3.5–3.6 Expected values and variance

Prof. Tesler

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Expected Winnings in a Game

Setup

- A simple game:
  1. Pay \$1 to play once.
  2. Flip two fair coins.
  3. Win \$5 if HH, nothing otherwise.
- The payoff is $X = \begin{cases} \$5 & \text{with probability } 1/4; \\ \$0 & \text{with probability } 3/4. \end{cases}$
- The net winnings are

$$Y = X - 1 = \begin{cases} \$5 - \$1 = \$4 & \text{with probability } 1/4; \\ \$0 - \$1 = -\$1 & \text{with probability } 3/4. \end{cases}$$

- Playing the game once is called a trial.
- Playing the game n times is an experiment with n trials.

Expected Winnings in a Game

Average payoff

- If you play the game n times, the payoff will be \$5 about $n/4$ times and \$0 about $3n/4$ times, totalling
$$\$5 \cdot n/4 + \$0 \cdot 3n/4 = \$5n/4$$
- The expected payoff (long term average payoff) per game is obtained by dividing by n:
$$E(X) = \$5 \cdot 1/4 + \$0 \cdot 3/4 = \$1.25$$
- For the expected winnings (long term average winnings), subtract off the bet:
$$E(Y) = E(X-1) = \$4 \cdot 1/4 - \$1 \cdot 3/4 = \$0.25$$
- That's good for you and bad for the house.
- A fair game has expected winnings = \$0.
A game favors the player if the expected winnings are positive.
A game favors the house if the expected winnings are negative.

Expected Value of a Random Variable

(Technical name for long term average)

- The expected value of a discrete random variable X is

$$E(X) = \sum_x x \cdot p_X(x)$$

- The expected value of a continuous random variable X is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

- $E(X)$ is often called the mean value of X and is denoted μ (or μ_X if there is more than one random variable in the problem).
- μ doesn't have to be a value in the range of X. The previous example had range $X = \$0$ or $\$5$, and mean $\$1.25$.

Expected Value of Binomial Distribution

- Consider a biased coin with probability $p = 3/4$ for heads.
- Flip it 10 times and record the number of heads, x_1.
  Flip it another 10 times, get x_2 heads.
  Repeat to get $x_1, \cdots, x_{1000}$.
- **Estimate the average of $x_1, \ldots, x_{1000}$:** $10(3/4) = 7.5$
  (Later we'll show $E(X) = np$ for the binomial distribution.)
- **An estimate based on the pdf:**
  About $1000 p_X(k)$ of the x_i's equal k for each $k = 0, \ldots, 10$, so

$$\text{average of } x_i\text{'s} = \frac{\sum_{i=1}^{1000} x_i}{1000} \approx \frac{\sum_{k=0}^{10} k \cdot 1000\, p_X(k)}{1000} = \sum_{k=0}^{10} k \cdot p_X(k)$$

which is the formula for $E(X)$ in this case.

Interpretation of the word “Expected”

- Although $E(X) = 7.5$, this is not a possible value for X.
- Expected value does *not* mean we anticipate observing that value.
- It means the long term average of many independent measurements of X will be approximately $E(X)$.

Function of a Random Variable

- Let X be the value of a roll of a biased die and $Z = (X-3)^2$.

x	$p_X(x)$	$z = (x-3)^2$	$p_Z(z)$
1	q_1	4	
2	q_2	1	
3	q_3	0	$p_Z(0) = q_3$
4	q_4	1	$p_Z(1) = q_2 + q_4$
5	q_5	4	$p_Z(4) = q_1 + q_5$
6	q_6	9	$p_Z(9) = q_6$

- **pdf of X:** Each $q_i \geqslant 0$ and $q_1 + \cdots + q_6 = 1$.
- **pdf of Z:** Each probability is also $\geqslant 0$, and the total sum is also 1.

Expected Value of a Function

- Let X be the value of a roll of a biased die and $Z = (X-3)^2$.

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1	q_1	4	
2	q_2	1	
3	q_3	0	$p_Z(0) = q_3$
4	q_4	1	$p_Z(1) = q_2 + q_4$
5	q_5	4	$p_Z(4) = q_1 + q_5$
6	q_6	9	$p_Z(9) = q_6$

- $E(Z)$, in terms of values of Z and the pdf of Z, is

$$E(Z) = \sum_z z \cdot p_Z(z) = 0(q_3) + 1(q_2 + q_4) + 4(q_1 + q_5) + 9(q_6)$$

- Regroup it in terms of X:

$$= 4q_1 + 1q_2 + 0q_3 + 1q_4 + 4q_5 + 9q_6 = \sum_{x=1}^{6} (x-3)^2 p_X(x)$$

Expected Value of a Function

- Let X be a discrete random variable, and $g(X)$ be a function, such as $(X-3)^2$.
- The expected value of $g(X)$ is

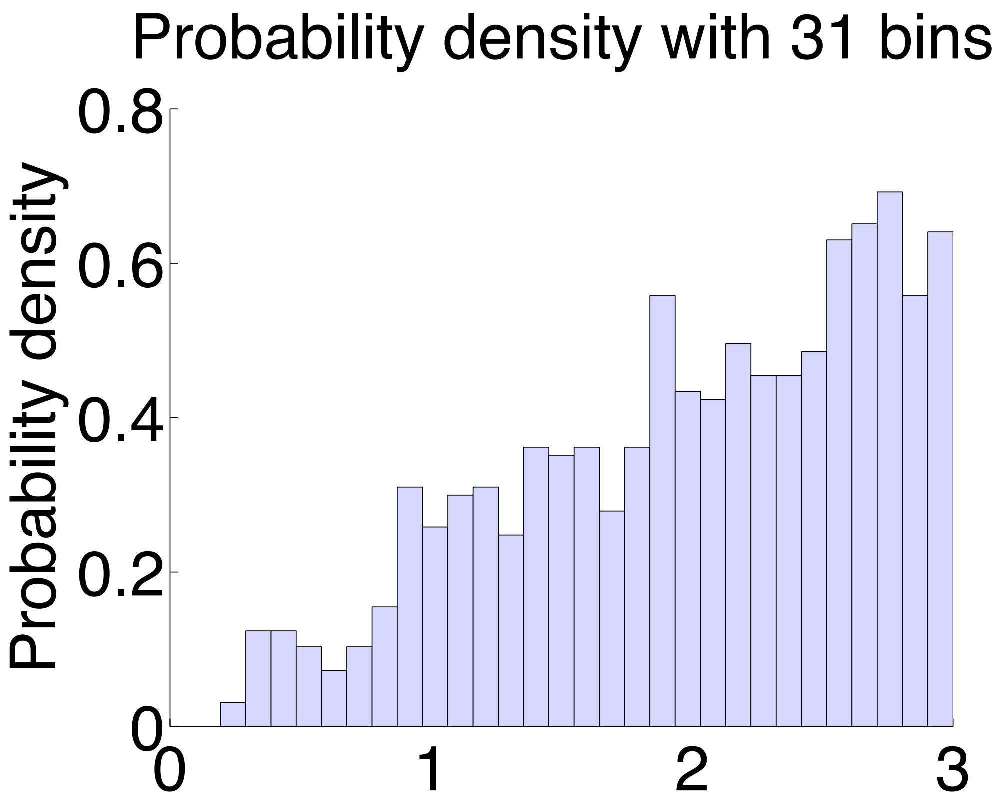
$$E(g(X)) = \sum_x g(x)\,p_X(x)$$

- For a continuous random variable,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\,f_X(x)\,dx$$

- Note that if $Z = g(X)$ then $E(Z) = E(g(X))$.

Expected Value of a Continuous distribution



- Consider the dartboard of radius 3 example, with pdf
$$f_R(r) = \begin{cases} 2r/9 & \text{if } 0 \leqslant r \leqslant 3; \\ 0 & \text{otherwise.} \end{cases}$$
- Throw n darts and make a histogram with k bins.
- $r_1, r_2, \ldots$ are representative values of R in each bin.
- The bin width is $\Delta r = 3/k$, the height is $\approx f_R(r_i)$, and the area is $\approx f_R(r_i)\,\Delta r$.
- The approximate number of darts in bin i is $n f_R(r_i)\,\Delta r$.

Expected Value of a Continuous Distribution

- The estimated average radius is

$$\frac{\sum_i r_i \cdot n f_R(r_i) \Delta r}{n} = \sum_i r_i \cdot f_R(r_i)\, \Delta r$$

- As $n, k \to \infty$, the histogram smoothes out and this becomes

$$\int_0^3 r \cdot f_R(r)\, dr$$

Mean of a continuous distribution

- Consider the dartboard of radius 3 example, with pdf

$$f_R(r) = \begin{cases} 2r/9 & \text{if } 0 \leqslant r \leqslant 3; \\ 0 & \text{otherwise.} \end{cases}$$

- The "average radius" (technically the mean radius or expected value of R) is

$$\begin{aligned} \mu = E(R) &= \int_{-\infty}^{\infty} r \cdot f_R(r)\,dr = \int_0^3 r \cdot \frac{2r}{9}\,dr = \int_0^3 \frac{2r^2}{9}\,dr \\ &= \left.\frac{2r^3}{27}\right|_0^3 = \frac{2(3^3 - 0^3)}{27} = 2 \end{aligned}$$

Expected Values — Properties

- The gambling slide earlier had $E(X-1)=E(X)-1$.

Theorem

$E(aX+b)=aE(X)+b$ *where* a,b *are constants.*

Proof (discrete case).

$$
\begin{aligned}
E(aX+b) &= \sum_x (ax+b)\cdot p_X(x) \\
&= a\sum_x x\cdot p_X(x) + b\sum_x p_X(x) \\
&= a\cdot E(X) + b\cdot 1 = aE(X)+b
\end{aligned}
$$

□

Expected Values — Properties

- The gambling slide earlier had $E(X-1) = E(X) - 1$.

Theorem

$E(aX+b) = aE(X)+b$ *where* a, b *are constants.*

Proof (continuous case).

$$\begin{aligned} E(aX+b) &= \int_{-\infty}^{\infty} (ax+b)\cdot f_X(x)\,dx \\ &= a\int_{-\infty}^{\infty} x\cdot f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx \\ &= a\cdot E(X) + b\cdot 1 = aE(X)+b \end{aligned}$$

$\square$

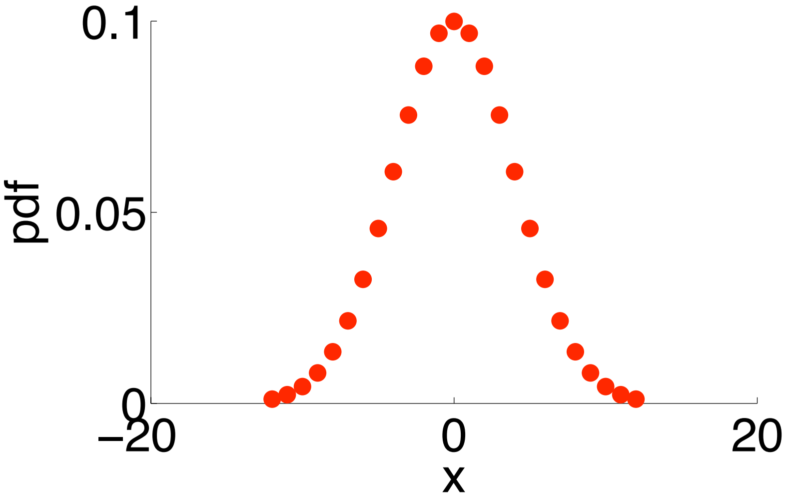
Expected Values — Properties

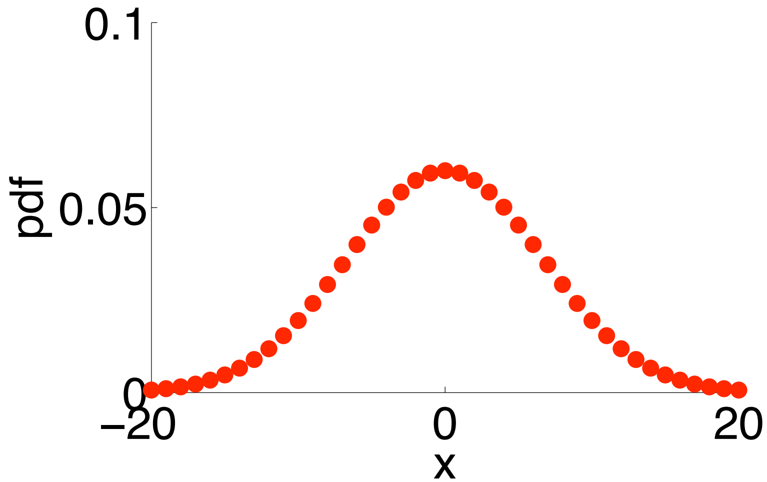
These properties hold for both discrete and continuous random variables:

- $E(aX + b) = a\,E(X) + b$ for any constants a, b.
- $E(aX) = a\,E(X)$
- $E(b) = b$
- $E\Big(g(X) + h(X)\Big) = E(g(X)) + E(h(X))$

Variance

- These distributions both have mean=0, but the right one is more spread out.





- The variance of X measures the square of the spread from the mean:

$$\sigma^2 = \operatorname{Var}(X) = E\left((X-\mu)^2\right)$$

- The standard deviation of X is $\sigma = \sqrt{\operatorname{Var}(X)}$ and measures how wide the curve is.

Variance

- The same definition $\operatorname{Var}(X) = E\big((X-\mu)^2\big)$ is used for both the discrete and continuous cases, but expected value is computed differently in the two cases.
- Why don't we use $E(X-\mu)$ or $E\big(|X-\mu|\big)$ to measure the spread?
  - $E(X-\mu) = E(X) - \mu = \mu - \mu = 0$, so it doesn't measure the spread.
  - Both $|X-\mu|$ and $(X-\mu)^2$ are nonnegative.
    We will see that $E\big((X-\mu)^2\big)$ leads to useful properties.
    It turns out that $E\big(|X-\mu|\big)$ does not have nice properties.

Variance formula $\sigma^2 = E\big((X-\mu)^2\big)$

- Consider the dartboard of radius 3 example, with pdf
$$f_R(r) = \begin{cases} 2r/9 & \text{if } 0 \leqslant r \leqslant 3; \\ 0 & \text{otherwise.} \end{cases}$$
- $\mu = 2$ from earlier slide.
- $$\begin{aligned} \sigma^2 = \operatorname{Var}(R) &= E\big((R-\mu)^2\big) = E\big((R-2)^2\big) \\ &= \int_{-\infty}^{\infty} (r-2)^2 f_R(r)\,dr = \int_0^3 \frac{(r-2)^2 \cdot 2r}{9}\,dr \\ &= \int_0^3 \frac{2r^3 - 8r^2 + 8r}{9}\,dr = \left(\frac{r^4}{18} - \frac{8r^3}{27} + \frac{4r^2}{9}\right)\Big|_0^3 \\ &= \left(\frac{3^4}{18} - \frac{8(3^3)}{27} + \frac{4(3^2)}{9}\right) - 0 = \frac{1}{2} \end{aligned}$$
- **Variance:** $\sigma^2 = \frac{1}{2}$ **Standard deviation:** $\sigma = \sqrt{1/2}$

Variance — Second formula

There are two equivalent formulas to compute variance.
In any problem, choose the easier one:

$$\sigma^2 = \mathrm{Var}(X) = E\big((X-\mu)^2\big) \quad \text{(Definition)}$$
$$= E(X^2) - \mu^2 \quad \text{(Sometimes easier to compute)}$$

Proof.

$$\begin{aligned}
\mathrm{Var}(X) &= E\big((X-\mu)^2\big) \\
&= E(X^2 - 2\mu X + \mu^2) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

□

Variance formula $\sigma^2 = E(R^2) - \mu^2$

- Consider the dartboard of radius 3 example, with pdf

$$f_R(r) = \begin{cases} 2r/9 & \text{if } 0 \leqslant r \leqslant 3; \\ 0 & \text{otherwise.} \end{cases}$$

- $\mu = E(R) = 2$

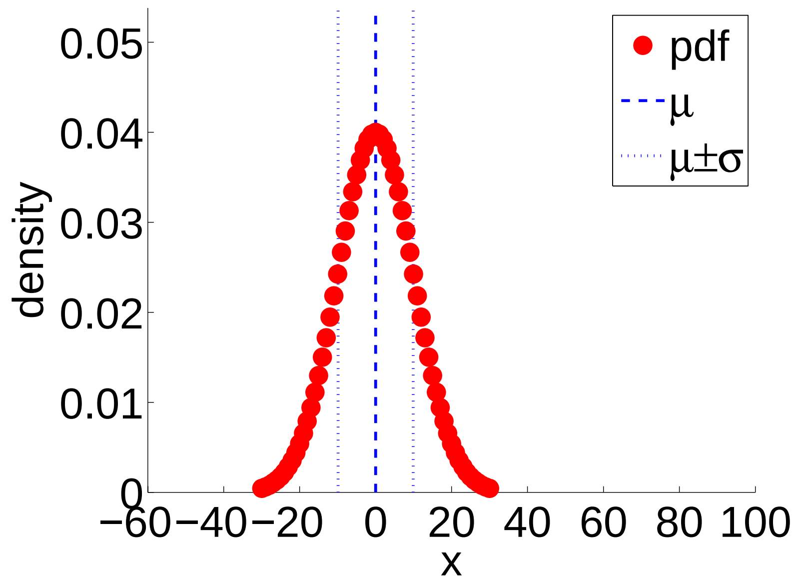
- $E(R^2) = \int_0^3 r^2 \cdot \frac{2r}{9}\,dr = \int_0^3 \frac{2r^3}{9}\,dr = \left.\frac{2r^4}{36}\right|_0^3 = \frac{2(81-0)}{36} = 9/2$

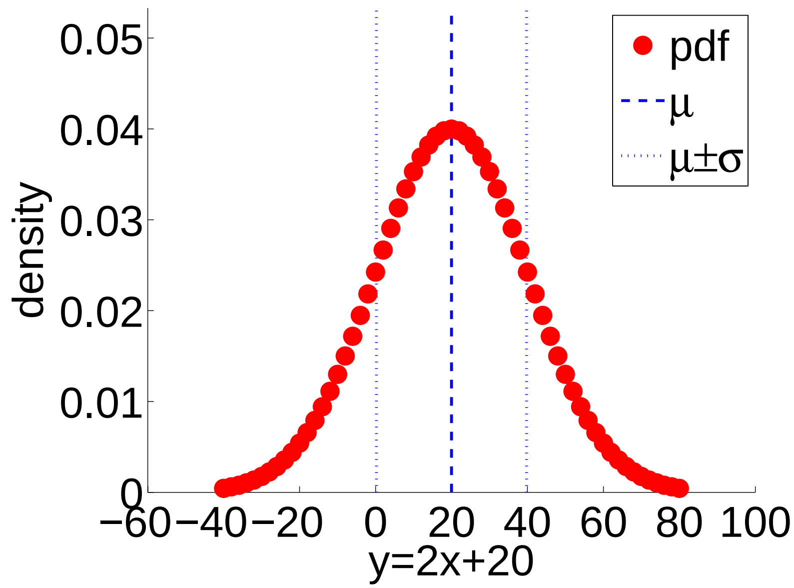
- **Variance:** $\sigma^2 = E(R^2) - \mu^2 = \frac{9}{2} - 2^2 = \frac{1}{2}$

  Standard deviation: $\sigma = \sqrt{1/2}$

Variance — Properties

- $\text{Var}(aX+b) = a^2\,\text{Var}(X)$



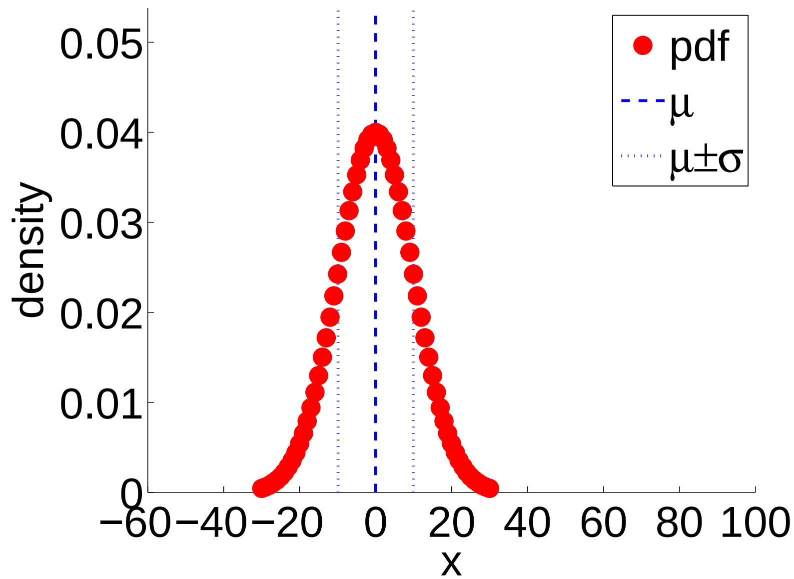


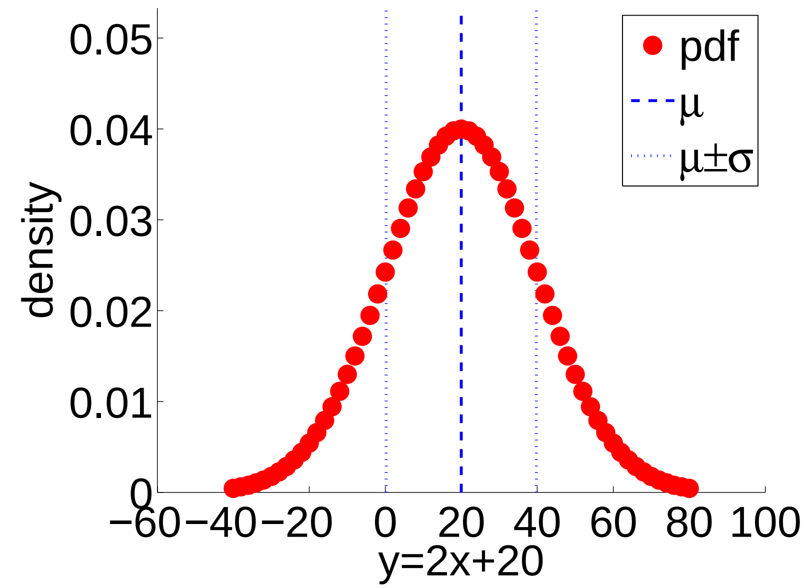
- Adding b shifts the curve without changing the width, so b disappears on the right side of the formula.
- Multiplying by a dilates the width a factor of a, so variance goes up a factor a^2.
- For $Y = aX + b$, we have $\sigma_Y = |a|\,\sigma_X$.
- **Example:** Convert measurements in $^\circ C$ to $^\circ F$:

  $F = (9/5)C + 32 \qquad \mu_F = (9/5)\mu_C + 32 \qquad \sigma_F = (9/5)\sigma_C$

Variance — Properties

- $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$





Proof of $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$.

$$\begin{aligned}
E((aX+b)^2) &= E(a^2X^2 + 2ab\,X + b^2) &&= a^2E(X^2) + 2ab\,E(X) + b^2\\
(E(aX+b))^2 &= (aE(X)+b)^2 &&= a^2(E(X))^2 + 2ab\,E(X) + b^2\\
\mathrm{Var}(aX+b) &= \text{difference} &&= a^2\left(E(X^2) - (E(X))^2\right)\\
& &&= a^2\,\mathrm{Var}(X) \qquad \square
\end{aligned}$$

Mean and Variance of the Binomial Distribution

- For the binomial distribution,

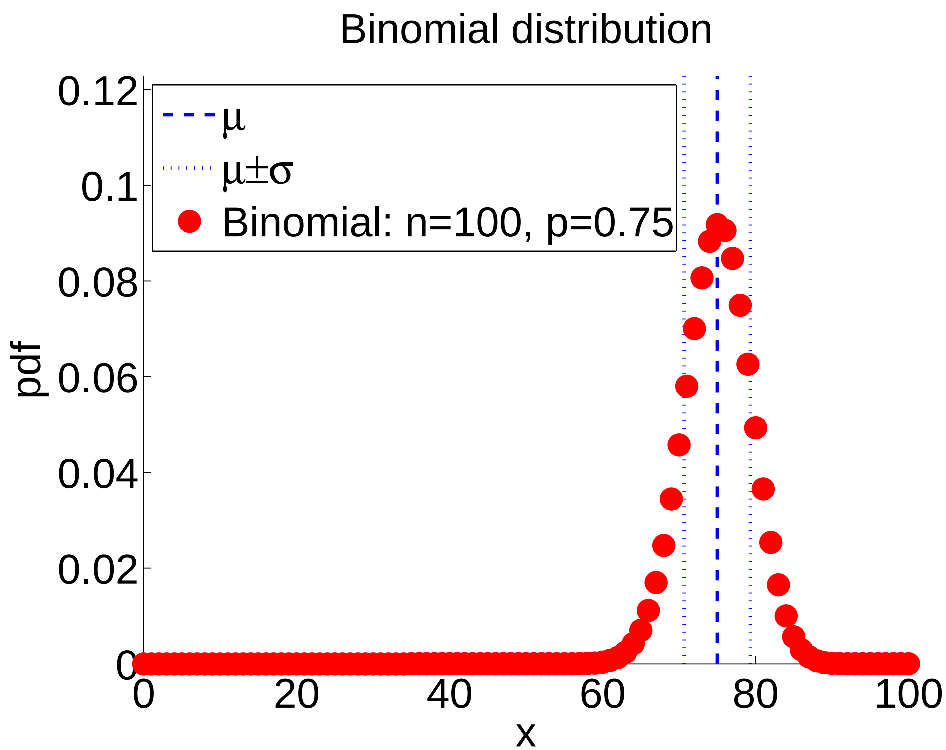
  Mean: $\mu = np$

  Standard deviation:

  $$\sigma = \sqrt{np(1-p)}$$

- At $n = 100$ and $p = 3/4$:

  $$\mu = 100(3/4) = 75$$
  $$\sigma = \sqrt{100(3/4)(1/4)} \approx 4.33$$



- Approximately 68% of the probability is for X between $\mu \pm \sigma$. Approximately 95% of the probability is for X between $\mu \pm 2\sigma$. More on that in Chapter 4.

Proof that $\mu = np$ for binomial distribution.

$$E(X) = \sum_k k \cdot p_X(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k q^{n-k}$$

Calculus Trick: $(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$

Differentiate: $\frac{\partial}{\partial p}(p+q)^n = \sum_{k=0}^{n} k\binom{n}{k} p^{k-1} q^{n-k}$

Times p: $p\frac{\partial}{\partial p}(p+q)^n = \sum_{k=0}^{n} k\binom{n}{k} p^{k} q^{n-k} = E(X)$

Evaluate left side: $p\frac{\partial}{\partial p}(p+q)^n = p \cdot n(p+q)^{n-1} = p \cdot n \cdot 1^{n-1} = np$ since $p+q=1$.

So $E(X) = np$. □

We'll do $\sigma = \sqrt{np(1-p)}$ later.