Ch. 18.1: Counting structures with symmetry

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Math 184A
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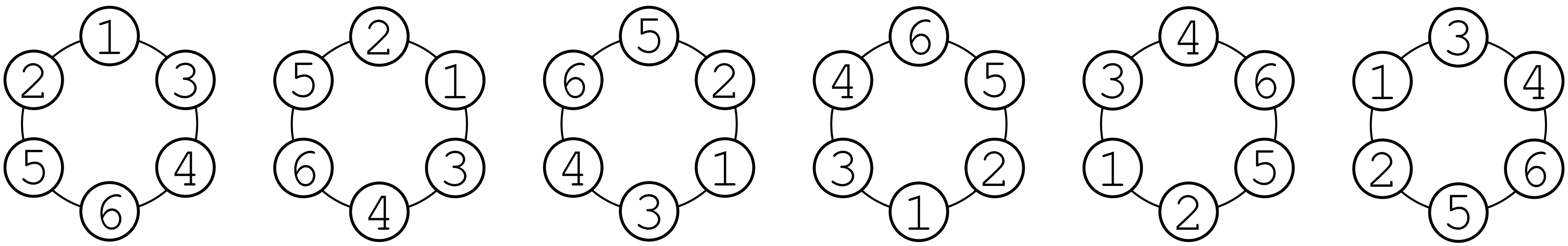
Counting circular permutations

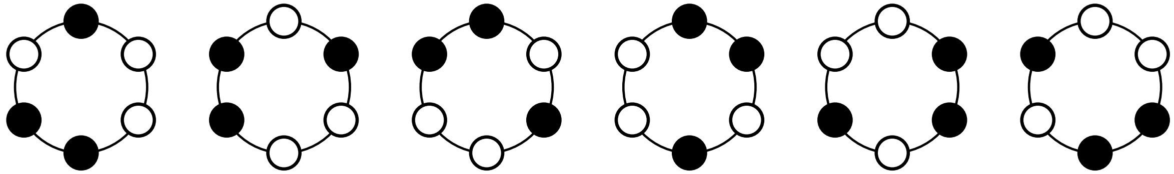
- Put n people $1, 2, \ldots, n$ on a Ferris wheel, one per seat.
- Rotations are regarded as equivalent:



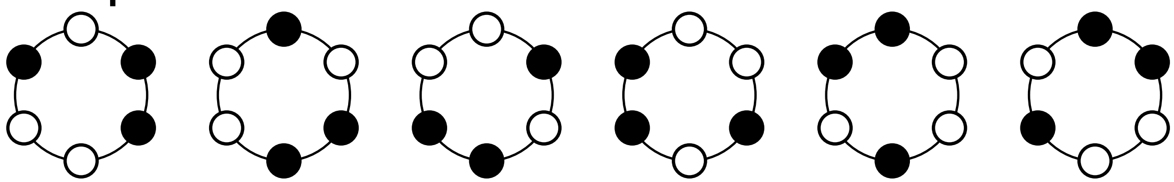
- For general n, how many distinct circular permutations are there?
- Read it clockwise starting at the 1: 134652.
  (n-1)! circular permutations.

Counting Ferris wheels and necklaces

- Consider a Ferris wheel with $n = 6$ seats, each black or white. We regard rotations of it as equivalent:



- Use the same drawings for necklaces with black and white beads. Ferris wheel only have rotations, but necklaces have both rotations and reflections (by flipping them over), so for necklaces, those 6 are equivalent to these:

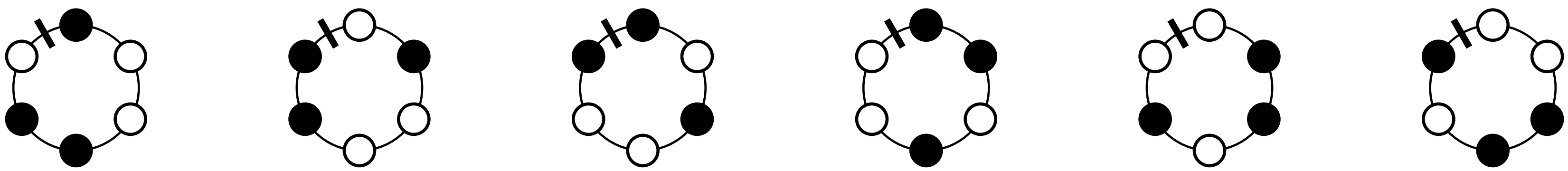


- Types of questions we can address:
  - How many colorings of Ferris wheels or necklaces are there with n seats/beads and k colors, using the above notions of equivalence?

    We'll use $n = 6$ seats/beads and $k = 2$ colors (black and white).
  - How many colorings with exactly 4 white and 2 black?

Representing the circular arrangements as strings

- Start at the top spot. Read off colors clockwise; `B`=black, `W`=white:



`BWWBBBW` `WBWWBB` `BWBWWB` `BBWBWW` `WBBWBW` `WWBBWB`

- If you have a large collection of Ferris wheels or necklaces of this sort, you could catalog them by choosing the alphabetically smallest string for each. This one is `BBWBWW`.
- This is an example of a *canonical representative*: given an object with multiple representations, apply a rule to choose a specific one.

Lexicographic Order on strings, lists, . . .

- *Lexicographic order* generalizes alphabetical order to strings, lists, sequences, . . . whose entries have a total ordering.
- Compare x and y position by position, left to right.
- $x < y$ if the first different position is smaller in x than in y, or if x is a prefix of y and is shorter than y.

Lex order on strings

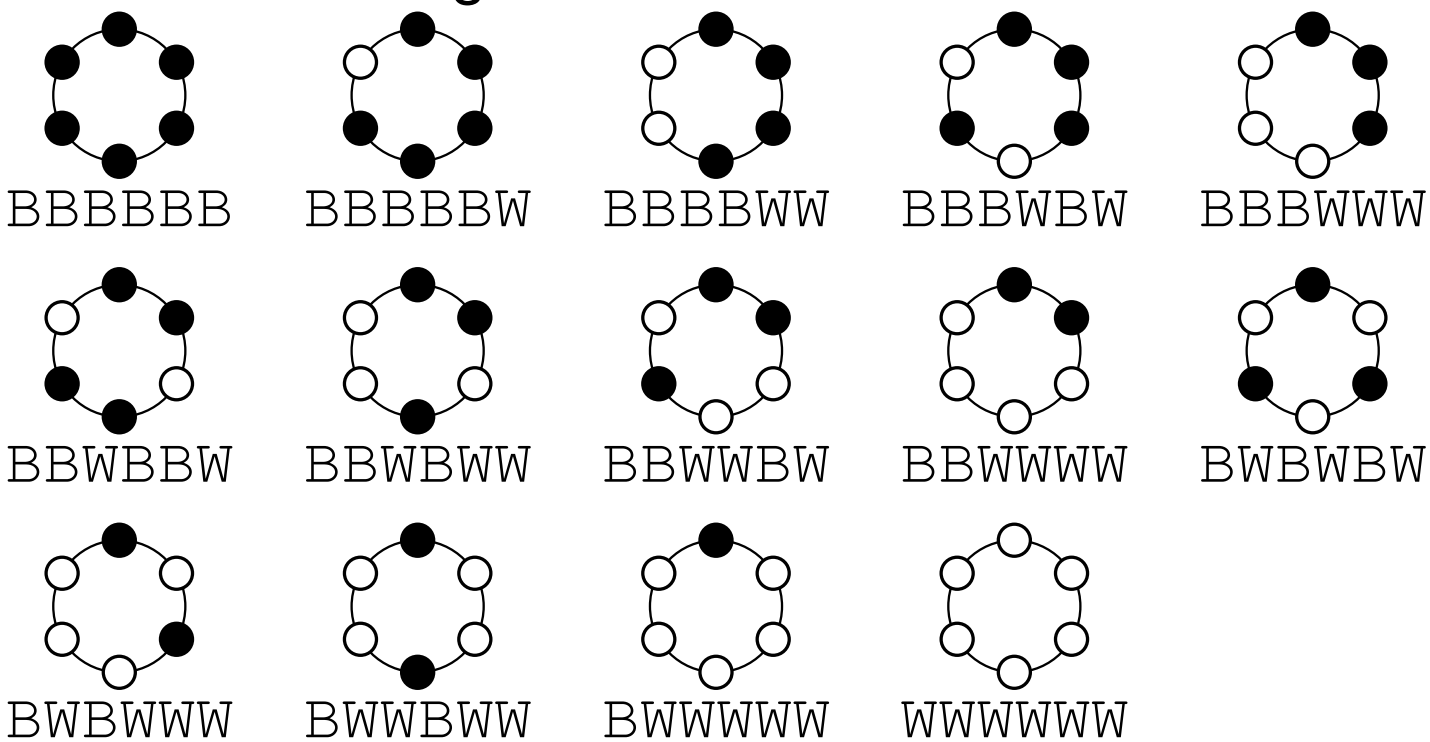
- $\texttt{CALIFORNIA} < \texttt{CALORIE}$:
  Both start $\texttt{CAL}$. In the next position, $\texttt{I} < \texttt{O}$.
- $\texttt{UC} < \texttt{UCSD}$: The left side is a prefix of the right.

Lex order on numeric lists

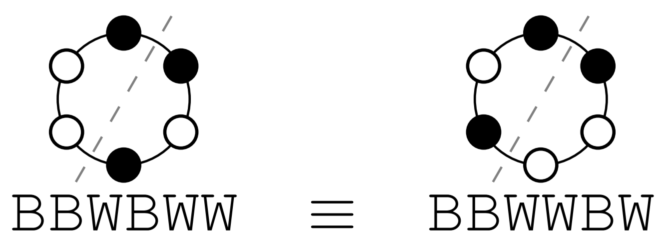
- $(10, 30, 20, 50, 60) < (10, 30, 20, 80, 5)$:
  Both start $10, 30, 20$. In the next position, $50 < 80$.
- $(10, 30, 20) < (10, 30, 20, 80, 5)$: The left side is a prefix of the right.

Distinct colorings of the Ferris wheel

- For a Ferris wheel with 6 seats, each colored black or white, there are 14 distinct colorings:



- In the necklace problem (reflections allowed), there are 13 distinct colorings because two of the above become equivalent:



String rotation

- Define a *rotation operation* ρ on strings that moves the last letter to the first:

$$\rho(x_1\,x_2\ldots x_n) = x_n\,x_1\,x_2\ldots x_{n-1}$$

$$\rho(\texttt{CALIFORNIA}) = \texttt{ACALIFORNI}$$
$$\rho^2(\texttt{CALIFORNIA}) = \texttt{IACALIFORN}$$

- For $m \geqslant 0$, ρ^m means to apply ρ consecutively m times.
  For $m = 0, 1, \ldots, n$, that moves the last m letters to the start.

- ρ^{-1} moves the first letter to the end, and ρ^{-m} moves the first m letters to the end:

$$\rho^{-1}(x_1\,x_2\ldots x_n) = x_2\ldots x_n\,x_1$$

$$\rho^{-1}(\texttt{CALIFORNIA}) = \texttt{ALIFORNIAC}$$
$$\rho^{-2}(\texttt{CALIFORNIA}) = \texttt{LIFORNIACA}$$

String rotation

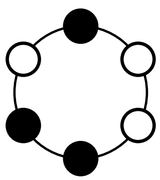
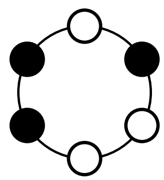
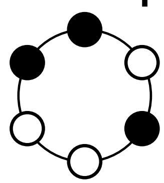
- CALIFORNIA has length $n = 10$ letters:

$$\rho^{10}(\texttt{CALIFORNIA}) = \texttt{CALIFORNIA} \qquad \text{so } \rho^{10} = \rho^0 = \text{identity}$$
$$\rho^{12}(\texttt{CALIFORNIA}) = \texttt{IACALIFORN} \qquad \rho^{12} = \rho^2$$
$$\rho^{-10}(\texttt{CALIFORNIA}) = \texttt{CALIFORNIA} \qquad \rho^{-10} = \rho^0 = \text{identity}$$
$$\rho^{-12}(\texttt{CALIFORNIA}) = \texttt{LIFORNIACA} \qquad \rho^{-12} = \rho^8$$

- For strings of length n,
  - ρ^n is the identity
  - $\rho^{nq+m} = \rho^m$ for any integer q.

String rotation on Ferris wheel

- ρ describes the rotations of the spots clockwise one position:

$\texttt{BWWBBW}$ $\quad \rho(\texttt{BWWBBW}) = \texttt{WBWWBB}$ $\quad \rho^2(\texttt{BWWBBW}) = \texttt{BWBWWB}$

- ρ^6 is the identity.

Cyclic group of order n

- For rotations of n letters, there are n different rotations,
$$\mathcal{C}_n = \{1, \rho, \rho^2, \ldots, \rho^{n-1}\}$$
- **Multiplication of group elements:**
$$\rho^a \cdot \rho^b = \rho^{a+b} = \rho^c \qquad \text{where } c = a + b \bmod n.$$
Note $\rho^0 = \rho^n = 1$ (identity), $\rho^{m+n} = \rho^m$, etc.

Group

- In abstract algebra (Math 100/103), a *group* G is a set of elements and an operation $x \cdot y$ obeying these axioms:
  - *Closure:* For all $x, y \in G$, we have $x \cdot y \in G$
  - *Associative:* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in G$
  - *Identity element:* There is a unique element $\mathrm{id} \in G$ (here, it's $\rho^0 = 1$) with $\mathrm{id} \cdot x = x \cdot \mathrm{id} = x$ for all $x \in G$
  - *Inverses:* For every $x \in G$, there is a $y \in G$ with $x{\cdot}y = y{\cdot}x = \mathrm{id}$ (One can prove y is unique; denote it $y = x^{-1}$.)
- $\mathcal{C}_n$ is a *commutative group* ($x \cdot y = y \cdot x$ for all $x, y \in G$).
Later in these slides, we'll have a noncommutative group.

Group action

- Let S be the set of n-long strings in `B`, `W`.
- Applying group $G = \mathcal{C}_n$ to S (or to directly rotate the Ferris wheels) is called a *group action*:
  - For $x \in S$ and $g \in G$, $\quad g(x)$ is an element of S.
  - For $x \in S$ and $g, h \in G$, $\quad g(h(x)) = (gh)(x)$.
  - E.g., $\rho^2(\rho^3(x)) = \rho^5(x)$ because rotating x by 3 and then rotating the result by 2, is the same as rotating x by 5 all at once.

Orbits and stabilizers

- Let G be a group acting on a set S.
  We'll use $G = \mathcal{C}_6$ and let S be 6-long strings of `B`, `W`.

- Let $x \in S$. The *orbit of x* is $\mathrm{Orb}(x) = \{\, g(x) \;:\; g \in G \,\} \subseteq S$

$$\begin{aligned}\mathrm{Orb}(\texttt{BWWBBW}) = \{&\texttt{BWWBBW}, \texttt{WBWWBB}, \texttt{BWBWWB},\\ &\texttt{BBWBWW}, \texttt{WBBWBW}, \texttt{WWBBWB}\}\end{aligned}$$

$$\mathrm{Orb}(\texttt{BWWBWW}) = \{\texttt{BWWBWW}, \texttt{WBWWBW}, \texttt{WWBWWB}\}$$

- The *stabilizer of x* is $\mathrm{Stab}(x) = \{\, g \in G \;:\; g(x) = x \,\} \subseteq G$

$$\mathrm{Stab}(\texttt{BWWBBW}) = \{1\} \qquad \mathrm{Stab}(\texttt{BWWBWW}) = \left\{1, \rho^3\right\}$$

- Notice $|\mathrm{Orb}(x)| \cdot |\mathrm{Stab}(x)| = 6 = |G|$ in both examples.

Orbits for the 6 seat, 2 color Ferris wheel

The $2^6 = 64$ strings split into 14 orbits.
The canonical representative (smallest alphabetically) is in **bold**.
The other elements represent rotations of it.

BBBBBB					
BBBBBW	WBBBBB	BWBBBB	BBWBBB	BBBWBB	BBBBWB
BBBBWW	WBBBBW	WWBBBB	BWWBBB	BBWWBB	BBBWWB
BBBWBW	WBBBWB	BWBBBW	WBWBBB	BWBWBB	BBWBWB
BBBWWW	WBBBWW	WWBBBW	WWWBBB	BWWWBB	BBWWWB
BBWBBW	WBBWBB	BWBBWB			
BBWBWW	WBBWBW	WWBBWB	BWWBBW	WBWWBB	BWBWWB
BBWWBW	WBBWWB	BWBBWW	WBWBBW	WWBWBB	BWWBWB
BBWWWW	WBBWWW	WWBBWW	WWWBBW	WWWWBB	BWWWWB
BWBWBW	WBWBWB				
BWBWWW	WBWBWW	WWBWBW	WWWBWB	BWWWBW	WBWWWB
BWWBWW	WBWWBW	WWBWWB			
BWWWWW	WBWWWW	WWBWWW	WWWBWW	WWWWBW	WWWWWB
WWWWWW					

Orbits and stabilizers

- If $y \in \operatorname{Orb}(x)$ then x and y have the same orbit:
$$\begin{aligned}\operatorname{Orb}(\texttt{BWWBWW}) &= \{\texttt{BWWBWW}, \texttt{WBWWBW}, \texttt{WWBWWB}\} \\ &= \operatorname{Orb}(\texttt{WBWWBW}) = \operatorname{Orb}(\texttt{WWBWWB})\end{aligned}$$

- Also $|\operatorname{Stab}(x)| = |\operatorname{Stab}(y)|$ (stabilizers have the same size, but are not necessarily the same set); here, each stabilizer equals $\{1, \rho^3\}$.

- For $x = \texttt{BWWBWW}$,
$$\operatorname{Orb}(x) = \left\{x, \rho(x), \rho^2(x)\right\} \qquad \operatorname{Stab}(x) = \left\{1, \rho^3\right\}$$

- Since $x = \rho^3(x)$, plug $x \mapsto \rho^3(x)$ into the $\operatorname{Orb}(x)$ formula above:
$$\begin{aligned}\operatorname{Orb}(\rho^3(x)) &= \left\{\rho^3(x), \rho(\rho^3(x)), \rho^2(\rho^3(x))\right\} \\ &= \left\{\rho^3(x), \rho^4(x), \rho^5(x)\right\}\end{aligned}$$
We've accounted for all 6 group elements $1, \rho, \ldots, \rho^5$ acting on x.

Theorem (Orbit-Stabilizier Theorem)

For all $x \in S$, $\qquad |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)| = |G|$.

Skip proof

Proof of Orbit-Stabilizer Theorem

Optional for students who took Abstract Algebra (Math 100/103)

$\mathrm{Stab}(x)$ is a subgroup of G

- **Identity:** $1x = x$ so $1 \in \mathrm{Stab}(x)$.
- **Closure:** If $g, h \in \mathrm{Stab}(x)$, then $(gh)(x) = g(h(x)) = g(x) = x$, so $gh \in \mathrm{Stab}(x)$.
- **Inverse:** If $g \in \mathrm{Stab}(x)$, then
$$\begin{aligned} g^{-1}(x) &= g^{-1}(g(x)) \\ &= (g^{-1}g)(x) = 1x = x \end{aligned}$$
so $g^{-1} \in \mathrm{Stab}(x)$.

Proof of Orbit-Stabilizer Theorem

Optional for students who took Abstract Algebra (Math 100/103)

- Write the elements of $\operatorname{Stab}(x)$ and $\operatorname{Orb}(x)$ as follows, with no repetitions:

$$\operatorname{Stab}(x) = \{s_1, s_2, \ldots, s_k\}$$
$$\operatorname{Orb}(x) = \{o_1(x), o_2(x), \ldots, o_m(x)\}$$

- We will show that the products $o_i\, s_j$ for $i = 1, \ldots, k$ and $j = 1, \ldots, m$ are distinct and give all elements of the group G.

- Thus, $km = |G|$; that is, $|\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)| = |G|$.

Proof of Orbit-Stabilizer Theorem

Optional for students who took Abstract Algebra (Math 100/103)

$$\mathrm{Stab}(x) = \{s_1, s_2, \ldots, s_k\}$$
$$\mathrm{Orb}(x) = \{o_1(x), o_2(x), \ldots, o_m(x)\}$$

The products $o_i\, s_j$ are all distinct

- Suppose $o_i\, s_j = o_p\, s_q$.
- Apply both sides to x:

	left:	$o_i\, s_j(x) = o_i(x)$
	right:	$o_p\, s_q(x) = o_p(x)$
	combined:	$o_i(x) = o_p(x)$

- Since elements of the orbit are only listed once, $o_i = o_p$.
- So $o_i\, s_j = o_p\, s_q$ becomes $o_i\, s_j = o_i\, s_q$.
- Multiply both sides on the left by o^{-1} to get $s_j = s_q$.
- Thus, if $o_i\, s_j = o_p\, s_q$, then $o_i = o_p$ and $s_j = s_q$.

Proof of Orbit-Stabilizer Theorem

Optional for students who took Abstract Algebra (Math 100/103)

$$\begin{aligned} \mathrm{Stab}(x) &= \{s_1, s_2, \ldots, s_k\} \\ \mathrm{Orb}(x) &= \{o_1(x), o_2(x), \ldots, o_m(x)\} \end{aligned}$$

Every element of G can be written $o_i\, s_j$

- Let $g \in G$.
- $g(x)$ is in the orbit of x, so it equals $o_i(x)$ for some i.
- $$\begin{aligned} o_i(x) = g(x) \quad &\Rightarrow \quad o_i^{-1}\, g(x) = x \\ &\Rightarrow \quad o_i^{-1}\, g \in \mathrm{Stab}(x) \\ &\Rightarrow \quad o_i^{-1}\, g = s_j \text{ for some } j \\ &\Rightarrow \quad g = o_i\, s_j. \end{aligned}$$

Fixed points

- Let $g \in G$. The *fixed points of g* are $\mathrm{Fix}(g) = \{x \in S \ : \ g(x) = x\} \subseteq S$.
  $$\mathrm{Fix}(\rho^2) = \{\mathtt{BBBBBB}, \mathtt{BWBWBW}, \mathtt{WBWBWB}, \mathtt{WWWWWW}\}$$

Systematic method to compute $\mathrm{Fix}(\rho^2)$ for strings of length 6:

Let $x = x_1\,x_2\,x_3\,x_4\,x_5\,x_6$ as 6 individual letters. Then $\rho^2(x)$ is
$$\rho^2(x_1\,x_2\,x_3\,x_4\,x_5\,x_6) = x_5\,x_6\,x_1\,x_2\,x_3\,x_4$$

- $\rho^2(x) = x$ gives $x_5\,x_6\,x_1\,x_2\,x_3\,x_4 = x_1\,x_2\,x_3\,x_4\,x_5\,x_6$
- $x_5 = x_1, \quad x_6 = x_2, \quad x_1 = x_3, \quad x_2 = x_4, \quad x_3 = x_5, \quad x_4 = x_6$
  which combine into $\quad x_1 = x_3 = x_5, \quad x_2 = x_4 = x_6.$
- So $\mathrm{Fix}(\rho^2)$ consists of words of the form $x = x_1\,x_2\,x_1\,x_2\,x_1\,x_2$.

- **For two colors $\mathtt{B}$, $\mathtt{W}$:**
  2 choices for x_1 times 2 choices for x_2 gives 4 fixed points.
- **For k colors:** $|\mathrm{Fix}(\rho^2)| = k^2$.

Second method

- Fill in one letter at a time and look at all the places it moves.
- $x = a - - - - -$
- $\rho^2(x) = x$ copies the a over 2 positions so $x = a - a - - -$
  Do it again and get $x = a - a - a -$
- Fill in another letter, $x = aba - a -$.
- $\rho^2(x) = x$ copies the b over 2 positions so $x = ababa -$
  and doing it again gives $x = ababab$.

Fixed points of ρ^4 for strings of length 6

- $$\rho^4(x_1\,x_2\,x_3\,x_4\,x_5\,x_6) = x_3\,x_4\,x_5\,x_6\,x_1\,x_2$$
  $\rho^2(x) = x$ gives $\quad x_1 = x_3 = x_5, \quad x_2 = x_4 = x_6$
  so $\mathrm{Fix}(\rho^4)$ also consists of words of the form $x_1\,x_2\,x_1\,x_2\,x_1\,x_2$.

First explanation

- $\rho^2 \cdot \rho^2 = \rho^4$ so elements fixed by ρ^2 are also fixed by ρ^4.
  $\rho^4 \cdot \rho^4 = \rho^2$ so elements fixed by ρ^4 are also fixed by ρ^2.
  Thus $\mathrm{Fix}(\rho^2) = \mathrm{Fix}(\rho^4)$.
- **General rule:** In $\mathcal{C}_n, \quad \mathrm{Fix}(\rho^m) = \mathrm{Fix}(\rho^d)$ where $d = \gcd(m, n)$.

Second explanation

- ρ^2 (rotate 2 forwards / clockwise) and ρ^4 (rotate 2 backwards / counterclockwise) are inverses.
- Suppose $g(x) = x$. Apply g^{-1} to both sides to get $x = g^{-1}(x)$.
- **General rule:** In any group $G, \quad \mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$ for all $g \in G$.

Fixed points of ρ^m for strings of length 6

- $\rho(x_1\,x_2\,x_3\,x_4\,x_5\,x_6) = x_6\,x_1\,x_2\,x_3\,x_4\,x_5$
  $\rho^5(x_1\,x_2\,x_3\,x_4\,x_5\,x_6) = x_2\,x_3\,x_4\,x_5\,x_6\,x_1$
  $\rho(x) = x$ and $\rho^5(x) = x$ both give $x_1 = \cdots = x_6$,
  so $\mathrm{Fix}(\rho) = \mathrm{Fix}(\rho^5)$ consists of words of the form $x_1\,x_1\,x_1\,x_1\,x_1\,x_1$.
- **For B,W:** this gives $\mathrm{Fix}(\rho) = \{\texttt{BBBBBB}, \texttt{WWWWWW}\}$.
  For k colors: there are k choices of x_1 so $|\mathrm{Fix}(\rho)| = k$.

- $\rho^3(x_1\,x_2\,x_3\,x_4\,x_5\,x_6) = x_4\,x_5\,x_6\,x_1\,x_2\,x_3$
  $\rho^3(x) = x$ gives $\quad x_1 = x_4, \quad x_2 = x_5, \quad x_3 = x_6,$
  so $\mathrm{Fix}(\rho^3)$ consists of words $x_1\,x_2\,x_3\,x_1\,x_2\,x_3$.
- **For B,W:** $|\mathrm{Fix}(\rho^3)| = 2^3 = 8$
  For k colors: $|\mathrm{Fix}(\rho^3)| = k^3$

Fixed points of ρ^m for strings of length 6

g	Form of words fixed by g	$\lvert\mathbf{Fix}(g)\rvert$ B,W	k colors
1	$x_1\,x_2\,x_3\,x_4\,x_5\,x_6$	$2^6 = 64$	k^6
ρ, ρ^5	$x_1\,x_1\,x_1\,x_1\,x_1\,x_1$	2	k
ρ^2, ρ^4	$x_1\,x_2\,x_1\,x_2\,x_1\,x_2$	$2^2 = 4$	k^2
ρ^3	$x_1\,x_2\,x_3\,x_1\,x_2\,x_3$	$2^3 = 8$	k^3

Lemma (Burnside's Lemma)

The number of orbits of G on X is

$$\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}(g)|$$

In other words, the number of orbits is the average number of fixed points per group element.

Ferris wheel with 6 seats and colors `B`, `W`

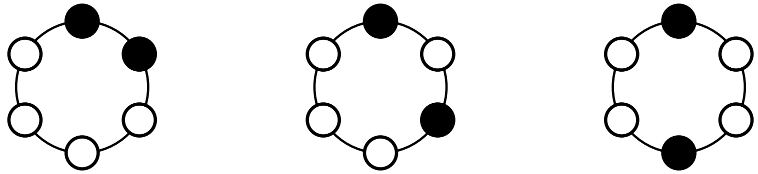
g	**Form of words**	$\lvert\mathbf{Fix}(g)\rvert$
1	$x_1\,x_2\,x_3\,x_4\,x_5\,x_6$	$2^6=64$
ρ	$x_1\,x_1\,x_1\,x_1\,x_1\,x_1$	2
ρ^2	$x_1\,x_2\,x_1\,x_2\,x_1\,x_2$	$2^2=4$
ρ^3	$x_1\,x_2\,x_3\,x_1\,x_2\,x_3$	$2^3=8$
ρ^4	$x_1\,x_2\,x_1\,x_2\,x_1\,x_2$	$2^2=4$
ρ^5	$x_1\,x_1\,x_1\,x_1\,x_1\,x_1$	2

The number of orbits is

$$\frac{64+2+4+8+4+2}{6}=\frac{84}{6}=14$$

Ferris wheel with 4 white seats and 2 black seats

- Consider Ferris wheels with 4 white seats and 2 black seats.
- These are represented by rearrangements of the string WWWWBB.
- There are $\binom{6}{4,2} = \frac{6!}{4!\,2!} = 15$ such strings.
- Some are equivalent upon rotation, leaving:



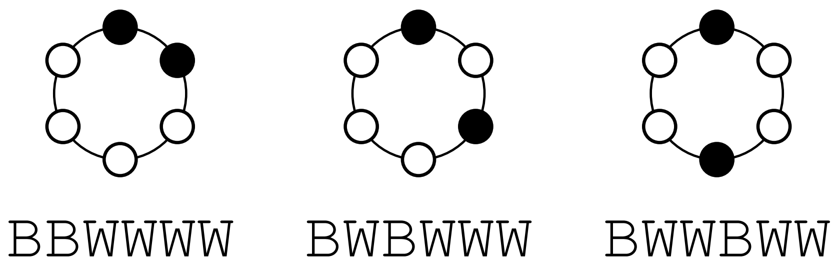
BBWWWW BWBWWW BWWBWW

Ferris wheel with 4 white seats and 2 black seats

g	**Form of words**	$\lvert \mathbf{Fix}(g) \rvert$
1	$x_1\,x_2\,x_3\,x_4\,x_5\,x_6$	$\binom{6}{2} = 15$ ways to choose 2 black
ρ	$x_1\,x_1\,x_1\,x_1\,x_1\,x_1$	0 since all 6 seats are same color
ρ^2	$x_1\,x_2\,x_1\,x_2\,x_1\,x_2$	0 since 3 seats are x_1 and 3 are x_2
ρ^3	$x_1\,x_2\,x_3\,x_1\,x_2\,x_3$	$\binom{3}{1} = 3$ ways to choose which x_i is black
ρ^4	$x_1\,x_2\,x_1\,x_2\,x_1\,x_2$	0 since 3 seats are x_1 and 3 are x_2
ρ^5	$x_1\,x_1\,x_1\,x_1\,x_1\,x_1$	0 since all 6 seats are same color

The number of orbits is

$$\frac{15+0+0+3+0+0}{6} = \frac{18}{6} = 3$$



Proof of Burnside's Lemma

We'll count the size of $A = \{\, (g, x) \;:\; g \in G,\ x \in S,\ g(x) = x \,\}$ in two ways; one based on each $g \in G$, one based on each $x \in S$.

Counting first by $g \in G$

For each $g \in G$, the values of x with $g(x) = x$ form $\operatorname{Fix}(g)$, so

$$|A| = \sum_{g \in G} |\operatorname{Fix}(g)|$$

Counting first by $x \in S$

For each x, the values of g with $g(x) = x$ form $\operatorname{Stab}(x)$, so

$$|A| = \sum_{x \in S} |\operatorname{Stab}(x)|$$

We'll show this equals the number of orbits times $|G|$.

Putting the two counts together

$$|A| = \sum_{g \in G} |\operatorname{Fix}(g)| = \sum_{x \in S} |\operatorname{Stab}(x)| = \text{number of orbits times } |G|$$

so the number of orbits is $\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|$

- $\sum_{x \in S} |\operatorname{Stab}(x)|$ organized by orbits (each row is a complete orbit):

BBBBBB						6
BBBBBW	WBBBBB	BWBBBB	BBWBBB	BBBWBB	BBBBWB	$+1+1+1+1+1+1$
BBBBWW	WBBBBW	WWBBBB	BWWBBB	BBWWBB	BBBWWB	$+1+1+1+1+1+1$
BBBWBW	WBBBWB	BWBBBW	WBWBBB	BWBWBB	BBWBWB	$+1+1+1+1+1+1$
BBBWWW	WBBBWW	WWBBBW	WWWBBB	BWWWBB	BBWWWB	$+1+1+1+1+1+1$
BBWBBW	WBBWBB	BWBBWB				$+2+2+2$
BBWBWW	WBBWBW	WWBBWB	BWWBBW	WBWWBB	BWBWWB	$+1+1+1+1+1+1$
BBWWBW	WBBWWB	BWBBWW	WBWBBW	WWBWBB	BWWBWB	$+1+1+1+1+1+1$
BBWWWW	WBBWWW	WWBBWW	WWWBBW	WWWWBB	BWWWWB	$+1+1+1+1+1+1$
BWBWBW	WBWBWB					$+3+3$
BWBWWW	WBWBWW	WWBWBW	WWWBWB	BWWWBW	WBWWWB	$+1+1+1+1+1+1$
BWWBWW	WBWWBW	WWBWWB				$+2+2+2$
BWWWWW	WBWWWW	WWBWWW	WWWBWW	WWWWBW	WWWWWB	$+1+1+1+1+1+1$
WWWWWW						$+6$
						$= 6 \cdot 14 = 84$

- By the Orbit-Stabilizer Theorem, in each row (orbit), all stabilizers have the same size and sum to $|$orbit$| \cdot |$stabilizer$| = |G|$.
- Summing $|\operatorname{Stab}(x)|$ over all $x \in S$ gives $|G|$ times the # of orbits.

Proof of Burnside's Lemma

$A = \{\, (g,x) \ : \ g \in G,\ x \in S,\ g(x) = x \,\}$

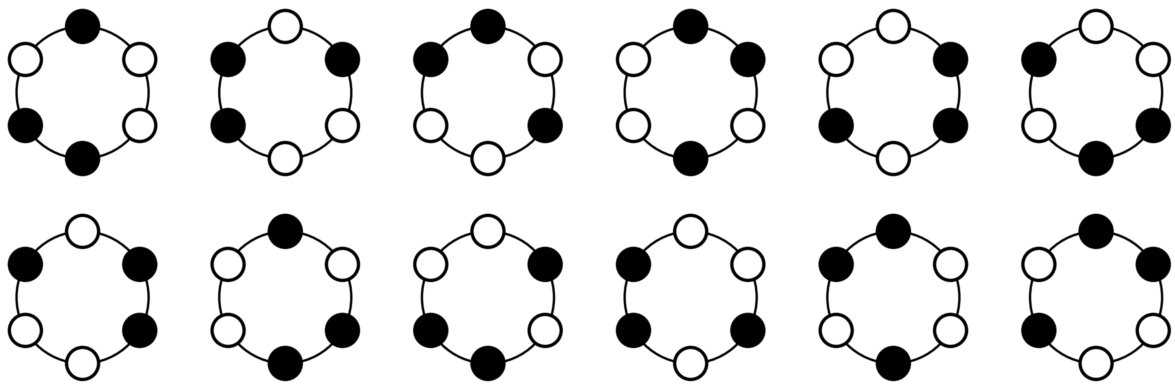
Counting first by $x \in S$

- Split S into orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$; these partition the set S.
- For each x, the values of g with $g(x) = x$ form $\operatorname{Stab}(x)$, so
$$|A| = \sum_{x \in S} |\operatorname{Stab}(x)|$$
- For each $x \in \mathcal{O}_i$, $\operatorname{Stab}(x) = \frac{|G|}{|\operatorname{Orb}(x)|} = \frac{|G|}{|\mathcal{O}_i|}$.
- $\displaystyle\sum_{x \in \mathcal{O}_i} |\operatorname{Stab}(x)| = \frac{|G|}{|\mathcal{O}_i|} \cdot |\mathcal{O}_i| = |G|$
- $\displaystyle |A| = \sum_{i=1}^{N} \sum_{x \in \mathcal{O}_i} |\operatorname{Stab}(x)| = \sum_{i=1}^{N} |G| = N\,|G|$

- Equating the two counts gives $|A| = \sum_{g \in G} |\operatorname{Fix}(g)| = N\,|G|$.
- Dividing by $|G|$ gives the number of orbits, $N = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|$.

Reflections

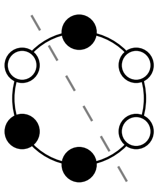
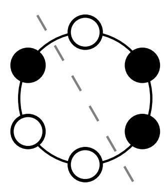
- Now we have rotations and reflections regarded as equivalent:



- Let $\sigma(x_1\, x_2\, \ldots\, x_n) = x_n\, \ldots\, x_2\, x_1$ (reverse a string):

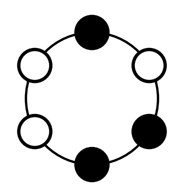
$$\sigma(\texttt{CALIFORNIA}) = \texttt{AINROFILAC}$$

- σ describes this mirror image:

 $\mapsto$ 

$\texttt{BWWBBBW}$ $\quad \sigma(\texttt{BWWBBBW}) = \texttt{WBBBWWB}$

$\rho\sigma$ is reflect and then rotate:



$\rho\sigma(\texttt{BWWBBBW}) = \texttt{BWBBBWW}$

- Note $\sigma^2 = 1$ and $\sigma\rho^m = \rho^{-m}\sigma$.

Mixing rotations and reflections

$$\sigma\rho(\texttt{ABCDE}) = \sigma(\texttt{EABCD}) = \texttt{DCBAE}$$

$$\rho^{-1}\sigma(\texttt{ABCDE}) = \rho^{-1}(\texttt{EDCBA}) = \texttt{DCBAE}$$

vs. $\rho\sigma(\texttt{ABCDE}) = \rho(\texttt{EDCBA}) = \texttt{AEDCB}$

- Notice $\sigma\rho = \rho^{-1}\sigma$, *NOT* $\rho\sigma$, because σ inverts the order of the characters.
- In general, $\sigma\rho^m = \rho^{-m}\sigma$ for any integer m.

Simplifying products

Simplify any product of ρ's, σ's, and powers

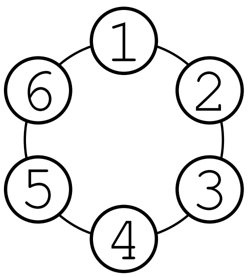
- Use $\sigma\rho^m = \rho^{-m}\sigma$ to move σ's to the right and ρ's to the left.
- Combine powers and simplify with $\sigma^2 = 1$ and $\rho^6 = 1$.
- Keep going until the final form: ρ^k or $\rho^k\sigma$ with $k = 0, \ldots, 5$.

$$\sigma\rho^2\sigma^3\rho^4\sigma\rho^{-1} = \sigma\rho^2\sigma^3\rho^4\rho\sigma$$
$$\sigma\rho^2\sigma^3\rho^4\rho\sigma = \sigma\rho^2\sigma^3\rho^5\sigma$$
$$\sigma\rho^2\sigma^3\rho^5\sigma = \sigma\rho^2\sigma\rho^5\sigma = \cdots$$

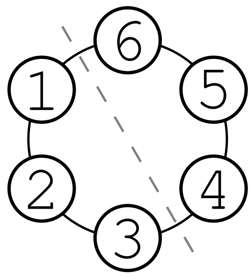
$$\cdots = \sigma\rho^2\sigma\rho^5\sigma = \sigma\rho^2\rho^{-5}\sigma\sigma$$
$$\sigma\rho^2\rho^{-5}\sigma\sigma = \sigma\rho^{-3}\cdot 1 = \sigma\rho^{-3}$$
$$\sigma\rho^{-3} = \boxed{\boldsymbol{\rho^3\sigma}}$$

Reflections $\rho^m\sigma$

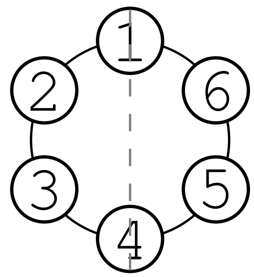
$\rho^m\sigma$ is a reflection across an axis at angle $(120-30m)^\circ$ (polar coords.).
These are all the reflections that keep the spots in the same positions.



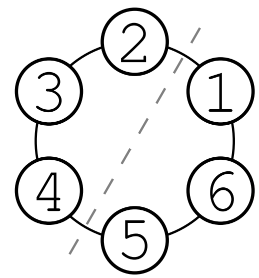
x



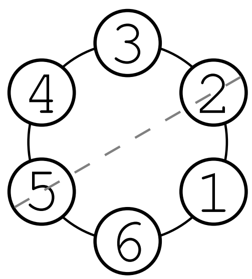
$\sigma(x),\ 120^\circ$



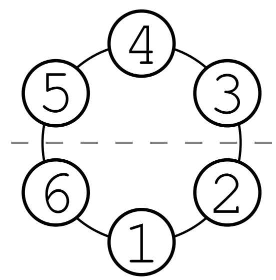
$\rho\sigma(x),\ 90^\circ$



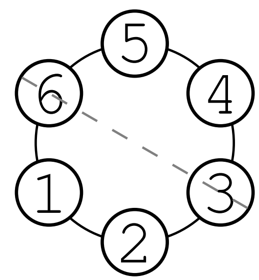
$\rho^2\sigma(x),\ 60^\circ$



$\rho^3\sigma(x),\ 30^\circ$



$\rho^4\sigma(x),\ 0^\circ$



$\rho^5\sigma(x),\ -30^\circ$

Dihedral group $\mathcal{D}_{2n}$ (for $n \geqslant 3$)

- Let $\mathcal{D}_{2n} = \{\underbrace{1, \rho, \rho^2, \ldots, \rho^{n-1}}_{\text{Rotations}}, \underbrace{\sigma, \rho\sigma, \rho^2\sigma, \ldots, \rho^{n-1}\sigma}_{\text{Reflections}}\}$
- This is a noncommutative group with $2n$ elements for $n \geqslant 3$ (some are degenerate for $n = 1, 2$).
- Simplify multiplications using $\sigma^2 = 1$, $\rho^n = 1$, and $\sigma\rho^m = \rho^{-m}\sigma$.
- Some disciplines and books use the notation $\mathcal{D}_n$ instead of $\mathcal{D}_{2n}$. Always check which definition is in use.

For $n = 6$ and $G = \mathcal{D}_{12}$,
$\mathrm{Orb}(\mathtt{BWWBBBW})$ has 12 elements, each stabilized only by the identity.

$\mathrm{Orb}(\mathtt{BWWBWW}) = \{\mathtt{BWWBWW}, \mathtt{WBWWBW}, \mathtt{WWBWWB}\}$

$$\mathrm{Stab}(\mathtt{BWWBWW}) = \{1, \rho^3, \rho\sigma, \rho^4\sigma\}$$
$$\mathrm{Stab}(\mathtt{WBWWBW}) = \{1, \rho^3, \sigma, \rho^3\sigma\}$$
$$\mathrm{Stab}(\mathtt{WWBWWB}) = \{1, \rho^3, \rho^2\sigma, \rho^5\sigma\}$$

The stabilizers are different but all have the same size, $\frac{|G|}{|\text{orbit}|} = \frac{12}{3} = 4$.

$\mathrm{Fix}(g)$ in dihedral group $\mathcal{D}_{12}$ on strings of length 6

$\mathrm{Fix}(\sigma)$

- $\sigma(x_1\, x_2\, x_3\, x_4\, x_5\, x_6) = x_6\, x_5\, x_4\, x_3\, x_2\, x_1$
- $x = \sigma(x)$ is $\quad x_1\, x_2\, x_3\, x_4\, x_5\, x_6 = x_6\, x_5\, x_4\, x_3\, x_2\, x_1$
  so $\quad x_1 = x_6, \quad x_2 = x_5, \quad x_3 = x_4$
- Elements of $\mathrm{Fix}(\sigma)$ have form $x = x_1\, x_2\, x_3\, x_3\, x_2\, x_1$.
- For 2 color necklaces: $2^3 = 8$ elements; for k colors, k^3 elements.

Second method

- Fill in one letter at a time and look at all the places it moves.
- $x = a - - - - -$
- $\sigma(x) = \ - - - - - a$ so $\sigma(x) = x$ gives $x = a - - - - a$.
- $\sigma(a - - - - a) = a - - - - a$, so a's are completed.
- $\sigma(ab - - - \ a) = a - - - \ ba$ so $\sigma(x) = x$ gives $x = ab - - ba$.
- $\sigma(abc - ba) = ab - \ cba$ so $x = abccba$.
- For 2 color necklaces: $2^3 = 8$ elements; for k colors, k^3 elements.

$\mathrm{Fix}(g)$ in dihedral group $\mathcal{D}_{12}$ on strings of length 6

$\mathrm{Fix}(\rho\sigma)$

- $\rho\sigma(x_1\,x_2\,x_3\,x_4\,x_5\,x_6) = x_1\,x_6\,x_5\,x_4\,x_3\,x_2$
- $x = \rho\sigma(x)$ is $\quad x_1\,x_2\,x_3\,x_4\,x_5\,x_6 = x_1\,x_6\,x_5\,x_4\,x_3\,x_2$
  giving $\quad x_1, x_4$ unrestricted, $\quad x_2 = x_6, \quad x_3 = x_5$
- Elements of $\mathrm{Fix}(\rho\sigma)$ have form $x = x_1\,x_2\,x_3\,x_4\,x_3\,x_2$.
- For 2 colors, $2^4 = 16$ elements; for k colors, k^4 elements.

Second method

- Fill in one letter at a time and look at all the places it moves.
- $x = a - - - - -$
- $\rho\sigma(x) = a - - - - -$, so a's are completed.
- $\rho\sigma(ab - - - -) = a - - - - b$, so $\rho\sigma(x) = x$ gives $x = ab - - - \; b$.
- $\rho\sigma(abc - - b) = ab - - cb$, so $x = abc - cb$.
- $x = abcdcb$.
- For 2 colors, $2^4 = 16$ elements; for k colors, k^4 elements.

Counting necklaces with 6 black and white beads

g	Form of words	$\lvert\mathbf{Fix}(g)\rvert$ `B,W`	k colors
1	$x_1\,x_2\,x_3\,x_4\,x_5\,x_6$	$2^6=64$	k^6
$\rho,\ \rho^5$	$x_1\,x_1\,x_1\,x_1\,x_1\,x_1$	2	k
$\rho^2,\ \rho^4$	$x_1\,x_2\,x_1\,x_2\,x_1\,x_2$	$2^2=4$	k^2
ρ^3	$x_1\,x_2\,x_3\,x_1\,x_2\,x_3$	$2^3=8$	k^3
σ	$x_1\,x_2\,x_3\,x_3\,x_2\,x_1$	$2^3=8$	k^3
$\rho\sigma$	$x_1\,x_2\,x_3\,x_4\,x_3\,x_2$	$2^4=16$	k^4
$\rho^2\sigma$	$x_1\,x_1\,x_3\,x_4\,x_4\,x_3$	$2^3=8$	k^3
$\rho^3\sigma$	$x_1\,x_2\,x_1\,x_4\,x_5\,x_4$	$2^4=16$	k^4
$\rho^4\sigma$	$x_1\,x_2\,x_2\,x_1\,x_5\,x_5$	$2^3=8$	k^3
$\rho^5\sigma$	$x_1\,x_2\,x_3\,x_2\,x_1\,x_6$	$2^4=16$	k^4

For 6 bead necklaces made from black and white beads, Burnside's Lemma gives the number of orbits:

$$\frac{64+2(2+4)+8+8+16+8+16+8+16}{12}=\frac{156}{12}=13$$

Differences in book's notation

	Slides	**Our textbook**
Composition	Right-to-left $(gh)(x) = g(h(x))$	Left-to-right $(gh)(x) = h(g(x))$
Stabilizer	$\mathrm{Stab}(x)$	G_x
Orbit	$\mathrm{Orb}(x)$	x^G
Fixed points	$\mathrm{Fix}(g)$	F_g
Subgroup	$H \subseteq G$	$H \leqslant G$

Orbit-Stabilizer Theorem	$\lvert\mathrm{Orb}(x)\rvert \cdot \lvert\mathrm{Stab}(x)\rvert = \lvert G\rvert$	$\lvert x^G\rvert \cdot \lvert G_x\rvert = \lvert G\rvert$

Burnside's Lemma # orbits =	$\frac{1}{\lvert G\rvert} \sum_{g\in G} \lvert\mathrm{Fix}(g)\rvert$	$\frac{1}{\lvert G\rvert} \sum_{g\in G} \lvert F_g\rvert$