

3.8 Functions of random variables

3.7, 3.9, 3.11 Multiple random variables (discrete)

Prof. Tesler

Math 186
February 1, 2008

3.8 One random variable as a function of another random variable

Function of a Discrete Random Variable

- Let $X =$ roll of a biased die, $Y = 10X + 2$, and $Z = (X - 3)^2$.

x	$p_X(x)$	$y = 10x + 2$	$p_Y(y)$	$z = (x - 3)^2$	$p_Z(z)$
1	q_1	12	q_1	4	
2	q_2	22	q_2	1	
3	q_3	32	q_3	0	$p_Z(0) = q_3$
4	q_4	42	q_4	1	$p_Z(1) = q_2 + q_4$
5	q_5	52	q_5	4	$p_Z(4) = q_1 + q_5$
6	q_6	62	q_6	9	$p_Z(9) = q_6$

- For $W = g(X)$, the pdf $p_W(w)$ is the sum of $p_X(x)$ over all possible inverses $x = g^{-1}(w)$, or $p_W(w) = 0$ if there are no inverses:

$$p_W(w) = \sum_{x : g(x) = w} p_X(x)$$

Function of a Discrete Random Variable

- Let $X =$ roll of a biased die, $Y = 10X + 2$, and $Z = (X - 3)^2$.

x	$p_X(x)$	$y = 10x + 2$	$p_Y(y)$	$z = (x - 3)^2$	$p_Z(z)$
1	q_1	12	q_1	4	
2	q_2	22	q_2	1	
3	q_3	32	q_3	0	$p_Z(0) = q_3$
4	q_4	42	q_4	1	$p_Z(1) = q_2 + q_4$
5	q_5	52	q_5	4	$p_Z(4) = q_1 + q_5$
6	q_6	62	q_6	9	$p_Z(9) = q_6$

- For $Y = aX + b$ with $a \neq 0$, the unique inverse is $X = \frac{Y-b}{a}$, so

$$p_Y(y) = p_X\left(\frac{y-b}{a}\right)$$

Function of a Discrete Random Variable

- Let $Y = X^2$.
- Inverses of $y = x^2$ are

$$x = \begin{cases} \pm\sqrt{y} & \text{if } y > 0; \\ 0 & \text{if } y = 0; \\ \text{none} & \text{if } y < 0. \end{cases}$$



$$P(Y = y) = P(X^2 = y) = \begin{cases} P(X = \sqrt{y}) + P(X = -\sqrt{y}) & \text{if } y > 0; \\ P(X = 0) & \text{if } y = 0; \\ 0 & \text{if } y < 0. \end{cases}$$

- Use pdf notation to express pdf of Y in terms of pdf of X :

$$p_Y(y) = \begin{cases} p_X(\sqrt{y}) + p_X(-\sqrt{y}) & \text{if } y > 0; \\ p_X(0) & \text{if } y = 0; \\ 0 & \text{if } y < 0. \end{cases}$$

Function of a Continuous Random Variable

- Let U be uniform on the real interval $[1, 7]$, so

$$f_U(u) = \begin{cases} 1/6 & \text{if } 1 \leq u \leq 7 \text{ (in real numbers);} \\ 0 & \text{otherwise} \end{cases}$$

- Let $V = 2U$. This is uniform on $[2, 14]$, so

$$f_V(v) = \begin{cases} 1/12 & \text{if } 2 \leq v \leq 14 \text{ (in real numbers);} \\ 0 & \text{otherwise} \end{cases}$$

- Even though there's just one inverse for each value, the probabilities of corresponding values are different!

Continuous Random Variables:

Computing $f_Y(y)$ from $f_X(x)$ when $Y = g(X)$

Procedure

- Let $Y = g(X)$.
- Compute $F_X(x)$.
- Compute $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$
 - Details depend on the function.
 - Typically it is expressed in terms of $F_X(x)$ at various values of x (the ones where $g(x) = y$).
- Compute $f_Y(y) = \frac{d}{dy}F_Y(y)$.

Example with one-to-one function

- Let $f_X(x) = 8e^{-8x}$ for $x \geq 0$.
- It's a valid pdf since it's ≥ 0 and the total probability is 1:

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x) dx &= \int_0^{\infty} 8e^{-8x} dx = \left. \frac{8e^{-8x}}{-8} \right|_{x=0}^{\infty} \\ &= -(e^{-\infty} - e^0) = -(0 - 1) = 1\end{aligned}$$

- The CDF is

$$\begin{aligned}F_X(x) &= \int_{-\infty}^x f_X(t) dt = \int_0^x 8e^{-8t} dt = \left. \frac{8e^{-8t}}{-8} \right|_{t=0}^x \\ &= -(e^{-8x} - e^0) = 1 - e^{-8x}\end{aligned}$$

provided $x \geq 0$, while $F_X(x) = 0$ if $x < 0$.

- We'll compute the pdf of $Y = 10X + 2$.

Example with one-to-one function

$$f_X(x) = \begin{cases} 8e^{-8x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases} \quad F_X(x) = \begin{cases} 1 - e^{-8x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases} \quad Y = 10X + 2$$

Wrong Way: $f_Y(y) = f_X(g^{-1}(y))$ as in the discrete case

- Try $f_Y(y) = f_X((y-2)/10)$:

$$f_Y(y) = \begin{cases} 8e^{-8(y-2)/10} & \text{if } y \geq 2; \\ 0 & \text{if } y < 2. \end{cases}$$

- Compute the total probability:

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y) dy &= \int_2^{\infty} 8e^{-8(y-2)/10} dy = \frac{8e^{-8(y-2)/10}}{-8/10} \Big|_{y=2}^{\infty} \\ &= -10(e^{-\infty} - e^0) = -10(0 - 1) \\ &= 10 \neq 1 \end{aligned}$$

- The total probability is not 1, so this is not a valid pdf.

Example with one-to-one function

$$f_X(x) = \begin{cases} 8e^{-8x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases} \quad F_X(x) = \begin{cases} 1 - e^{-8x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases} \quad Y = 10X + 2$$

Right way: Compute CDF $F_Y(y)$ and differentiate $f_Y(y) = F_Y'(y)$

- Compute CDF $F_Y(y)$:

$$F_Y(y) = P(Y \leq y) = P(10X + 2 \leq y)$$

$$= P\left(X \leq \frac{y-2}{10}\right) = F_X\left(\frac{y-2}{10}\right) = \begin{cases} 1 - e^{-8(y-2)/10} & \text{if } y \geq 2; \\ 0 & \text{if } y < 2. \end{cases}$$

- Differentiate to get PDF $f_Y(y)$:

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{8}{10}e^{-8(y-2)/10} & \text{if } y > 2; \\ 0 & \text{if } y < 2. \end{cases}$$

- In general, if $f_X(x)$ is any continuous pdf and $Y = aX + b$ with $a \neq 0$, then $f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$.

Example with general function $Y = g(X)$

- Compute CDF and PDF of $Y = X^2$ where X is a continuous R.V.
- When $y \leq 0$,
 - $F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = 0$
 - $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} 0 = 0$
- When $y > 0$,
 - $F_Y(y) = P(Y \leq y) = P(X^2 \leq y)$
 $= P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$
 - $f_Y(y) = \frac{d}{dy} (F_X(\sqrt{y}) - F_X(-\sqrt{y}))$
 $= F_X'(\sqrt{y}) \left(\frac{1}{2\sqrt{y}} \right) - F_X'(-\sqrt{y}) \left(-\frac{1}{2\sqrt{y}} \right)$
 $= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$

3.7, 3.9, 3.11 Multiple random variables (discrete)

Multinomial distribution

- Consider a biased 6-sided die where q_i is the probability of rolling i for $i = 1, 2, \dots, 6$. (6 sides is an example, it could be any # sides.) Each q_i is between 0 and 1, and $q_1 + \dots + q_6 = 1$.
- The probability of a sequence of independent rolls is

$$P(1131326) = q_1 q_1 q_3 q_1 q_3 q_2 q_6 = q_1^3 q_2 q_3^2 q_6 = \prod_{i=1}^6 q_i^{\# i\text{'s}}$$

- Roll the die n times ($n = 0, 1, 2, 3, \dots$).

Let X_1 be the number of 1's, X_2 be the number of 2's, etc.

$$p_{X_1, X_2, \dots, X_6}(k_1, k_2, \dots, k_6) = P(X_1 = k_1, X_2 = k_2, \dots, X_6 = k_6)$$

$$= \begin{cases} \binom{n}{k_1, k_2, \dots, k_6} q_1^{k_1} q_2^{k_2} \dots q_6^{k_6} & \text{if } k_1, \dots, k_6 \text{ are integers } \geq 0 \text{ adding up to } n; \\ 0 & \text{otherwise.} \end{cases}$$

Genetics example

Consider a $TtRR \times TtRr$ cross of pea plants:

Punnett Square

	TR (1/2)	tR (1/2)
TR (1/4)	$TTRR$ (1/8)	$TtRR$ (1/8)
Tr (1/4)	$TTRr$ (1/8)	$TtRr$ (1/8)
tR (1/4)	$TtRR$ (1/8)	$ttRR$ (1/8)
tr (1/4)	$TtRr$ (1/8)	$ttRr$ (1/8)

Genotype	Prob.
$TTRR$	1/8
$TtRR$	2/8 = 1/4
$TTRr$	1/8
$TtRr$	2/8 = 1/4
$ttRR$	1/8
$ttRr$	1/8

Genetics example

- If there are 27 offspring, what is the probability that 9 offspring have genotype $TTRR$, 2 have genotype $TtRR$, 3 have genotype $TTRr$, 5 have genotype $TtRr$, 7 have genotype $ttRR$, and 1 has genotype $ttRr$?
- Use the multinomial distribution:

Genotype	Probability	Frequency
$TTRR$	$1/8$	9
$TtRR$	$1/4$	2
$TTRr$	$1/8$	3
$TtRr$	$1/4$	5
$ttRR$	$1/8$	7
$ttRr$	$1/8$	1
Total	1	27

$$P = \frac{27!}{9! 2! 3! 5! 7! 1!} \left(\frac{1}{8}\right)^9 \left(\frac{1}{4}\right)^2 \left(\frac{1}{8}\right)^3 \left(\frac{1}{4}\right)^5 \left(\frac{1}{8}\right)^7 \left(\frac{1}{8}\right)^1 \approx 2.19 \cdot 10^{-7}$$

Genetics example

- If there are **25** offspring, what is the probability that 9 offspring have genotype TTRR, 2 have genotype TtRR, 3 have genotype TTRr, 5 have genotype TtRr, 7 have genotype ttRR, and 1 has genotype ttRr?
- $P = 0$ because the numbers 9, 2, 3, 5, 7, 1 do not add up to 25.

Joint distribution — Example (Discrete Case)

- Roll a fair 6-sided die n times.
- Let $X = \#$ of 1's. X has a binomial distribution with probability $\frac{1}{6}$:

$$p_X(x) = \begin{cases} \binom{n}{x} (1/6)^x (5/6)^{n-x} & \text{if } x = 0, 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $Y = \#$ of 2's and 3's. Y has a binomial distribution with probability $\frac{2}{6} = \frac{1}{3}$:

$$p_Y(y) = \begin{cases} \binom{n}{y} (1/3)^y (2/3)^{n-y} & \text{if } y = 0, 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- There are x 1's, y 2's and 3's, and $n - x - y$ other numbers (4,5,6), so the joint pdf of X and Y is multinomial:

$$p_{X,Y}(x, y) = \begin{cases} \binom{n}{x, y, n-x-y} (1/6)^x (2/6)^y (3/6)^{n-x-y} & \text{provided } x, y \text{ are integers } \geq 0 \text{ and } x + y \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

Joint distribution — Probability Density Function

We will work with the joint pdf of X and Y for 3 rolls:

$$p_{X,Y}(x, y) = \begin{cases} \binom{3}{x,y,3-x-y} (1/6)^x (2/6)^y (3/6)^{3-x-y} \\ \quad \text{provided } x, y \text{ are integers } \geq 0 \text{ and } x + y \leq 3; \\ 0 \quad \text{otherwise.} \end{cases}$$

$p_{X,Y}(x, y)$	$x = 0$	1	2	3
$y = 0$	1/8	1/8	1/24	1/216
1	1/4	1/6	1/36	0
2	1/6	1/18	0	0
3	1/27	0	0	0

Joint distribution — Marginal densities

- | $p_{X,Y}(x, y)$ | $x = 0$ | 1 | 2 | 3 | Total
$p_Y(y)$ |
|-----------------|---------|-------|------|-------|-------------------|
| $y = 0$ | 1/8 | 1/8 | 1/24 | 1/216 | 8/27 |
| 1 | 1/4 | 1/6 | 1/36 | 0 | 4/9 |
| 2 | 1/6 | 1/18 | 0 | 0 | 2/9 |
| 3 | 1/27 | 0 | 0 | 0 | 1/27 |
| Total $p_X(x)$ | 125/216 | 25/72 | 5/72 | 1/216 | 1 |

- Total of column x :** $p_X(x) = \sum_y p_{X,Y}(x, y)$

- Total of row y :** $p_Y(y) = \sum_x p_{X,Y}(x, y)$

- $p_X(x)$ and $p_Y(y)$ are the **marginal densities** of $p_{X,Y}(x, y)$ because they are in the margins of the table.

- Grand total of table:** $\sum_x \sum_y p_{X,Y}(x, y) = 1$

Joint distribution — Marginal densities

- With multiple variables, the marginal density for each variable is obtained by summing over the others:

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x, y, z)$$

$$p_Y(y) = \sum_x \sum_z p_{X,Y,Z}(x, y, z)$$

$$p_Z(z) = \sum_x \sum_y p_{X,Y,Z}(x, y, z)$$

- Summing over one or more variables gives the joint PDF in the remaining variable(s):

$$p_{X,Y}(x, y) = \sum_z p_{X,Y,Z}(x, y, z)$$

Joint distribution — Functions

- The pdf of $Z = g(X, Y)$ is

$$p_Z(z) = \sum_{(x,y): g(x,y)=z} p_{X,Y}(x, y)$$

- For $Z = X + Y$, this is

$$p_Z(z) = \sum_x p_{X,Y}(x, z - x)$$

- $p_Z(2) = p_{X,Y}(0, 2) + p_{X,Y}(1, 1) + p_{X,Y}(2, 0) = \frac{1}{6} + \frac{1}{6} + \frac{1}{24} = \frac{3}{8}$

$p_{X,Y}(x, y)$	$x = 0$	1	2	3
$y = 0$	1/8	1/8	1/24	1/216
1	1/4	1/6	1/36	0
2	1/6	1/18	0	0
3	1/27	0	0	0

Joint distribution — Conditional densities

- Suppose you know that $Y = 2$.

$p_{X,Y}(x, y)$	$x = 0$	1	2	3	Total $p_Y(y)$
$y = 0$	1/8	1/8	1/24	1/216	8/27
1	1/4	1/6	1/36	0	4/9
2	1/6	1/18	0	0	2/9
3	1/27	0	0	0	1/27
Total $p_X(x)$	125/216	25/72	5/72	1/216	1

$$P(X = 0|Y = 2) = \frac{P(X=0,Y=2)}{P(Y=2)} = \frac{1/6}{2/9} = \frac{3}{4}$$

$$P(X = 1|Y = 2) = \frac{P(X=1,Y=2)}{P(Y=2)} = \frac{1/18}{2/9} = \frac{1}{4}$$

$$P(X = x|Y = 2) = \frac{P(X=x,Y=2)}{P(Y=2)} = \frac{0}{2/9} = 0 \quad \text{if } x \neq 0, 1$$

- **Our book's notation:**

$$\begin{aligned} p_{X|Y=y}(x) = p_{X|y}(x) &= P(X = x|Y = y) \\ &= \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)} \end{aligned}$$

$$p_{X|2}(0) = 3/4 \quad p_{X|2}(1) = 1/4 \quad p_{X|2}(x) = 0 \text{ if } x \neq 0, 1$$

- Conditional density in column x instead of row y :

$$p_{Y|X=x}(y) = p_{Y|x}(y) = \frac{p_{X,Y}(x, y)}{p_X(x)}$$

Joint distribution — Independence

Discrete random variables X, Y, Z, \dots are **independent** if

$$p_{X,Y,Z}(x, y, z) = p_X(x) p_Y(y) p_Z(z) \quad \text{for all values of } x, y, z.$$

If there are any exceptions, they are not independent.

Example (Repeated trials)

- A biased die has probability q_i of showing $i = 1, \dots, 6$.
- U, V are the values of two independent rolls.
- $p_{U,V}(i, j) = \begin{cases} q_i q_j & \text{for } i \text{ and } j \text{ in } 1, \dots, 6; \\ 0 & \text{otherwise.} \end{cases}$
- U and V are independent.
- For n independent rolls U_1, \dots, U_n , the probability is multiplicative, so U_1, \dots, U_n are independent random variables.

Joint distribution — Independence

- In our running example, check $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all (x, y) :

$p_{X,Y}(x, y)$	$x = 0$	1	2	3	Total $p_Y(y)$
$y = 0$	1/8	1/8	1/24	1/216	8/27
1	1/4	1/6	1/36	0	4/9
2	1/6	1/18	0	0	2/9
3	1/27	0	0	0	1/27
Total $p_X(x)$	125/216	25/72	5/72	1/216	1

- $p_X(3)p_Y(2) = (1/216)(2/9) = 1/1072$ but $p_{X,Y}(3, 2) = 0$.
(There are other exceptions too, but it only takes one.)
So X and Y are *not* independent.

Joint Cumulative Distribution Function (CDF)

The **joint cumulative distribution function** (cdf) for X, Y, Z is

$$\begin{aligned} F_{X,Y,Z}(x, y, z) &= P(X \leq x, Y \leq y, Z \leq z) \\ &= \sum_{u \leq x} \sum_{v \leq y} \sum_{w \leq z} p_{X,Y,Z}(u, v, w) \end{aligned}$$

Joint Cumulative Distribution Function (CDF)

- $F_{X,Y}(1, 2) = 8/9$ by summing $p_{X,Y}(x, y)$ for $x = 0, 1, y = 0, 1, 2$:

pdf $p_{X,Y}(x, y)$	$x = 0$	1	2	3
$y = 0$	1/8	1/8	1/24	1/216
1	1/4	1/6	1/36	0
2	1/6	1/18	0	0
3	1/27	0	0	0

cdf $F_{X,Y}(x, y)$	$x = 0$	1	2	3
$y = 0$	1/8	1/4	7/24	8/27
1	3/8	2/3	53/72	20/27
2	13/24	8/9	23/24	26/27
3	125/216	25/27	215/216	1

Joint Cumulative Distribution Function (CDF)

Values of X, Y not in the range

- | cdf $F_{X,Y}(x, y)$ | $x = 0$ | 1 | 2 | 3 |
|---------------------|---------|-------|---------|-------|
| $y = 0$ | 1/8 | 1/4 | 7/24 | 8/27 |
| 1 | 3/8 | 2/3 | 53/72 | 20/27 |
| 2 | 13/24 | 8/9 | 23/24 | 26/27 |
| 3 | 125/216 | 25/27 | 215/216 | 1 |

- The pdf is 0 for values of (X, Y) not in the range:

$$p_{X,Y}(1.5, 2.3) = 0$$

But the cdf is usually NOT 0:

$$F_{X,Y}(1.5, 2.3) = P(X \leq 1.5, Y \leq 2.3) = P(X \leq 1, Y \leq 2) = F_{X,Y}(1, 2) = \frac{8}{9}$$

- $F_{X,Y}(-1, 2.5) = P(X \leq -1, Y \leq 2.5) = 0$
- $F_{X,Y}(10, 0) = P(X \leq 10, Y \leq 0) = P(X \leq 3, Y \leq 0) = F_{X,Y}(3, 0) = 8/27$

Expected Values for Joint pdfs

- Let $Z = g(X, Y)$ be a function of discrete random variables X, Y . The **expected value** of $g(X, Y)$ is

$$E(g(X, Y)) = \sum_x \sum_y g(X, Y) p_{X,Y}(x, y)$$

- If there are more random variables, use more Σ 's.
- $E(g(X, Y))$ has the same value as $E(Z)$:

$$E(Z) = \sum_z z p_Z(z)$$

- $E(g(X, Y))$ estimates the average

$$\frac{g(x_1, y_1) + \cdots + g(x_n, y_n)}{n}$$

where $(x_1, y_1), \dots, (x_n, y_n)$ are randomly chosen with probabilities given by the joint distribution.

Expected Values for Joint pdfs



pdf $p_{X,Y}(x, y)$	$x = 0$	1	2	3
$y = 0$	1/8	1/8	1/24	1/216
1	1/4	1/6	1/36	0
2	1/6	1/18	0	0
3	1/27	0	0	0



$$E(X + Y) =$$

$$\begin{aligned} & (0 + 0)(1/8) + (1 + 0)(1/8) + (2 + 0)(1/24) + (3 + 0)(1/216) \\ & + (0 + 1)(1/4) + (1 + 1)(1/6) + (2 + 1)(1/36) + (3 + 1)(0) \\ & + (0 + 2)(1/6) + (1 + 2)(1/18) + (2 + 2)(0) + (3 + 2)(0) \\ & + (0 + 3)(1/27) + (1 + 3)(0) + (2 + 3)(0) + (3 + 3)(0) \\ & = 3/2 \end{aligned}$$

- We will see next that this could be computed as

$$\begin{aligned} E(X + Y) &= E(X) + E(Y) \\ &= 3(1/6) + 3(2/6) = (1/2) + 1 = 3/2. \end{aligned}$$

Expected Values for Joint pdfs — Properties

Theorem

$E(X + Y) = E(X) + E(Y)$, provided $E(X)$ and $E(Y)$ are defined and finite.

Proof (discrete case).

$$\begin{aligned} E(X + Y) &= \sum_x \sum_y (x + y) p_{X,Y}(x, y) \\ &= \sum_x x \left(\sum_y p_{X,Y}(x, y) \right) + \sum_y y \left(\sum_x p_{X,Y}(x, y) \right) \\ &\quad \text{(inside sums evaluate to marginal densities)} \\ &= \sum_x x p_X(x) + \sum_y y p_Y(y) \\ &= E(X) + E(Y) \quad \square \end{aligned}$$

Expected Values for Joint pdfs — Properties

These addition properties hold for both discrete and continuous random variables:

- $E(X + Y + Z + \dots) = E(X) + E(Y) + E(Z) + \dots$
- $E(aX + bY + cZ + \dots) = aE(X) + bE(Y) + cE(Z) + \dots$
for constants a, b, c, \dots
- $E(g_1(X, Y, Z) + g_2(X, Y, Z) + \dots)$
 $= E(g_1(X, Y, Z)) + E(g_2(X, Y, Z)) + \dots$

***If X, Y, Z are independent,* there are two more properties:**

- $E(XYZ) = E(X)E(Y)E(Z)$
- $\text{Var}(X + Y + Z) = \text{Var}(X) + \text{Var}(Y) + \text{Var}(Z)$
- Those formulas extend to any number of independent variables.

If they are not independent, those properties may or may not hold.

Expected Value of a Product — Dependent Variables

Example (Dependent)

- Let U be the roll of a fair 6-sided die.
- Let V be the value of the exact same roll of the die ($U = V$).
- $E(U) = E(V) = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$ and $E(U)E(V) = \frac{49}{4}$.
- $$E(UV) = \frac{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 6}{6} = \frac{91}{6}$$

Example (Independent)

- Now let U, V be the values of two independent rolls of a fair 6-sided die.
- $$E(UV) = \sum_{x=1}^6 \sum_{y=1}^6 \frac{x \cdot y}{36} = \frac{441}{36} = \frac{49}{4}$$

and $E(U)E(V) = (7/2)(7/2) = 49/4$

Expected Value of a Product — Independent Variables

Proof of $E(XY) = E(X)E(Y)$ provided X, Y are independent:

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy p_{X,Y}(x, y) \\ &= \sum_x \sum_y xy p_X(x) p_Y(y) \quad \text{provided } X, Y \text{ are independent} \\ &= \left(\sum_x x p_X(x) \right) \left(\sum_y y p_Y(y) \right) \\ &= E(X)E(Y) \end{aligned}$$



Variance of a Sum — Dependent Variables

- We will show that if X, Y are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

Example (Dependent)

First consider this dependent example:

Let X be any non-constant random variable and $Y = -X$.

$$\text{Var}(X + Y) = \text{Var}(0) = 0$$

$$\begin{aligned}\text{Var}(X) + \text{Var}(Y) &= \text{Var}(X) + \text{Var}(-X) \\ &= \text{Var}(X) + (-1)^2 \text{Var}(X) = 2 \text{Var}(X)\end{aligned}$$

but usually $\text{Var}(X) \neq 0$ (the only exception would be if X is a constant).

Variance of a Sum — Independent Variables

Theorem

If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Proof.

$$\begin{aligned} E((X + Y)^2) &= E(X^2 + 2XY + Y^2) = E(X^2) + 2E(XY) + E(Y^2) \\ (E(X + Y))^2 &= (E(X) + E(Y))^2 = (E(X))^2 + 2E(X)E(Y) + (E(Y))^2 \end{aligned}$$

$$\begin{aligned} \text{Var}(X + Y) &= E((X + Y)^2) - (E(X + Y))^2 \\ &= (E(X^2) - (E(X))^2) \\ &\quad + 2(E(XY) - E(X)E(Y)) \\ &\quad + (E(Y^2) - (E(Y))^2) \\ &= \text{Var}(X) + 2(E(XY) - E(X)E(Y)) + \text{Var}(Y) \end{aligned}$$

If X, Y are independent then $E(XY) = E(X)E(Y)$ so the middle term vanishes. □

Mean and Variance of the Binomial Distribution

- A **Bernoulli trial** is a single flip of a coin, heads with probability p .
- Do n coin flips (n Bernoulli trials).

Let X_i be the number of heads on flip i : $X_i = \begin{cases} 1 & \text{flip } i \text{ is heads;} \\ 0 & \text{flip } i \text{ is tails.} \end{cases}$

- The total number of heads in all flips is $X = X_1 + X_2 + \cdots + X_n$.
- The variables X_1, \dots, X_n are independent and have the same pdfs. They are called **i.i.d. random variables**, for **independent identically distributed**.

- $$E(X_1) = 0(1 - p) + 1p = p$$
$$E(X_1^2) = 0^2(1 - p) + 1^2p = p$$
$$\text{Var}(X_1) = E(X_1^2) - (E(X_1))^2 = p - p^2 = p(1 - p)$$

- $E(X_i) = p$ and $\text{Var}(X_i) = p(1 - p)$ for all $i = 1, \dots, n$ because they are identically distributed.

Mean and Variance of the Binomial Distribution

- The total number of heads in all flips is $X = X_1 + X_2 + \cdots + X_n$.
- $E(X_i) = p$ and $\text{Var}(X_i) = p(1 - p)$ for all $i = 1, \dots, n$.

Mean:

$$\begin{aligned}\mu_X = E(X) &= E(X_1 + \cdots + X_n) \\ &= E(X_1) + \cdots + E(X_n) \\ &= p + \cdots + p = np \quad \text{identically distributed}\end{aligned}$$

Variance:

$$\begin{aligned}\sigma_X^2 = \text{Var}(X) &= \text{Var}(X_1 + \cdots + X_n) \\ &= \text{Var}(X_1) + \cdots + \text{Var}(X_n) \quad \text{by independence} \\ &= p(1 - p) + \cdots + p(1 - p) \quad \text{identically distributed} \\ &= np(1 - p) = npq\end{aligned}$$

Standard deviation:

$$\sigma_X = \sqrt{np(1 - p)} = \sqrt{npq}$$