Consider the SAT math scores again. Secretly, the mean is 500 and the standard deviation is 100.

**Chapter 5:** We assumed $\sigma = 100$ was known. We estimated $\mu$ from data as a confidence interval centered on the sample mean.

**Chapter 6:** We did hypothesis tests about $\mu$ under the same circumstances.

**Chapter 7:** Both $\mu$ and $\sigma$ are unknown. We estimate both of them from data, either for confidence intervals or hypothesis tests.

### Data

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<tr>
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</table>
Number of standard deviations $m$ is away from $\mu$ when $\mu = 500$ and $\sigma = 100$, for sample mean of $n = 6$ points

### Number of standard deviations if $\sigma$ is known:

The $z$-score of $m$ is

$$z = \frac{m - \mu}{\sigma / \sqrt{n}} = \frac{m - 500}{100 / \sqrt{6}}$$

### Estimating number of standard deviations if $\sigma$ is unknown:

The $t$-score of $m$ is

$$t = \frac{m - \mu}{s / \sqrt{n}} = \frac{m - 500}{s / \sqrt{6}}$$

- It uses sample standard deviation $s$ in place of $\sigma$.
- Note that $s$ is computed from the same data as $m$.
- $t$ has the same degrees of freedom as $s$; here, $df = n - 1 = 5$.
- The random variable is called $T_5$ ($T$ distribution with 5 degrees of freedom).
Number of standard deviations $m$ is away from $\mu$

### Data

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#1: $z = \frac{496.67 - 500}{100/\sqrt{6}} \approx -0.082$

$\quad t = \frac{496.67 - 500}{103.28/\sqrt{6}} \approx -0.079$

Close

#2: $z = \frac{468.33 - 500}{100/\sqrt{6}} \approx -0.776$

$\quad t = \frac{468.33 - 500}{66.76/\sqrt{6}} \approx -1.162$

Far

#3: $z = \frac{428.33 - 500}{100/\sqrt{6}} \approx -1.756$

$\quad t = \frac{428.33 - 500}{66.76/\sqrt{6}} \approx -2.630$

Far
In $z = \frac{m - \mu}{\sigma/\sqrt{n}}$, the numerator depends on $x_1, \ldots, x_n$ while the denominator is constant.

But in $t = \frac{m - \mu}{s/\sqrt{n}}$, both the numerator and denominator are functions of $x_1, \ldots, x_n$ (since $m$ and $s$ are functions of them).

The pdf of $t$ is no longer the standard normal distribution, but instead is a new distribution, $T_{n-1}$, the $t$-distribution with $n - 1$ degrees of freedom. ($d.f. = n - 1$)

The pdf is still symmetric and “bell-shaped,” but not the same “bell” as the normal distribution.

Degrees of freedom $d.f. = n - 1$ match here and in the $s^2$ formula.

As $d.f.$ rises, the curves get closer to the standard normal curve; the curves are really close for $d.f. \geq 30$.

This was developed in 1908 by William Gosset under the pseudonym “Student.” He worked at Guinness Brewery with small sample sizes, such as $n = 3$. 

---

Prof. Tesler

Ch. 7: One sample hypoth. tests for $\mu$, $\sigma$

Math 186 / Winter 2016
The curves from bottom to top (at $t = 0$) are for $d.f. = 1, 2, 10, 30$. The top one is the standard normal curve.
For the \( t \)-distribution with \( df \) degrees of freedom (random variable \( T_{df} \)), define \( t_{\alpha, df} \) so that

\[
P(T_{df} \geq t_{\alpha, df}) = \alpha.
\]

This is analogous to the standard normal distribution, where \( z_{\alpha} \) was defined so the area right of \( z_{\alpha} \) is \( \alpha \):

\[
P(Z \geq z_{\alpha}) = \alpha.
\]
Part of $t$-table in back of book (Larsen & Marx, p. 699)

Look up $t_{0.025,5} = 2.5706$

**TABLE A.2: Upper Percentiles of Student $t$ Distributions**

<table>
<thead>
<tr>
<th>df</th>
<th>0.20</th>
<th>0.15</th>
<th>0.10</th>
<th>0.05</th>
<th>0.025</th>
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<td>1.061</td>
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</table>
Confidence intervals for estimating $\mu$ from $m$

- In Chapter 5, we made 95% confidence intervals for $\mu$ from $m$ assuming we knew $\sigma$ (and it works for any $n$):

$$\left( m - 1.96 \frac{\sigma}{\sqrt{n}}, m + 1.96 \frac{\sigma}{\sqrt{n}} \right)$$

- We now replace $\sigma$ by the estimate $s$ from the data. 1.96 is replaced by a cutoff for $t$ for $6 - 1 = 5$ degrees of freedom.

To put 95% of the area in the center, 2.5% on the left, and 2.5% on the right, look up $t_{0.025,5} = 2.5706$ in the table in the book.

$$\left( m - \frac{2.5706 s}{\sqrt{6}}, m + \frac{2.5706 s}{\sqrt{6}} \right)$$

- Note that the cutoff 2.5706 depended on $df = n - 1 = 5$ and would change for other $n$’s; also, we still divide by $\sqrt{n} = \sqrt{6}$. 
Confidence intervals for estimating $\mu$ from $m$

Formulas for 2-sided $100(1 - \alpha)$% confidence interval for $\mu$

When $\sigma$ is known, use normal distribution

$$
\left( m - \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}, m + \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} \right)
$$

95% confidence interval ($\alpha = 0.05$) with $\sigma = 100$,
$z_{0.05} = 1.96$:

$$
\left( m - \frac{1.96(100)}{\sqrt{n}}, m + \frac{1.96(100)}{\sqrt{n}} \right)
$$

When $\sigma$ is not known, and $m, s$ estimated from same $n$ points

$$
\left( m - \frac{t_{\alpha/2,n-1} \cdot s}{\sqrt{n}}, m + \frac{t_{\alpha/2,n-1} \cdot s}{\sqrt{n}} \right)
$$

A 95% confidence interval ($\alpha = .05$) when $n = 6$;
$t_{0.05,5} = 2.5706$

$$
\left( m - \frac{2.5706 s}{\sqrt{6}}, m + \frac{2.5706 s}{\sqrt{6}} \right)
$$

- The cutoff $z = 1.96$ doesn’t depend on $n$, but $t = 2.5706$ does ($df = n - 1 = 5$) and would change for other values of $n$.
- In both versions, we divide by $\sqrt{n} = 6$. 
95% confidence intervals for \( \mu \)

<table>
<thead>
<tr>
<th>Exp. #</th>
<th>Data ( x_1, \ldots, x_6 )</th>
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When \( \sigma \) known (say \( \sigma = 100 \)), use normal distribution

\#1: \( (496.67 - \frac{1.96(100)}{\sqrt{6}}, 496.67 + \frac{1.96(100)}{\sqrt{6}}) = (416.65, 576.69) \)

\#2: \( (468.33 - \frac{1.96(100)}{\sqrt{6}}, 468.33 + \frac{1.96(100)}{\sqrt{6}}) = (388.31, 548.35) \)

\#3: \( (428.33 - \frac{1.96(100)}{\sqrt{6}}, 428.33 + \frac{1.96(100)}{\sqrt{6}}) = (348.31, 508.35) \)

When \( \sigma \) not known, estimate \( \sigma \) by \( s \) and use \( t \)-distribution

\#1: \( (496.67 - \frac{2.5706(103.28)}{\sqrt{6}}, 496.67 + \frac{2.5706(103.28)}{\sqrt{6}}) = (388.28, 605.06) \)

\#2: \( (468.33 - \frac{2.5706(66.76)}{\sqrt{6}}, 468.33 + \frac{2.5706(66.76)}{\sqrt{6}}) = (398.27, 538.39) \)

\#3: \( (428.33 - \frac{2.5706(66.76)}{\sqrt{6}}, 428.33 + \frac{2.5706(66.76)}{\sqrt{6}}) = (358.27, 498.39) \) (missing 500)
Hypothesis tests for $\mu$

Test $H_0: \mu = 500$ vs. $H_1: \mu \neq 500$ at significance level $\alpha = .05$

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When $\sigma$ is known (say $\sigma = 100$)

Reject $H_0$ when $|z| \geq z_{\alpha/2} = z_{.025} = 1.96$.

- #1: $z = -.082$, $|z| < 1.96$ so accept $H_0$.
- #2: $z = -.776$, $|z| < 1.96$ so accept $H_0$.
- #3: $z = -1.756$, $|z| < 1.96$ so accept $H_0$.

When $\sigma$ is not known, but is estimated by $s$

Reject $H_0$ when $|t| \geq t_{\alpha/2, n-1} = t_{.025,5} = 2.5706$.

- #1: $t = -.079$, $|t| < 2.5706$ so accept $H_0$.
- #2: $t = -1.162$, $|t| < 2.5706$ so accept $H_0$.
- #3: $t = -2.630$, $|t| \geq 2.5706$ so reject $H_0$. 
Use to compute approx. 2-sided $P$-values for $t = -0.079, -1.162, -2.630$, $d.f. = 5$

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$t$-test using $P$-values

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Test $H_0: \mu = 500$ vs. $H_1: \mu \neq 500$ at significance level $\alpha = .05$.

- **#1:** $P = P(T \leq -.079) + P(T \geq .079) > 2(.20) = .40$
  so $P > \alpha (P > .40 > .05)$ and we accept $H_0$.

- **#2:** $P = P(T \leq -1.16) + P(T \geq 1.16) \approx 2(.15) = .30$.
  Since $P > \alpha (.30 > .05)$, accept $H_0$.

- **#3:** $P = P(T \leq -2.63) + P(T \geq 2.63)$
  
  $P$ is between $2(.025) = .05$ and $2(.01) = .02$ based on the table.
  So $P \leq .05$ and we reject $H_0$.

- **On a calculator:** $\#1: P = .9401 \quad \#2: P = .2977 \quad \#3: P = .0465$
The $\chi^2$ ("Chi-squared") distribution (Chapter 7.5)

- Used for confidence intervals and hypothesis tests on the unknown parameter $\sigma^2$ of the normal distribution, based on the test statistic $s^2$ (sample variance):

$$\chi^2 = \frac{(n - 1)s^2}{\sigma_0^2} = \sum_{i=1}^{n} \frac{(x_i - m)^2}{\sigma_0^2} \quad d.f. = n - 1 \text{ (same as for } t)$$

**Point these out on the graphs:**

- The chi-squared distribution with $k$ degrees of freedom has
  
  **Range** $[0, \infty)$
  
  **Mean** $\mu = k$
  
  **Variance** $\sigma^2 = 2k$
  
  **Mode** $\chi^2 = k - 2$ (the pdf is maximum for $\chi^2 = k - 2$)
  
  **Median** Between $k$ and $k - \frac{2}{3}$.
  
  Asymptotically decreases $\rightarrow k - \frac{2}{3}$ as $k \rightarrow \infty$.

- Unlike $z$ and $t$, the pdf for $\chi^2$ is NOT symmetric, and the mean, median, and mode are different.
\( \chi^2 \) (“Chi-squared”) distribution — pdf graphs

The graphs for 1 and 2 degrees of freedom are decreasing:

The rest are “hump” shaped and skewed to the right:
Define $\chi^2_{\alpha, df}$ as the number where the cdf (area left of it) is $\alpha$:

$$P(\chi^2_{df} \leq \chi^2_{\alpha, df}) = \alpha$$

This is different than how our book did it for the $z$ and $t$-distributions, because this pdf isn’t symmetric.

We still put 95% of the area in the middle and 2.5% at each end, but the lower and upper cutoffs are determined separately instead of ± each other.
χ^2 table from book (Larsen & Marx, p. 702)

For two-sided test with \( \alpha = .05 \) and \( n = 6 \), look up
\[
\chi^2_{\alpha/2, n-1} = \chi^2_{.025, 5} = .831 \quad \text{and} \quad \chi^2_{1-\alpha/2, n-1} = \chi^2_{.975, 5} = 12.832
\]

<table>
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<tr>
<th>df</th>
<th>0.010</th>
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<th>0.050</th>
<th>0.10</th>
<th>0.90</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
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<td>0.00393</td>
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<td>3.841</td>
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<td>5.991</td>
<td>7.378</td>
<td>9.210</td>
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<td>0.216</td>
<td>0.352</td>
<td>0.584</td>
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</table>
Two-sided hypothesis test for variance

Test  \( H_0 : \sigma^2 = \sigma_0^2 \) \( \text{vs.} \)  \( H_1 : \sigma^2 \neq \sigma_0^2 \)

Decision procedure

Test  \( H_0 : \sigma^2 = 10000 \) \( \text{vs.} \)  \( H_1 : \sigma^2 \neq 10000 \) at sig. level  \( \alpha = .05 \) (so  \( \sigma_0 = 100 \))

1. Get a sample  \( x_1, \ldots, x_n \).
   650, 510, 470, 570, 410, 370 \quad \text{with} \quad n = 6

2. Calculate  \( m = \frac{x_1 + \cdots + x_n}{n} \) and  \( s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - m)^2 \).
   \( m = 496.67, \quad s^2 = 10666.67, \quad s = 103.28 \)

3. Calculate the test-statistic  \( \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \sum_{i=1}^{n} \frac{(x_i - m)^2}{\sigma_0^2} \)
   \( \chi^2 = \frac{(6-1)(10666.67)}{10000} = 5.33 \)

4. Accept  \( H_0 \) if  \( \chi^2 \) is between  \( \chi^2_{\alpha/2,n-1} \) and  \( \chi^2_{1-\alpha/2,n-1} \).
   Reject  \( H_0 \) otherwise.
   \( \chi^2_{.025,5} = .831 \quad \text{and} \quad \chi^2_{.975,5} = 12.832 \).
   Since  \( \chi^2 = 5.33 \) is between these, we accept  \( H_0 \).
   (Or, there is insufficient evidence to reject  \( \sigma^2 = 10000 \).)
Properties of Chi-squared distribution

1. **Definition of Chi-squared distribution:**

Let $Z_1, \ldots, Z_n$ be independent standard normal variables.

Let $\chi^2_n = Z_1^2 + \cdots + Z_n^2$.

The pdf of the random variable $\chi^2_n$ is the “chi-squared distribution with $n$ degrees of freedom.”

The book has the exact formula of the pdf (but you don’t need to know it).

2. **Pooling property:** If $X$ and $Y$ are independent $\chi^2$ random variables with $n$ and $m$ degrees of freedom respectively, then $X + Y$ is a $\chi^2$ random variable with $n + m$ degrees of freedom.