Ch. 18.1: Counting structures with symmetry

Prof. Tesler

Math 184A
Fall 2014
Put \( n \) people 1, 2, \ldots, \( n \) on a Ferris wheel, one per seat.

Rotations are regarded as equivalent:

For general \( n \), how many distinct circular permutations are there?

Read it clockwise starting at the 1: 134652.

\[(n-1)!\] circular permutations.
Consider a Ferris wheel with \( n = 6 \) seats, each black or white. We regard rotations of it as equivalent:

\[
\begin{array}{ccccccc}
\text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} & \text{Diagram 4} & \text{Diagram 5} & \text{Diagram 6} \\
\end{array}
\]

Use the same drawings for necklaces with black and white beads. Ferris wheel only have rotations, but necklaces have both rotations and reflections (by flipping them over), so for necklaces, those 6 are equivalent to these:

\[
\begin{array}{ccccccc}
\text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} & \text{Diagram 4} & \text{Diagram 5} & \text{Diagram 6} \\
\end{array}
\]

Types of questions we can address:

- How many colorings of Ferris wheels or necklaces are there with \( n \) seats/beads and \( k \) colors, using the above notions of equivalence?
- We’ll use \( n = 6 \) seats/beads and \( k = 2 \) colors (black and white).
- How many colorings with exactly 4 white and 2 black?
Representing the circular arrangements as strings

- Start at the top spot. Read off colors clockwise; \( B = \text{black}, \ W = \text{white} \):
  
  - BWWBBW
  - WBWWBB
  - BWBWWB
  - BBWBWW
  - WBBWBW
  - WWBBWB

- If you have a large collection of Ferris wheels or necklaces of this sort, you could catalog them by choosing the alphabetically smallest string for each. This one is \( \text{BBWBWW} \).

- This is an example of a **canonical representative**: given an object with multiple representations, apply a rule to choose a specific one.
Lexicographic order generalizes alphabetical order to strings, lists, sequences, ... whose entries have a total ordering.

Compare $x$ and $y$ position by position, left to right.

$x < y$ if the first different position is smaller in $x$ than in $y$, or if $x$ is a prefix of $y$ and is shorter than $y$.

**Lex order on strings**

- **CALIFORNIA < CALORIE:**
  Both start **CAL**. In the next position, I < O.

- **UC < UCSD:** The left side is a prefix of the right.

**Lex order on numeric lists**

- **(10, 30, 20, 50, 60) < (10, 30, 20, 80, 5):**
  Both start 10, 30, 20. In the next position, 50 < 80.

- **(10, 30, 20) < (10, 30, 20, 80, 5):** The left side is a prefix of the right.
Distinct colorings of the Ferris wheel

For a Ferris wheel with 6 seats, each colored black or white, there are 14 distinct colorings:

- $\text{BBBBBB}$
- $\text{BBBBBW}$
- $\text{BBBBWW}$
- $\text{BBBWBW}$
- $\text{BBBWWW}$
- $\text{BBWBBW}$
- $\text{BBWBWW}$
- $\text{BBWWBW}$
- $\text{BBWWWW}$
- $\text{BWBWBW}$
- $\text{BWBWWW}$
- $\text{BWWBWW}$
- $\text{BWWWWW}$
- $\text{WWWWWW}$

In the necklace problem (reflections allowed), there are 13 distinct colorings because two of the above become equivalent:

- $\text{BBWBWW} \equiv \text{BBWWBW}$
Define a *rotation operation* \( \rho \) on strings that moves the last letter to the first:

\[
\rho(x_1 x_2 \ldots x_n) = x_n x_1 x_2 \ldots x_{n-1}
\]

\( \rho(\text{CALIFORNIA}) = \text{ACALIFORNI} \)

\( \rho^2(\text{CALIFORNIA}) = \text{IACALIFORN} \)

\( \rho^m \) means to apply \( \rho \) consecutively \( m \) times, which moves the last \( m \) letters to the start.

\( \rho \) describes the rotations of the spots clockwise one position:

\[
\begin{align*}
\text{BWWBBW} & \quad \rho(\text{BWWBBW}) = \text{WBWWBB} & \rho^2(\text{BWWBBW}) = \text{BWBWWB}
\end{align*}
\]
Cyclic group of order $n$

- For rotations of $n$ letters, there are $n$ different rotations,
  \[ C_n = \{ 1, \rho, \rho^2, \ldots, \rho^{n-1} \} \]

- **Multiplication of group elements:**
  \[ \rho^a \cdot \rho^b = \rho^{a+b} = \rho^c \]
  where $c = a + b \mod n$.
  Note $\rho^0 = \rho^n = 1$ (identity), $\rho^{m+n} = \rho^m$, etc.

---

Group

- In abstract algebra (Math 100/103), a **group** $G$ is a set of elements and an operation $x \cdot y$ obeying these axioms:
  - **Closure:** For all $x, y \in G$, we have $x \cdot y \in G$
  - **Associative:** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in G$
  - **Identity element:** There is a unique element $\text{id} \in G$ (here, it’s $\rho^0 = 1$) with $\text{id} \cdot x = x \cdot \text{id} = x$ for all $x \in G$
  - **Inverses:** For every $x \in G$, there is a $y \in G$ with $x \cdot y = y \cdot x = \text{id}$
    (One can prove $y$ is unique; denote it $y = x^{-1}$.)

- $C_n$ is a **commutative group** ($x \cdot y = y \cdot x$ for all $x, y \in G$).
  Later in these slides, we’ll have a noncommutative group.
Let $S$ be the set of $n$-long strings in $\mathbb{B}, \mathbb{W}$.

Applying group $G = C_n$ to $S$ (or to directly rotate the Ferris wheels) is called a group action:

- For $x \in S$ and $g \in G$, $g(x)$ is an element of $S$.
- For $x \in S$ and $g, h \in G$, $g(h(x)) = (gh)(x)$.
- E.g., $\rho^2(\rho^3(x)) = \rho^5(x)$ because rotating $x$ by 3 and then rotating the result by 2, is the same as rotating $x$ by 5 all at once.
Let \( G \) be a group acting on a set \( S \). We’ll use \( G = \mathbb{C}_6 \) and let \( S \) be 6-long strings of \( B, W \).

Let \( x \in S \). The **orbit of** \( x \) is \( \text{Orb}(x) = \{ g(x) : g \in G \} \subseteq S \)

- \( \text{Orb}(BWWBBW) = \{BWWBBW, WBWWBB, BWWBWB, BBWBWW, WBBWBW, WWBBWB\} \)
- \( \text{Orb}(BWBWBB) = \{BWBWBB, WBWBWB, BBWBWW, BBWBWW, WWBBWB\} \)

The **stabilizer of** \( x \) is \( \text{Stab}(x) = \{ g \in G : g(x) = x \} \subseteq G \)

- \( \text{Stab}(BWWBBW) = \{1\} \)
- \( \text{Stab}(BWBWBB) = \{1, \rho^3\} \)

Notice \( |\text{Orb}(x)| \cdot |\text{Stab}(x)| = 6 = |G| \) in both examples.
Orbits for the 6 seat, 2 color Ferris wheel

The $2^6 = 64$ strings split into 14 orbits.
The canonical representative (smallest alphabetically) is in **bold**.
The other elements represent rotations of it.

| BBB BBBB | WBB BBBB | BWBB BB | BBW BB | BBBW BB | BBBWB |
| BBB BWW | WBB BWW | WWW BB | BWW BB | BBW BB | BBBWB |
| BBB WWB | WBB WB | BWW BB | BWB BB | BBW BB | BBBWB |
| BBBWWW | WBB WW | WWW BB | BWW BB | BWB WB | BBBWB |
| BBWWW | WBBWW | WWW BB | BWW BB | BWBWB | BBBWB |
| BBWWW | WBBWW | WWW BB | BWW BB | BWBWB | BBBWB |
| BBWWWWW | WBBWW | WWW BB | BWW BB | BWBWB | BBBWB |
| BWWW | WBW | WWW BB | BWW BB | BWBWB | BBBWB |
| BWBWWW | WBWWW | WWW BB | BWW BB | BWBWB | BBBWB |
| BWBWWW | WBWWW | WWW BB | BWW BB | BWBWB | BBBWB |
| WWWWWW | WBWWW | WWW BB | BWW BB | BWBWB | BBBWB |

Prof. Tesler

Ch. 18.1: Structures with symmetry

Math 184A / Fall 2014
Orbits and stabilizers

- If \( y \in \text{Orb}(x) \) then \( x \) and \( y \) have the same orbit:
  \[
  \text{Orb}(BWWBWW) = \{BWWBWW, WBWWBW, WWBWWB\} = \text{Orb}(WBWWBW) = \text{Orb}(WWBWWB)
  \]

- Also \( |\text{Stab}(x)| = |\text{Stab}(y)| \) (stabilizers have the same size, but are not necessarily the same set); here, each stabilizer equals \( \{1, \rho^3\} \).

- For \( x = BWWBWW \),
  \[
  \text{Orb}(x) = \{x, \rho(x), \rho^2(x)\} \quad \text{Stab}(x) = \{1, \rho^3\}
  \]

- Since \( x = \rho^3(x) \), plug \( x \mapsto \rho^3(x) \) into the \( \text{Orb}(x) \) formula above:
  \[
  \text{Orb}(\rho^3(x)) = \{\rho^3(x), \rho(\rho^3(x)), \rho^2(\rho^3(x))\} = \{\rho^3(x), \rho^4(x), \rho^5(x)\}
  \]
  We’ve accounted for all 6 group elements \( 1, \rho, \ldots, \rho^5 \) acting on \( x \).

**Theorem (Orbit-Stabilizer Theorem)**

For all \( x \in S \), \( |\text{Orb}(x)| \cdot |\text{Stab}(x)| = |G| \)
Fixed points

Let \( g \in G \). The **fixed points of** \( g \) are \( \text{Fix}(g) = \{ x \in S : g(x) = x \} \subseteq S \).

\[
\text{Fix}(\rho^2) = \{ \text{BBBBBB}, \text{BWBWBW}, \text{WBWBWB}, \text{WWWWWW} \}
\]

**Systematic method to compute** \( \text{Fix}(\rho^2) \) **for strings of length 6:**

Let \( x = x_1 x_2 x_3 x_4 x_5 x_6 \) as 6 individual letters. Then \( \rho^2(x) \) is

\[
\rho^2(x_1 x_2 x_3 x_4 x_5 x_6) = x_5 x_6 x_1 x_2 x_3 x_4
\]

- \( \rho^2(x) = x \) gives \( x_5 x_6 x_1 x_2 x_3 x_4 = x_1 x_2 x_3 x_4 x_5 x_6 \)
- \( x_5 = x_1, \ x_6 = x_2, \ x_1 = x_3, \ x_2 = x_4, \ x_3 = x_5, \ x_4 = x_6 \)
  which combine into \( x_1 = x_3 = x_5, \ x_2 = x_4 = x_6. \)
- So \( \text{Fix}(\rho^2) \) consists of words of the form \( x = x_1 x_2 x_1 x_2 x_1 x_2 \).

- **For two colors** B, W:
  2 choices for \( x_1 \) times 2 choices for \( x_2 \) gives 4 fixed points.
- **For** \( k \) **colors:** \( |\text{Fix}(\rho^2)| = k^2. \)
Second method

- Fill in one letter at a time and look at all the places it moves.
- \( x = a -- -- -- -- \)
- \( \rho^2(x) = x \) copies the \( a \) over 2 positions so \( x = a - a -- -- -- \)
  Do it again and get \( x = a - a - a - \)
- Fill in another letter, \( x = aba - a - \).
- \( \rho^2(x) = x \) copies the \( b \) over 2 positions so \( x = ababa - \)
  and doing it again gives \( x = ababab \).
Fixed points of $\rho^4$ for strings of length 6

- $\rho^4(x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6) = x_3 \ x_4 \ x_5 \ x_6 \ x_1 \ x_2$

$\rho^2(x) = x$ gives $x_1 = x_3 = x_5$, $x_2 = x_4 = x_6$
so $\text{Fix}(\rho^4)$ also consists of words of the form $x_1 \ x_2 \ x_1 \ x_2 \ x_1 \ x_2$.

First explanation

- $\rho^2 \cdot \rho^2 = \rho^4$ so elements fixed by $\rho^2$ are also fixed by $\rho^4$.
- $\rho^4 \cdot \rho^4 = \rho^2$ so elements fixed by $\rho^4$ are also fixed by $\rho^2$.
  Thus $\text{Fix}(\rho^2) = \text{Fix}(\rho^4)$.

- **General rule:** In $C_n$, $\text{Fix}(\rho^m) = \text{Fix}(\rho^d)$ where $d = \gcd(m, n)$.

Second explanation

- $\rho^2$ (rotate 2 forwards / clockwise) and $\rho^4$ (rotate 2 backwards / counterclockwise) are inverses.
- Suppose $g(x) = x$. Apply $g^{-1}$ to both sides to get $x = g^{-1}(x)$.
- **General rule:** In any group $G$, $\text{Fix}(g) = \text{Fix}(g^{-1})$ for all $g \in G$. 

Fixed points of $\rho^m$ for strings of length 6

- $\rho(x_1 x_2 x_3 x_4 x_5 x_6) = x_6 x_1 x_2 x_3 x_4 x_5$
- $\rho^5(x_1 x_2 x_3 x_4 x_5 x_6) = x_2 x_3 x_4 x_5 x_6 x_1$
- $\rho(x) = x$ and $\rho^5(x) = x$ both give $x_1 = \cdots = x_6$, so $\text{Fix}(\rho) = \text{Fix}(\rho^5)$ consists of words of the form $x_1 x_1 x_1 x_1 x_1 x_1$.

- For $k$ colors: there are $k$ choices of $x_1$ so $|\text{Fix}(\rho)| = k$.

- $\rho^3(x_1 x_2 x_3 x_4 x_5 x_6) = x_4 x_5 x_6 x_1 x_2 x_3$
- $\rho^3(x) = x$ gives $x_1 = x_4$, $x_2 = x_5$, $x_3 = x_6$, so $\text{Fix}(\rho^3)$ consists of words $x_1 x_2 x_3 x_1 x_2 x_3$.

- For B,W: $|\text{Fix}(\rho^3)| = 2^3 = 8$
- For $k$ colors: $|\text{Fix}(\rho^3)| = k^3$
Fixed points of $\rho^m$ for strings of length 6

<table>
<thead>
<tr>
<th>$g$</th>
<th>Form of words fixed by $g$</th>
<th>$\text{Fix}(g)$</th>
<th>$k$ colors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_1 x_2 x_3 x_4 x_5 x_6$</td>
<td>$2^6 = 64$</td>
<td>$k^6$</td>
</tr>
<tr>
<td>$\rho$, $\rho^5$</td>
<td>$x_1 x_1 x_1 x_1 x_1 x_1$</td>
<td>2</td>
<td>$k$</td>
</tr>
<tr>
<td>$\rho^2$, $\rho^4$</td>
<td>$x_1 x_2 x_1 x_2 x_1 x_2$</td>
<td>$2^2 = 4$</td>
<td>$k^2$</td>
</tr>
<tr>
<td>$\rho^3$</td>
<td>$x_1 x_2 x_3 x_1 x_2 x_3$</td>
<td>$2^3 = 8$</td>
<td>$k^3$</td>
</tr>
</tbody>
</table>
Lemma (Burnside's Lemma)

The number of orbits of $G$ on $X$ is

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

In other words, the number of orbits is the average number of fixed points per group element.

Ferris wheel with 6 seats and colors $B, W$:

| $g$  | Form of words | $|\text{Fix}(g)|$ |
|------|--------------|------------------|
| $1$  | $x_1 x_2 x_3 x_4 x_5 x_6$ | $2^6 = 64$ |
| $\rho$ | $x_1 x_1 x_1 x_1 x_1 x_1$ | $2$ |
| $\rho^2$ | $x_1 x_2 x_1 x_2 x_1 x_2$ | $2^2 = 4$ |
| $\rho^3$ | $x_1 x_2 x_3 x_1 x_2 x_3$ | $2^3 = 8$ |
| $\rho^4$ | $x_1 x_2 x_1 x_2 x_1 x_2$ | $2^2 = 4$ |
| $\rho^5$ | $x_1 x_1 x_1 x_1 x_1 x_1$ | $2$ |

The number of orbits is

$$\frac{64 + 2 + 4 + 8 + 4 + 2}{6} = \frac{84}{6} = 14$$
Ferris wheel with 4 white seats and 2 black seats

| $g$   | Form of words | $|\text{Fix}(g)|$                           |
|-------|--------------|-------------------------------------------|
| 1     | $x_1 x_2 x_3 x_4 x_5 x_6$ | $\binom{6}{2} = 15$ ways to choose 2 black |
| $\rho$ | $x_1 x_1 x_1 x_1 x_1 x_1$ | 0 since all 6 seats are same color         |
| $\rho^2$ | $x_1 x_2 x_1 x_2 x_1 x_2$ | 0 since 3 seats are $x_1$ and 3 are $x_2$ |
| $\rho^3$ | $x_1 x_2 x_3 x_1 x_2 x_3$ | $\binom{3}{1} = 3$ ways to choose which $x_i$ is black |
| $\rho^4$ | $x_1 x_2 x_1 x_2 x_1 x_2$ | 0 since 3 seats are $x_1$ and 3 are $x_2$ |
| $\rho^5$ | $x_1 x_1 x_1 x_1 x_1 x_1$ | 0 since all 6 seats are same color         |

The number of orbits is

$$\frac{15 + 0 + 0 + 3 + 0 + 0}{6} = \frac{18}{6} = 3$$

```
BBWWWW
BWBWWW
BWWBWW
```
Proof of Burnside’s Lemma

We’ll count the size of \( A = \{ (g, x) : g \in G, x \in S, g(x) = x \} \) in two ways; one based on each \( g \in G \), one based on each \( x \in S \).

**Counting first by \( g \in G \)**

For each \( g \in G \), the values of \( x \) with \( g(x) = x \) form \( \text{Fix}(g) \), so

\[
|A| = \sum_{g \in G} |\text{Fix}(g)|
\]

**Counting first by \( x \in S \)**

For each \( x \), the values of \( g \) with \( g(x) = x \) form \( \text{Stab}(x) \), so

\[
|A| = \sum_{x \in S} |\text{Stab}(x)|
\]

We’ll show this equals the number of orbits times \( |G| \).

**Putting the two counts together**

\[
|A| = \sum_{g \in G} |\text{Fix}(g)| = \sum_{x \in S} |\text{Stab}(x)| = \text{number of orbits times } |G|\]

so the number of orbits is

\[
\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|
\]
\[ \sum_{x \in S} |\text{Stab}(x)| \] organized by orbits (each row is a complete orbit):

<table>
<thead>
<tr>
<th>BBBBWB</th>
<th>WBBBBB</th>
<th>BWBBBB</th>
<th>BBWBBB</th>
<th>BBBWBB</th>
<th>BBBWBW</th>
<th>+1 + 1 + 1 + 1 + 1 + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>BBBBWB</td>
<td>WBBBBB</td>
<td>WWBBBB</td>
<td>BWBBBB</td>
<td>BBWBBB</td>
<td>BBBWBW</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BBBWBB</td>
<td>WBBBBB</td>
<td>BWBBBB</td>
<td>WBBBBB</td>
<td>BBBBBB</td>
<td>BBWBBW</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BBBWBB</td>
<td>WBBBBB</td>
<td>BWBBBB</td>
<td>WBBBBB</td>
<td>BBBBBB</td>
<td>BBWBBW</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BBBWWW</td>
<td>WBBBBW</td>
<td>WWBBBB</td>
<td>WWWBBB</td>
<td>BWWWBB</td>
<td>BBWWWB</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BBWBBW</td>
<td>WBBWBB</td>
<td>BWBBWB</td>
<td>WBWBBB</td>
<td>BWBBWB</td>
<td>BBWBWB</td>
<td>+2 + 2 + 2</td>
</tr>
<tr>
<td>BBWBBW</td>
<td>WBBWBB</td>
<td>WWBBWB</td>
<td>BBWBBB</td>
<td>WBWBBB</td>
<td>BBWBBW</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BBWWWW</td>
<td>WBBBBW</td>
<td>WWBBBB</td>
<td>WWWBBB</td>
<td>BWWWBB</td>
<td>BBWWWB</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BBWBWB</td>
<td>WBBBBB</td>
<td>BWBBBB</td>
<td>BBWBBB</td>
<td>BBBWBB</td>
<td>BBBWBW</td>
<td>+2 + 2 + 2</td>
</tr>
<tr>
<td>BBWWWW</td>
<td>WBBBBB</td>
<td>WWBBBB</td>
<td>BBBBWW</td>
<td>BBBBBB</td>
<td>BBWWWB</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>BWWWWW</td>
<td>WBBBBB</td>
<td>WWBBBB</td>
<td>BBBBWW</td>
<td>BBBBBB</td>
<td>BBWWWB</td>
<td>+1 + 1 + 1 + 1 + 1 + 1</td>
</tr>
<tr>
<td>WWWW</td>
<td>WBBBBB</td>
<td>WWBBBB</td>
<td>WWWBBB</td>
<td>BWWWBB</td>
<td>BBWWWB</td>
<td>+6</td>
</tr>
</tbody>
</table>

\[ = 6 \cdot 14 = 84 \]

- By the Orbit-Stabilizer Theorem, in each row (orbit), all stabilizers have the same size and sum to \(|\text{orbit}| \cdot |\text{stabilizer}| = |G|\).

- Summing \(|\text{Stab}(x)|\) over all \(x \in S\) gives \(|G|\) times the # of orbits.
Proof of Burnside’s Lemma

\[ A = \{ (g, x) : g \in G, x \in S, g(x) = x \} \]

Counting first by \( x \in S \)

- Split \( S \) into orbits \( O_1, O_2, \ldots, O_N \); these partition the set \( S \).
- For each \( x \), the values of \( g \) with \( g(x) = x \) form \( \text{Stab}(x) \), so
  \[ |A| = \sum_{x \in S} |\text{Stab}(x)| \]
- For each \( x \in O_i \), \( \text{Stab}(x) = \frac{|G|}{|\text{Orb}(x)|} = \frac{|G|}{|O_i|} \).
- \[ \sum_{x \in O_i} |\text{Stab}(x)| = \frac{|G|}{|O_i|} \cdot |O_i| = |G| \]
- \[ |A| = \sum_{i=1}^{N} \sum_{x \in O_i} |\text{Stab}(x)| = \sum_{i=1}^{N} |G| = N |G| \]

- Equating the two counts gives \( |A| = \sum_{g \in G} |\text{Fix}(g)| = N |G| \).
- Dividing by \( |G| \) gives the number of orbits, \( N = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| \).
Now we have rotations and reflections regarded as equivalent:

Let $\sigma(x_1 x_2 \ldots x_n) = x_n \ldots x_2 x_1$ (reverse a string):

$\sigma(\text{CALIFORNIA}) = \text{AINROFILAC}$

- $\sigma$ describes this mirror image:

  $\sigma(\text{BWWBBW}) = \text{WBBWWB}$

- $\rho\sigma$ is reflect and then rotate:

  $\rho\sigma(\text{BWWBBW}) = \text{BWBBWW}$

Note $\sigma^2 = 1$ and $\sigma \rho^m = \rho^{-m} \sigma$. 
Simplifying products

Simplify any product of $\rho$’s, $\sigma$’s, and powers

- Use $\sigma \rho^m = \rho^{-m} \sigma$ to move $\sigma$’s to the right and $\rho$’s to the left.
- Combine powers and simplify with $\sigma^2 = 1$ and $\rho^6 = 1$.
- Keep going until the final form: $\rho^k$ or $\rho^k \sigma$ with $k = 0, \ldots, 5$.

\[
\sigma \rho^2 \sigma^3 \rho^4 \sigma \rho^{-1} = \sigma \rho^2 \sigma^3 \rho^4 \rho \sigma \\
\sigma \rho^2 \sigma^3 \rho^4 \rho \sigma = \sigma \rho^2 \sigma^3 \rho^5 \sigma \\
\sigma \rho^2 \sigma^3 \rho^5 \sigma = \sigma \rho^2 \sigma \rho^5 \sigma = \cdots
\]

\[
\cdots = \sigma \rho^2 \sigma \rho^5 \sigma = \sigma \rho^2 \rho^{-5} \sigma \sigma \\
\sigma \rho^2 \rho^{-5} \sigma \sigma = \sigma \rho^{-3} \cdot 1 = \sigma \rho^{-3} \\
\sigma \rho^{-3} = \boxed{\rho^3 \sigma}
\]
Reflections $\rho^m \sigma$

$\rho^m \sigma$ is a reflection across an axis at angle $(120 - 30m)^\circ$ (polar coords.). These are all the reflections that keep the spots in the same positions.
Dihedral group $\mathcal{D}_{2n}$ (for $n \geq 3$)

- Let $\mathcal{D}_{2n} = \{1, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma, \rho \sigma, \rho^2 \sigma, \ldots, \rho^{n-1} \sigma\}$
  - Rotations
  - Reflections

- This is a noncommutative group with $2n$ elements for $n \geq 3$ (some are degenerate for $n = 1, 2$).

- Simplify multiplications using $\sigma^2 = 1$, $\rho^n = 1$, and $\sigma \rho^m = \rho^{-m} \sigma$.

- Some disciplines use the notation $\mathcal{D}_n$ instead of $\mathcal{D}_{2n}$.

For $n = 6$ and $G = \mathcal{D}_{12}$,

Orb($B\overline{W}WBB\overline{W}$) has 12 elements, each stabilized only by the identity.

Orb($B\overline{W}WB\overline{W}$) = \{BWWBWW, WBWWBW, WWBWWB\}

Stab($B\overline{W}WB\overline{W}$) = \{1, $\rho^3$, $\rho \sigma$, $\rho^4 \sigma$\}
Stab($WB\overline{W}BW$) = \{1, $\rho^3$, $\sigma$, $\rho^3 \sigma$\}
Stab($WWB\overline{W}B$) = \{1, $\rho^3$, $\rho^2 \sigma$, $\rho^5 \sigma$\}

The stabilizers are different but all have the same size, $\frac{|G|}{\text{orbit}} = \frac{12}{3} = 4$. 
Fix\((g)\) in dihedral group \(D_{12}\) on strings of length 6

**Fix\((\sigma)\)**

- \(\sigma(x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6) = x_6 \ x_5 \ x_4 \ x_3 \ x_2 \ x_1\)
- \(x = \sigma(x)\) gives \(x_1 = x_6, \ x_2 = x_5, \ x_3 = x_4\)
- Elements have form \(x_1 \ x_2 \ x_3 \ x_3 \ x_2 \ x_1\).
- For 2 color necklaces: \(2^3 = 8\) elements; for \(k\) colors, \(k^3\) elements.

**Second method**

- Fill in one letter at a time and look at all the places it moves.
- \(x = a -- -- -- --\)
- \(\sigma(x) = -- -- -- -- a\) so \(\sigma(x) = x\) gives \(x = a -- -- -- a\).
- \(\sigma(a -- -- -- a) = a -- -- -- a\), so \(a's\) are completed.
- \(\sigma(ab -- -- a) = a -- -- ba\) so \(\sigma(x) = x\) gives \(x = ab -- ba\).
- \(\sigma(abc -- ba) = ab -- cba\) so \(x = abccba\).
- For 2 color necklaces: \(2^3 = 8\) elements; for \(k\) colors, \(k^3\) elements.
**Fix(\(g\)) in dihedral group \(D_{12}\) on strings of length 6**

**Fix(\(\rho \sigma\))**
- \(\rho \sigma(x_1 x_2 x_3 x_4 x_5 x_6) = x_1 x_6 x_5 x_4 x_3 x_2\)
- \(x = \rho \sigma(x)\) gives \(x_1, x_4\) unrestricted, \(x_2 = x_6, x_3 = x_5\),
- Elements have form \(x_1 x_2 x_3 x_4 x_3 x_2\).
- For 2 colors, \(2^4 = 16\) elements; for \(k\) colors, \(k^4\) elements.

**Second method**
- Fill in one letter at a time and look at all the places it moves.
- \(x = a --- --- ---\)
- \(\rho \sigma(x) = a --- --- ---\), so \(a\)'s are completed.
- \(\rho \sigma(ab --- ---) = a --- --- b\), so \(\rho \sigma(x) = x\) gives \(x = ab --- b\).
- \(\rho \sigma(abc -- b) = ab -- cb\), so \(x = abc -- cb\).
- \(x = abcdcb\).
- For 2 colors, \(2^4 = 16\) elements; for \(k\) colors, \(k^4\) elements.
Counting necklaces with 6 black and white beads

| $g$       | Form of words | $|\text{Fix}(g)|_{B,W}$ | $k$ colors |
|-----------|---------------|--------------------------|------------|
| $1$       | $x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6$ | $2^6 = 64$ | $k^6$      |
| $\rho, \rho^5$ | $x_1 \ x_1 \ x_1 \ x_1 \ x_1 \ x_1$ | $2$ | $k$        |
| $\rho^2, \rho^4$ | $x_1 \ x_2 \ x_1 \ x_2 \ x_1 \ x_2$ | $2^2 = 4$ | $k^2$      |
| $\rho^3$ | $x_1 \ x_2 \ x_3 \ x_1 \ x_2 \ x_3$ | $2^3 = 8$ | $k^3$      |
| $\sigma$ | $x_1 \ x_2 \ x_3 \ x_3 \ x_2 \ x_1$ | $2^3 = 8$ | $k^3$      |
| $\rho\sigma$ | $x_1 \ x_2 \ x_3 \ x_4 \ x_3 \ x_2$ | $2^4 = 16$ | $k^4$      |
| $\rho^2\sigma$ | $x_1 \ x_1 \ x_3 \ x_4 \ x_4 \ x_3$ | $2^3 = 8$ | $k^3$      |
| $\rho^3\sigma$ | $x_1 \ x_2 \ x_1 \ x_4 \ x_5 \ x_4$ | $2^4 = 16$ | $k^4$      |
| $\rho^4\sigma$ | $x_1 \ x_2 \ x_2 \ x_1 \ x_5 \ x_5$ | $2^3 = 8$ | $k^3$      |
| $\rho^5\sigma$ | $x_1 \ x_2 \ x_3 \ x_2 \ x_1 \ x_6$ | $2^4 = 16$ | $k^4$      |

For 6 bead necklaces made from black and white beads, Burnside’s Lemma gives the number of orbits:

$$\frac{64 + 2(2 + 4) + 8 + 8 + 16 + 8 + 16 + 8 + 16}{12} = \frac{156}{12} = 13$$
## Differences in book’s notation

<table>
<thead>
<tr>
<th></th>
<th>Slides</th>
<th>Our textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Composition</strong></td>
<td>Right-to-left</td>
<td>Left-to-right</td>
</tr>
<tr>
<td></td>
<td>((gh)(x) = g(h(x)))</td>
<td>((gh)(x) = h(g(x)))</td>
</tr>
<tr>
<td><strong>Stabilizer</strong></td>
<td>(\text{Stab}(x))</td>
<td>(G_x)</td>
</tr>
<tr>
<td><strong>Orbit</strong></td>
<td>(\text{Orb}(x))</td>
<td>(x^G)</td>
</tr>
<tr>
<td><strong>Fixed points</strong></td>
<td>(\text{Fix}(g))</td>
<td>(F_g)</td>
</tr>
<tr>
<td><strong>Subgroup</strong></td>
<td>(H \subseteq G)</td>
<td>(H \leq G)</td>
</tr>
</tbody>
</table>

### Orbit-Stabilizer Theorem
\[
| \text{Orb}(x)| \cdot | \text{Stab}(x)| = |G| \\
| \text{x}^G| \cdot | \text{G}_x| = |G|
\]

### Burnside’s Lemma
\[
\#\text{ orbits} = \frac{1}{|G|} \sum_{g \in G} | \text{Fix}(g)|
\]
\[
\frac{1}{|G|} \sum_{g \in G} |F_g|
\]