$r = 0, 1, 2, 3, 4, 5$	$s = 0, 1, 2$
Weight (¢)	$1r = 0, 1, 2, 3, 4, 5$	$5s = 0, 5, 10$

- Let $a_n =$ # ways to make n ¢ using pennies (Column A).
  $a_n = 1$ for $0 \leqslant n \leqslant 5$ and $a_n = 0$ otherwise.
  $$A(x) = \sum_{n=0}^{5} a_n\, x^n = 1 + x + x^2 + x^3 + x^4 + x^5$$
- Let $b_n =$ # ways to make n ¢ using nickels (Column B).
  $b_n = 1$ for $n = 0, 5, 10$ and $b_n = 0$ otherwise.
  $$B(x) = \sum_{n=0}^{10} b_n\, x^n = 1 + x^5 + x^{10}$$

$$\begin{array}{rlr}
A(x)B(x) = & 1+x+\cdots+x^4+x^5 & \text{0 nickels} \\
 & +\quad x^5+x^6+\cdots+x^9+x^{10} & \text{1 nickel} \\
 & +\quad x^{10}+x^{11}+\cdots+x^{15} & \text{2 nickels} \\
\hline
C(x) = & 1+x+\cdots+x^4+2x^5+x^6+\cdots+x^9+2x^{10}+x^{11}+\cdots+x^{15} & \text{Total}
\end{array}$$

Divisibility notation, for integers a and b

- $a \mid b$ means that a divides into b; that is, $\frac{b}{a}$ is an integer.
- $a \nmid b$ means that a does not divide into b.
- $5 \mid 20$ but $5 \nmid 19$.

How many ways to make change with unlimited nickels?

- How many ways can we make change for j cents with an unlimited number of nickels allowed?
- One way if j is a multiple of 5, and no ways otherwise:

$$b_j = \begin{cases} 1 & \text{if } 5 \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

Gen. fn.: $B(x) = \sum_{j=0}^{\infty} b_j\, x^j$
$= 1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5}$

- The set of numbers of nickels permitted is $\mathcal{B} = \mathbb{N} = \{0, 1, 2, \ldots\}$.
- The weight of s nickels is the value in cents, $5s$.
- The generating function is also

$$B(x) = \sum_{s \in \mathcal{B}} x^{5s} = \sum_{s=0}^{\infty} x^{5s} = 1 + x^5 + x^{10} + \cdots = \frac{1}{1 - x^5}$$

How many ways can we make change from any combination of pennies, nickels, dimes, and quarters?

- Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ denote the sets of allowed numbers of pennies, nickels, dimes, and quarters. Set them all to $\mathbb{N}$.
- The *weight* is the value in cents.
  The generating functions for the weights for each kind of coin are

$$A(x) = \frac{1}{1-x} \quad B(x) = \frac{1}{1-x^5} \quad C(x) = \frac{1}{1-x^{10}} \quad D(x) = \frac{1}{1-x^{25}}$$

- Let $\mathcal{T} = \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D}$.
  The generating function for weights of combining coins is

$$T(x) = A(x)B(x)C(x)D(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})} = \sum_{n \geqslant 0} t_n\, x^n$$

  where t_n is the number of ways to make change for n¢ by choosing one option in each of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

How many ways can we make change?

- How many ways can we make change for $13¢$ from pennies, nickels, dimes, and quarters?
- It's the # solutions of $i + 5j + 10k + 25\ell = 13$ in integers $i, j, k, \ell \geqslant 0$, with i, j, k, ℓ the numbers of pennies, nickels, dimes, and quarters.
- Expand $T(x)$ and compute the coefficient of x^{13}:

$$T(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$
$$= (1 + x + \cdots + x^{13} + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + \cdots)(1 + \cdots)$$

- Only terms in each factor of degree $\leqslant 13$ contribute to x^{13}, so terms with higher exponents are ignored above. Then

$$\underbrace{x^{13}\cdot 1\cdot 1\cdot 1}_{\text{13 pennies}} + \underbrace{x^{8}\cdot x^{5}\cdot 1\cdot 1}_{\substack{\text{8 pennies \&} \\ \text{1 nickel}}} + \underbrace{x^{3}\cdot x^{10}\cdot 1\cdot 1}_{\substack{\text{3 pennies \&} \\ \text{2 nickels}}} + \underbrace{x^{3}\cdot 1\cdot x^{10}\cdot 1}_{\substack{\text{3 pennies \&} \\ \text{1 dime}}} = 4x^{13}$$

so $t_{13} = 4$.

Alternative formulation

- The *weight* of an integer partition is the sum of its parts.
  It's also the number of dots in its Ferrers diagram.

$$w\big((5,5,1,1)\big) = w\left(\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet \\ \bullet \end{array}\right) = 12$$

- Reformulate problem: use integer partitions instead of coin counts:

$$\mathcal{A} = \{\, \underbrace{(1,\ldots,1)}_{i \text{ ones, weight } i} \;:\; i \geqslant 0\} \qquad \mathcal{C} = \{\, \underbrace{(10,\ldots,10)}_{k \text{ tens, weight } 10k} \;:\; k \geqslant 0\}$$

$$\mathcal{B} = \{\, \underbrace{(5,\ldots,5)}_{j \text{ fives, weight } 5j} \;:\; j \geqslant 0\} \qquad \mathcal{D} = \{\, \underbrace{(25,\ldots,25)}_{\ell \text{ 25's, weight } 25\ell} \;:\; \ell \geqslant 0\}$$

$$\mathcal{T} = \{\, (\underbrace{25,\ldots,25}_{\ell}, \underbrace{10,\ldots,10}_{k}, \underbrace{5,\ldots,5}_{j}, \underbrace{1,\ldots,1}_{i}) \;:\; i,j,k,\ell \geqslant 0\}$$

$$\cong \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \qquad \text{weight } i + 5j + 10k + 25\ell$$

- E.g., $((1,1,1),(5),(),(25)) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \quad \leftrightarrow \quad (25,5,1,1,1) \in \mathcal{T}$.
- The bijection respects the weights.

Notation for products

- $\prod$ notation for products is like $\sum$ notation for sums:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \qquad \prod_{i=1}^{n} a_i = a_1\, a_2\, \cdots\, a_n \qquad n! = \prod_{i=1}^{n} i$$

- An empty sum is 0: $\displaystyle\sum_{i=1}^{0} a_i = 0$

  An empty product is 1: $\displaystyle\prod_{i=1}^{0} a_i = 1$

Number of partitions of an integer

- In place of coins of value i cents, we use parts of all sizes $i \geqslant 1$. Each part size i can be repeated any number of times.
- Partitions using just i are

$$(), \ (i), \ (i,i), \ (i,i,i), \ \ldots$$

(where $()$ is a partition of 0.)

- The number of partitions of $n \geqslant 0$ using parts of size i is 1 if $i \mid n$ and 0 otherwise. The generating function for this is

$$1 + x^i + x^{2i} + \cdots = \frac{1}{1 - x^i}$$

- The generating function for $p(n) =$ number of partitions of n is

$$\sum_{n=0}^{\infty} p(n)\,x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

How many integer partitions does 4 have?

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{i=1}^{\infty} \tfrac{1}{1-x^i} \\ &= (1+x+x^2+x^3+x^4+\cdots) && \text{from } \tfrac{1}{1-x} \\ &\quad \cdot (1+x^2+x^4+\cdots) && \text{from } \tfrac{1}{1-x^2} \\ &\quad \cdot (1+x^3+\cdots) && \text{from } \tfrac{1}{1-x^3} \\ &\quad \cdot (1+x^4+\cdots) && \text{from } \tfrac{1}{1-x^4} \\ &\quad \cdot (1+\cdots) && \text{from } \tfrac{1}{1-x^i} \text{ for } i \geqslant 5\end{aligned}$$

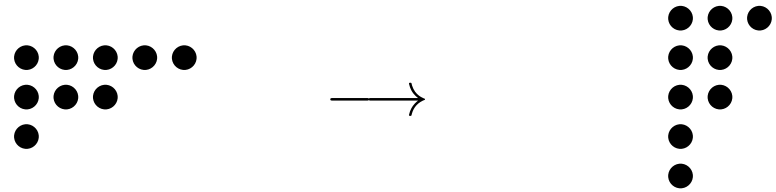
x^4 arises from

$$\underbrace{x^4\cdot 1\cdot 1\cdot 1}_{(1,1,1,1)} + \underbrace{x^2\cdot x^2\cdot 1\cdot 1}_{(1,1,2)} + \underbrace{x\cdot 1\cdot x^3\cdot 1}_{(1,3)} + \underbrace{1\cdot x^4\cdot 1\cdot 1}_{(2,2)} + \underbrace{1\cdot 1\cdot 1\cdot x^4}_{(4)} = 5x^4 \quad \Rightarrow \quad p(4) = 5$$

- It's a finite computation even though there's an infinite number of factors.
- To match the order of the factors, parts are shown in increasing order instead of decreasing order, e.g., $(1,3)$ instead of $(3,1)$.

Number of partitions with exactly k parts

- Recall that integer partition π has k parts iff π' has largest part k.



$$\pi = (5,3,1) \qquad\qquad \pi' = (3,2,2,1,1)$$

- Let $p_k(n) =$ # partitions of n with exactly k parts
  $=$ # partitions of n whose largest part is k (we'll use this).
- Form partitions with any # (including zero) of 1's, . . . , $(k-1)$'s;
  at least one k;
  no larger part sizes.
- For $1 \leqslant i \leqslant k-1$:
  The g.f. of zero or more i's is $1 + x^i + x^{2i} + \cdots = \frac{1}{1-x^i}$.
- The g.f. for one or more k's is $x^k + x^{2k} + x^{3k} + \cdots = \frac{x^k}{1-x^k}$.
- This gives

$$\sum_{n=0}^{\infty} p_k(n)\, x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdots \frac{1}{1-x^{k-1}} \cdot \frac{x^k}{1-x^k} = \frac{x^k}{(1-x)\cdots(1-x^k)}$$

Partitions with odd parts or distinct parts

- Let $p_{\text{odd}}(n) =$ # partitions of n into odd parts (meaning all part sizes are odd). E.g.,

$$\begin{array}{lll} n=5: & (5),(3,1,1),(1,1,1,1,1) & p_{\text{odd}}(5)=3 \\ n=6: & (5,1),(3,3),(3,1,1,1),(1,1,1,1,1,1) & p_{\text{odd}}(6)=4 \end{array}$$

- Let $p_d(n) =$ # partitions of n with all parts distinct. E.g.,

$$\begin{array}{lll} n=5: & (5),(4,1),(3,2) & p_d(5)=3 \\ n=6: & (6),(5,1),(4,2),(3,2,1) & p_d(6)=4 \end{array}$$

- **Theorem:** For all n, $\quad p_{\text{odd}}(n) = p_d(n)$.

- A bijection is known, but is rather complicated.
  We'll give a proof using generating functions.

Partitions with odd parts or distinct parts

- The generating functions for $p_{\text{odd}}(n)$ and $p_d(n)$ are

$$F(x) = \sum_{n=0}^{\infty} p_{\text{odd}}(n)\,x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

$$G(x) = \sum_{n=0}^{\infty} p_d(n)\,x^n = (1+x)(1+x^2)(1+x^3)\cdots$$

- We'll show $F(x) = G(x)$, which proves $p_{\text{odd}}(n) = p_d(n)$ for all n.
- Note that $1 - x^{2i} = (1-x^i)(1+x^i)$ gives $1 + x^i = \frac{1-x^{2i}}{1-x^i}$.
- In $G(x)$, for each i, replace $1 + x^i$ by $\frac{1-x^{2i}}{1-x^i}$, and cancel like terms:

$$G(x) = \prod_{i=1}^{\infty}(1+x^i) = \prod_{i=1}^{\infty}\frac{1-x^{2i}}{1-x^i} = \frac{\cancel{(1-x^2)}\cancel{(1-x^4)}(1-x^6)(1-x^8)\cdots}{(1-x)\cancel{(1-x^2)}(1-x^3)\cancel{(1-x^4)}\cdots}$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = F(x)$$

- Terms $\frac{1-x^n}{1-x^n}$ with n even cancel on top and bottom!
- Since $F(x){=}G(x)$, their Taylor series are equal, so $p_{\text{odd}}(n){=}p_d(n)$.

Generating functions for strict compositions

- Solutions (i_1, i_2, i_3) of $i_1 + i_2 + i_3 = 12$ with $i_1, i_2, i_3 \in \{2, 5, 10\}$:

$$(2,5,5), (5,2,5), (5,5,2)$$

- The # such solutions is the coefficient of x^{12} in

$$(x^2 + x^5 + x^{10})(x^2 + x^5 + x^{10})(x^2 + x^5 + x^{10}) = (x^2 + x^5 + x^{10})^3$$

- In the expansion of this product, x^{12} arises from

$$x^2 \cdot x^5 \cdot x^5 + x^5 \cdot x^2 \cdot x^5 + x^5 \cdot x^5 \cdot x^2 = 3x^{12}$$

  so there are 3 solutions.

Generating functions for strict compositions

- Let $\mathcal{A} \subseteq \mathbb{N}$ and $k, n \in \mathbb{N}$.
- The # solutions $(i_1, \ldots, i_k)$ of $i_1 + \cdots i_k = n$ with $i_1, \ldots, i_k \in \mathcal{A}$ is the coefficient of x^n in $\left(\sum_{j \in \mathcal{A}} x^j\right)^k$.
- For strict compositions of n into k parts:

$$\mathcal{A} = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

$$A(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

$$A(x)^k = \frac{x^k}{(1-x)^k} = x^k \sum_{m=k}^{\infty} \binom{m-1}{k-1} x^{m-k} = \sum_{m=k}^{\infty} \binom{m-1}{k-1} x^m$$

- The coefficient of x^n is at $m = n$, so $\binom{n-1}{k-1}$, as expected.

Total # strict compositions of n

Now count the total number of strict compositions of n.
E.g., there are 8 strict compositions of $n = 4$:

# Parts	Compositions	Count
1	(4)	1
2	$(3,1), (1,3), (2,2)$	3
3	$(2,1,1), (1,2,1), (1,1,2)$	3
4	$(1,1,1,1)$	1
	Total:	8

Total # strict compositions of n

- $A(x)^k = \frac{x^k}{(1-x)^k}$ is the g.f. for the # strict compositions of n into k parts.
- Sum it over k to get the total # strict compositions of n:

$$\sum_{k=0}^{\infty} A(x)^k = \frac{1}{1-A(x)} = \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x} = \frac{1}{1-2x} - \frac{x}{1-2x}$$

$$= \sum_{n=0}^{\infty} 2^n x^n - x \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} 2^n x^{n+1}$$

- To collect coefficients, express these in terms of x^n rather than mixing x^n and x^{n+1}.
  - In the rightmost summation, substitute $m = n+1$, so $n = m-1$:

$$\sum_{n=0}^{\infty} 2^n x^{n+1} = \sum_{m=1}^{\infty} 2^{m-1} x^m$$

  - Then rename m back to n.

Total # strict compositions of n

- Continuing:

$$\begin{aligned}\sum_{k=0}^{\infty} A(x)^k = \cdots &= \sum_{n=0}^{\infty} 2^n\, x^n - \sum_{n=1}^{\infty} 2^{n-1}\, x^n \\ &= 2^0\, x^0 + \sum_{n=1}^{\infty} (2^n - 2^{n-1}) x^n \\ &= 1\, x^0 + \sum_{n=1}^{\infty} 2^{n-1}\, x^n\end{aligned}$$

- Thus, for $n = 0$, there is 1 strict composition of 0, and for $n > 0$, there are 2^{n-1} strict compositions of n.

Summary of examples

Object	**Weight**
Sets	Set size
Integer partitions	Sum of part sizes or number of dots in Ferrers diagram
Integer compositions	Sum of part sizes
Coins	Value in cents

Averages

- What is the average size of a subset of $[n]$?
- For $n = 2$, the subsets are

$$\emptyset \qquad \{1\} \qquad \{2\} \qquad \{1,2\}$$

  The average size is $(0 + 1 + 1 + 2)/4 = 1$.
- We'll use two methods. The first is specific to this example of subsets of $[n]$. The second uses generating functions and can be applied to other structures.

Average size of a subset of $[n]$

First method: specific to subsets of $[n]$

- For $n = 0$, there is just $\emptyset$, with size 0, so the average size is 0.
- For $n > 0$, pair all subsets of $[n]$ using set complements: $\{S, S^c\}$.
- E.g., for $n = 2$:

S	S^c	sum of sizes
$\emptyset$	$\{1, 2\}$	$0 + 2 = 2$
$\{1\}$	$\{2\}$	$1 + 1 = 2$

- We get 2^{n-1} pairs $\{S, S^c\}$, each with $|S| + |S^c| = n$.
- There are 2^{n-1} pairs, so the sum of sizes of subsets is $n \cdot 2^{n-1}$.
- Divide by the number of subsets, 2^n, to get average $\frac{n\,2^{n-1}}{2^n} = \frac{n}{2}$.

Average size of a subset of $[n]$

Second method: more general, based on generating functions

- Let $\mathcal{A} \neq \emptyset$ be a finite set of structures,
  $a_k =$ # of structures in $\mathcal{A}$ with weight k, and
  $A(x) = \sum_{k=0}^{\infty} a_k\, x^k$. This is a polynomial since $\mathcal{A}$ is finite.
- Average weight $= \dfrac{\text{sum of all weights}}{\text{number of structures}} = \dfrac{\sum_{s \in \mathcal{A}} w(s)}{|\mathcal{A}|}$.
- The numerator is sum of a_0 0's, a_1 1's, a_2 2's, $\ldots$, so it equals

$$\sum_{s \in \mathcal{A}} w(s) = 0\,a_0 + 1\,a_1 + 2\,a_2 + \cdots = \sum_{k=0}^{\infty} k\,a_k$$

  Notice that $A'(x) = \sum_{k=0}^{\infty} k\,a_k\,x^{k-1}$, so numerator $= A'(1) = \sum_{k=0}^{\infty} k\,a_k$.
- The denominator is $|\mathcal{A}| = \sum_{k=0}^{\infty} a_k = A(1)$ since $A(1) = \sum_{k=0}^{\infty} a_k\,1^k$.
- Thus, the average is $A'(1)/A(1)$.
  No convergence issues since $A(x)$ is a polynomial.

Average size of a subset of $[n]$

Second method: more general, based on generating functions

- The generating function for # subsets of $[n]$ by weight is

$$G(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

- For $n > 0$:
  - $G'(x) = n(1+x)^{n-1}$ and $G'(1) = n \cdot 2^{n-1}$
  - $G(1) = (1+1)^n = 2^n$
  - Average $= G'(1)/G(1) = n \cdot 2^{n-1}/2^n = n/2$.

- For $n = 0$, the above is still valid, but we'll do it separately:
  - $G(x) = 1$ so $G'(x) = 0$
  - Average $= G'(1)/G(1) = 0/1 = 0$.

Generating functions in probability

- Keep flipping a fair coin until you get heads.
  Let U = # flips until the first heads.
  The probability in this scenario is called the *Geometric Distribution*.

Flips	U	Probability
H	1	$1/2$
TH	2	$1/4$
TTH	3	$1/8$
$\cdots$	$\cdots$	$\cdots$
$\mathrm{T}^{n-1}\mathrm{H}$	n	$1/2^n$ for $n \geqslant 1$

- **Generating function:**

$$G(x) = \sum_{n=0}^{\infty} P(U=n)\,x^n = \sum_{n=1}^{\infty} \frac{1}{2^n}\,x^n = \frac{x/2}{1-(x/2)} = \frac{x}{2-x}$$

  It converges in $|x/2| < 1$, which is $|x| < 2$.

- **Total probability:** $\sum_{n=0}^{\infty} P(U=n) = G(1) = \frac{1/2}{1-(1/2)} = 1.$

Generating functions in probability

- **Theoretical average of U:**
  $U{=}n$ a fraction $1/2^n$ of the time, giving
  $$(1/2)1+(1/4)2+(1/8)3+\cdots=\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{1/2}{(1-(1/2))^2}=2.$$
- In general, this theoretical average is $\sum_{n=0}^{\infty} P(U=n)\cdot n$.

Generating function	$G(x)$	$=\sum_{n=0}^{\infty} P(U=n)\cdot x^n$	$=x/(2-x)$
Derivative	$G'(x)$	$=\sum_{n=0}^{\infty} P(U=n)\cdot n\,x^{n-1}$	$=2/(2-x)^2$
Theoretical average	$G'(1)$	$=\sum_{n=0}^{\infty} P(U=n)\cdot n$	$=2/1^2=2$

- Since $G(1)$ and $G'(1)$ involve plugging $x=1$ into an infinite series, convergence does have to be considered.
- In probability, this type of average is called the *expected value*, $E(U)$. Repeat an experiment a huge number of times and average the results together. This experimental average varies since it's a random process, but is usually close to the expected value.