4.4–4.5 Geometric and Negative Binomial Distributions

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Geometric Distribution

- Consider a biased coin with probability p of heads.
- Flip it repeatedly (potentially ∞ times).
- Let X be the number of flips until the first head.
- **Example:** $TTTHTTHHT$ has $X = 4$.
- The pdf is

$$p_X(k) = \begin{cases} (1-p)^{k-1}p & \text{for } k = 1, 2, 3, \ldots; \\ 0 & \text{otherwise} \end{cases}$$

- **Mean:** $\mu = \frac{1}{p}$ **Variance:** $\sigma^2 = \frac{1-p}{p^2}$ **Std dev:** $\sigma = \frac{\sqrt{1-p}}{p}$

Negative Binomial Distribution

- Consider a biased coin with probability p of heads.
- Flip it repeatedly (potentially ∞ times).
- Let X be the number of flips until the r^{th} head ($r = 1, 2, 3, \ldots$ is a fixed parameter).
- For $r = 3$, $TTTHTHHTTH$ has $X = 7$.
- $X = k$ when
  - **first $k-1$ flips:** $r-1$ heads and $k-r$ tails in any order;
  - **k^{th} flip:** heads

  so the pdf is
  $$p_X(k) = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r} \cdot p = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$
  provided $k = r, r+1, r+2, \ldots$;
  $$p_X(k) = 0 \quad \text{otherwise.}$$

Negative Binomial Distribution – mean and variance

- Consider the sequence of flips $TTTHTHHTTH$.
- Break it up at each heads:

$$\underbrace{TTTH}_{X_1=4}\,/\,\underbrace{TH}_{X_2=2}\,/\,\underbrace{H}_{X_3=1}\,/\,\underbrace{TTH}_{X_4=3}$$

- X_1 is the number of flips until the 1st heads;
  X_2 is the number of additional flips until the 2nd heads;
  X_3 is the number of additional flips until the 3rd heads; ...
- The X_i's are i.i.d. geometric random variables with parameter p, and $X = X_1 + \cdots + X_r$.
- **Mean:** $E(X) = E(X_1) + \cdots + E(X_r) = \frac{1}{p} + \cdots + \frac{1}{p} = \frac{r}{p}$

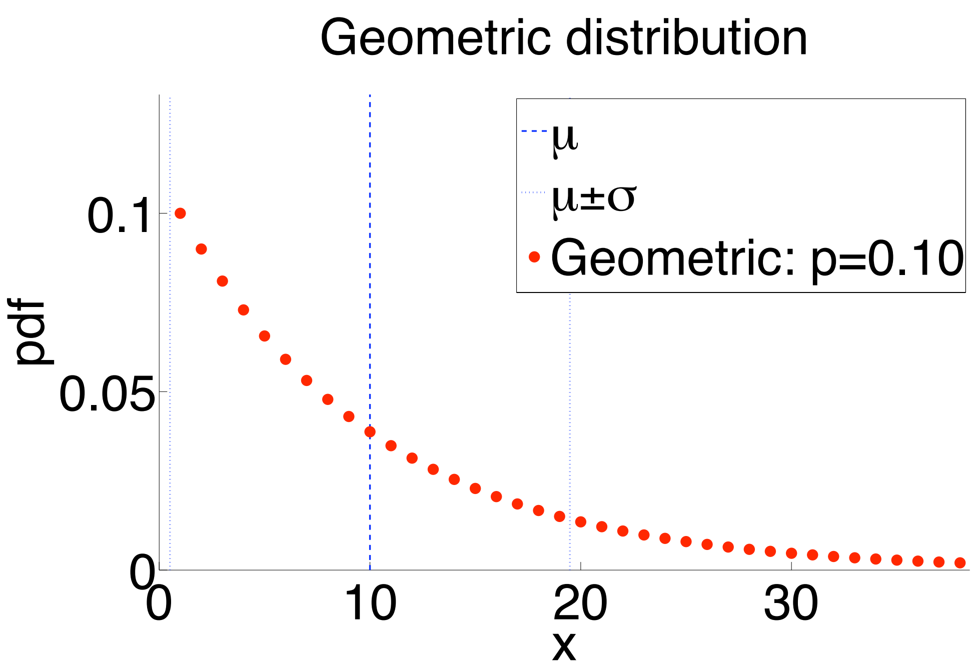
  Variance: $\sigma^2 = \frac{1-p}{p^2} + \cdots + \frac{1-p}{p^2} = \frac{r(1-p)}{p^2}$

  Standard deviation: $\sigma = \frac{\sqrt{r(1-p)}}{p}$

Geometric Distribution – example

- About 10% of the population is left-handed.
- Look at the handedness of babies in birth order in a hospital.
- **Number of births until first left-handed baby:**
  Geometric distribution with $p = .1$:

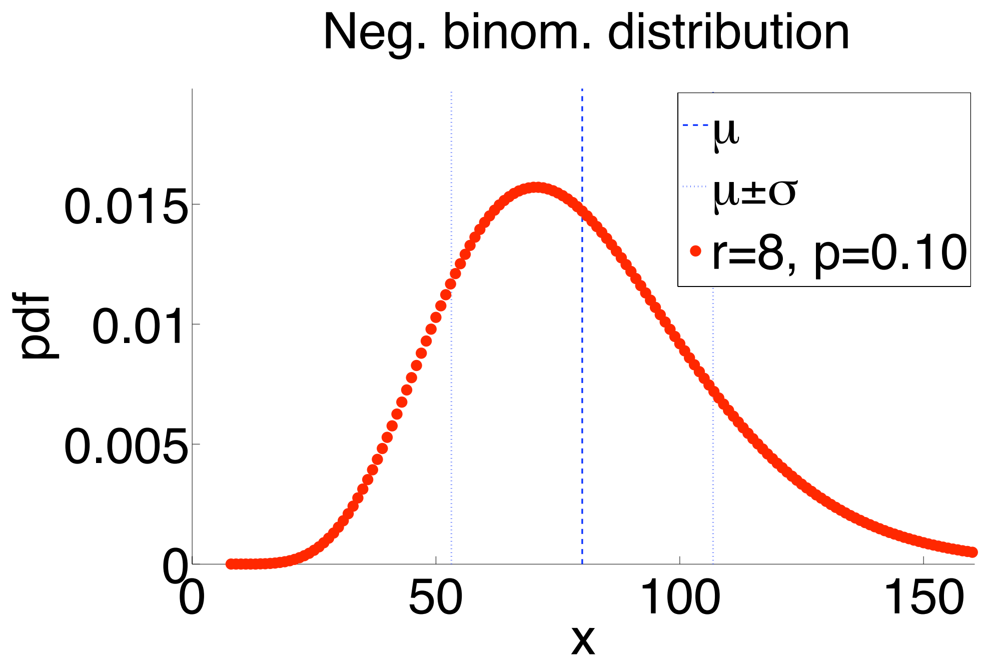
$$p_X(x) = .9^{x-1} \cdot .1 \qquad \text{for } x = 1, 2, 3, \ldots$$



- **Mean:** $\frac{1}{p} = \frac{1}{.1} = 10.$
  Standard deviation: $\sigma = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{.9}}{.1} \approx 9.487$, which is HUGE!

Negative Binomial Distribution – example

- **Number of births until 8th left-handed baby:**
  Negative binomial, $r = 8$, $p = .1$.
  $$p_X(x) = \binom{x-1}{8-1}(.1)^8(.9)^{x-8} \qquad \text{for } x = 8, 9, 10, \ldots$$



- **Mean:** $r/p = 8/.1 = 80$.
  Standard deviation: $\frac{\sqrt{r(1-p)}}{p} = \frac{\sqrt{8(.9)}}{.1} \approx 26.833$.
- **Probability the 50th baby is the 8th left-handed one:**
  $$p_X(50) = \binom{50-1}{8-1}(.1)^8(.9)^{50-8} = \binom{49}{7}(.1)^8(.9)^{42} \approx 0.0103$$

Where do the distribution names come from?

The PDFs correspond to the terms in certain Taylor series

Geometric series

- For real a, x with $|x| < 1$,
$$\frac{a}{1-x} = \sum_{i=0}^{\infty} a\,x^i$$
$$= a + ax + ax^2 + \cdots$$
- Total probability for the geometric distribution:
$$\sum_{k=1}^{\infty} (1-p)^{k-1} p$$
$$= \frac{p}{1-(1-p)}$$
$$= \frac{p}{p} = 1$$

Negative binomial series

- For integer $r > 0$ and real x with $|x| < 1$,
$$\frac{1}{(1-x)^r} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} x^{k-r}$$
- Total probability for the negative binomial distribution:
$$\sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r}$$
$$= p^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} (1-p)^{k-r}$$
$$= p^r \cdot \frac{1}{(1-(1-p))^r} = 1$$

Geometric and negative binomial – versions

Unfortunately, there are 4 versions of the definitions of these distributions, so you may see them defined differently elsewhere:

- Version 1: the definitions we already did (call the variable X).
- Version 2 (geometric): Let Y be the number of tails before the first heads, so $TTTHTTHHT$ has $Y = 3$.

  $$\text{pdf:} \quad p_Y(k) = \begin{cases} (1-p)^k p & \text{for } k = 0, 1, 2, \ldots; \\ 0 & \text{otherwise} \end{cases}$$

  Since $Y = X - 1$, we have $E(Y) = \frac{1}{p} - 1$, $\text{Var}(Y) = \frac{1-p}{p^2}$.
- Version 2 (negative binomial): Let Y be the number of tails before the r^{th} heads, so $Y = X - r$.

  $$p_Y(k) = \begin{cases} \binom{k+r-1}{r-1} p^r (1-p)^k & \text{for } k = 0, 1, 2, \ldots; \\ 0 & \text{otherwise} \end{cases}$$
- Versions 3 and 4: switch the roles of heads and tails in the first two versions (so p and $1 - p$ are switched).