4.4–4.5 Geometric and Negative Binomial Distributions

Prof. Tesler

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Geometric Distribution

- Consider a biased coin with probability p of heads.
- Flip it repeatedly (potentially ∞ times).
- Let X be the number of flips until the first head.
- **Example:** $TTTHTTHHT$ has $X = 4$.
- The pdf is

$$p_X(k) = \begin{cases} (1-p)^{k-1}p & \text{for } k = 1, 2, 3, \ldots; \\ 0 & \text{otherwise} \end{cases}$$

- Mean: $\mu = 1/p$
  Variance: $\sigma^2 = (1-p)/p^2$
  Standard deviation: $\sigma = \sqrt{1-p}/p$

Negative binomial distribution

- Consider a biased coin with probability p of heads.
- Flip it repeatedly (potentially ∞ times).
- Let X be the number of flips until the rth head ($r = 1, 2, 3, \ldots$ is a fixed parameter).
- For $r = 3$, $TTTHTHHTTH$ has $X = 7$.
- $X = k$ when
  - **first $k - 1$ flips:** $r - 1$ heads and $k - r$ tails in any order;
  - **kth flip:** heads

  so the pdf is

$$p_X(k) = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r} \cdot p = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

provided $k = r, r+1, r+2, \ldots$;

$$p_X(k) = 0 \text{ otherwise.}$$

Negative binomial distribution – mean and variance

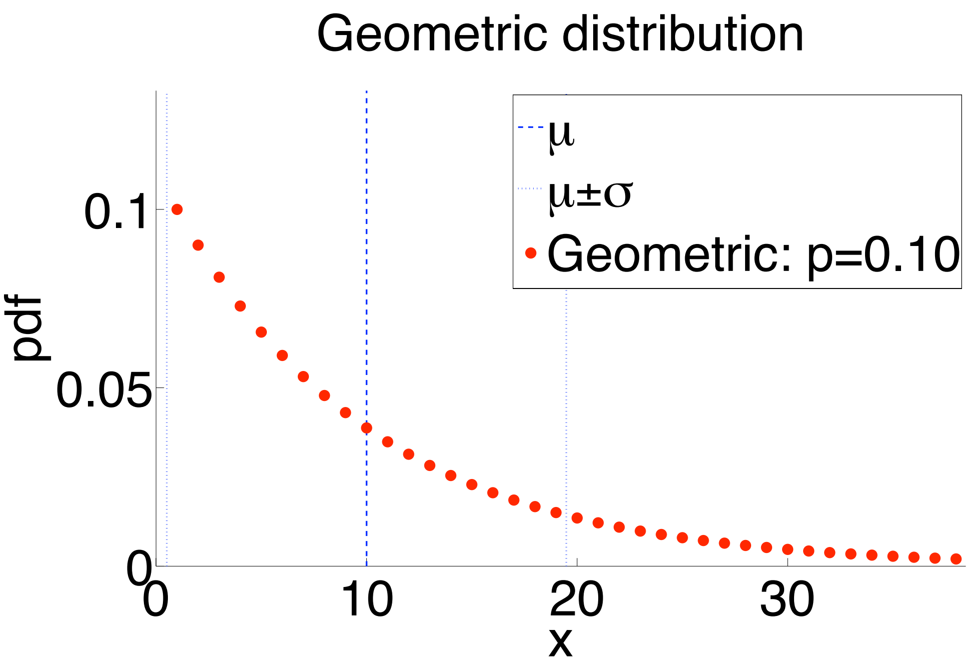
- Consider the sequence of flips *TTTHTHHTTH*.
- Break it up at each heads:

$$\underbrace{TTTH}_{X_1=4} / \underbrace{TH}_{X_2=2} / \underbrace{H}_{X_3=1} / \underbrace{TTH}_{X_4=3}$$

- X_1 is the number of flips until the first heads;
  X_2 is the number of additional flips until the 2nd heads;
  X_3 is the number of additional flips until the 3rd heads; ...
- The X_i's are i.i.d. geometric random variables with parameter p, and $X = X_1 + \cdots + X_r$.
- Mean: $E(X) = E(X_1) + \cdots + E(X_r) = 1/p + \cdots + 1/p = r/p$
  Variance: $\sigma^2 = \frac{1-p}{p^2} + \cdots + \frac{1-p}{p^2} = \frac{r(1-p)}{p^2}$
  Standard deviation: $\sigma = \sqrt{r(1-p)}/p$

Geometric distribution – example

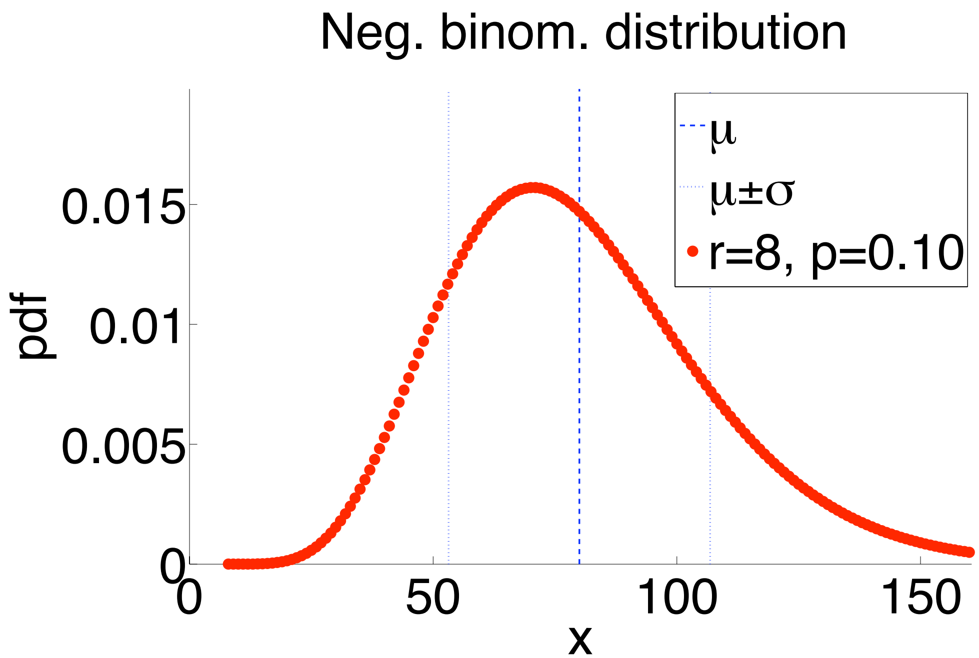
- About 10% of the population is left-handed.
- Look at the handedness of babies in birth order in a hospital.
- **Number of births until first left-handed baby:**
  Geometric distribution with $p = .1$.



- The mean is $1/p = 10$.
  The standard deviation is $\sigma = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{.9}}{.1} \approx 9.487$, which is HUGE!

Negative binomial distribution – examples

- **Number of births until 8th left-handed baby:**
  Negative binomial, $r = 8$, $p = .1$.



- The mean is $r/p = 8/.1 = 80$.
  The standard deviation is $\sqrt{r(1-p)}/p = \sqrt{8(.9)}/.1 \approx 26.833$.
- **Probability the 50th baby is the 8th left-handed one:**
  Use the negative binomial pdf:

$$\binom{50-1}{8-1}(.1)^8(.9)^{50-8} = \binom{49}{7}(.1)^8(.9)^{42} \approx 0.0103$$

Where the distribution names come from

Geometric series

- For real a, x with $|x| < 1$,

$$\frac{a}{1-x} = \sum_{i=0}^{\infty} a\,x^i$$
$$= a + ax + ax^2 + \cdots$$

- The total probability for the geometric distribution is

$$\sum_{k=1}^{\infty} (1-p)^{k-1} p$$
$$= \frac{p}{1-(1-p)}$$
$$= \frac{p}{p} = 1$$

Negative binomial series

- For integer $r > 0$ and real x with $|x| < 1$,

$$\frac{1}{(1-x)^r} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} x^{k-r}$$

- The total probability for the negative binomial distribution is

$$\sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r}$$
$$= p^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} (1-p)^{k-r}$$
$$= p^r \cdot \frac{1}{(1-(1-p))^r} = 1$$

Geometric and negative binomial – versions

Unfortunately, there are 4 versions of the definitions of these distributions, so you may see them defined differently elsewhere:

- Version 1: the definitions we already did.
- Version 2 (geometric): Let Y be the number of tails before the first heads, so $TTTHTTHHT$ has $Y = 3$.

$$\text{pdf:} \quad p_Y(k) = \begin{cases} (1-p)^k p & \text{for } k = 0, 1, 2, \ldots; \\ 0 & \text{otherwise} \end{cases}$$

  Since $Y = X - 1$, we have $E(Y) = \frac{1}{p} - 1$, $\mathrm{Var}(Y) = \frac{1-p}{p^2}$.
- Version 2 (negative binomial): Let Y be the number of tails before the rth heads, so $Y = X - r$.

$$p_Y(k) = \begin{cases} \binom{k+r-1}{r-1} p^r (1-p)^k & \text{for } k = 0, 1, 2, \ldots; \\ 0 & \text{otherwise} \end{cases}$$

- Versions 3 and 4: switch the roles of heads and tails in the first two versions (so p and $1-p$ are switched).