3.7, 3.8, 3.9, 3.11 Functions of multiple random variables (continuous)

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Two-dimensional version:

- Consider a shape $B \subset \mathbb{R}^2$.
- Make very thin horizontal and vertical cuts.
- Let $\rho(x, y)$ be the \textit{density} at $(x, y)$. This is the mass per unit area.
- It can be measured in g/cm$^2$. In 3D, it would be g/cm$^3$.
- $\rho(x, y) \geq 0$ everywhere.
- The area of a differential patch is $dA = dx \, dy = dy \, dx$.
- The mass of a differential patch is $\rho(x, y) \, dA$ (density times area).
- The total mass of $B$ is $\int \int_B \rho(x, y) \, dA$.
Continuous joint probability density function

Joint probability density function of two variables

We require:

1. \( f_{X,Y}(x, y) \geq 0 \) for all points \((x, y)\).
2. \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = 1 \)

Probability of an event

The probability of event \( B \subseteq \mathbb{R}^2 \) is

\[
P(B) = \int_{B} f_{X,Y}(x, y) \, dA
\]
Uniform probability on a region $C$

- **Uniform probability** on a region $C$ means that all points inside $C$ have equal probability density, and all points outside $C$ have probability density 0:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area}(C)} & \text{if } (x,y) \in C \\ 0 & \text{otherwise} \end{cases}$$

- Let $C$ be the disk of radius 2 centered at the origin:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4\pi} & \text{if } x^2 + y^2 \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

- Total probability:

$$\int\int_C \frac{1}{4\pi} \, dA = \frac{1}{4\pi} \cdot \text{area}(C) = \frac{1}{4\pi} \cdot 4\pi = 1$$
Probability of an event

\[ P(X > 0) = \int \int_D \frac{1}{4\pi} \, dA = \frac{1}{4\pi} \text{area}(D) = \frac{1}{4\pi} \cdot \frac{4\pi}{2} = \frac{1}{2} \]
Marginal densities

The marginal density at \( x \):
Form an \( x\)-strip.
Hold \( x \) constant and integrate over all \( y \).

\[
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy
\]

\[
= \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{4\pi} \, dy
\]

\[
= \frac{2\sqrt{4-x^2}}{4\pi}
\]

\[
f_X(x) = \begin{cases} 
\frac{\sqrt{4-x^2}}{2\pi} & \text{if } -2 \leq x \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]
Marginal densities

The marginal density at $y$ is similar:
Form a $y$-strip.
Hold $y$ constant and integrate over all $x$.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx$$
$$= \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{4\pi} \, dx$$
$$= \frac{2}{4\pi} \sqrt{4-y^2}$$

$$f_Y(y) = \begin{cases} \frac{\sqrt{4-y^2}}{2\pi} & \text{if } -2 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$
Random variables $X, Y, Z, \ldots$ are independent if their joint pdf factorizes as follows, for all $x, y, z, \ldots$:

$$f_{X,Y,Z,\ldots}(x, y, z, \ldots) = f_X(x)f_Y(y)f_Z(z) \cdots$$

**Technicality**: Exceptions are allowed, as long as the probability of an exception is 0. E.g., for a continuous distribution, the probability of a point is 0; in 2D, the probability of a discrete set of points or curves is 0; etc. This doesn’t happen for discrete distributions.
Independence

Summary of previous formulas

\[ f_{X,Y}(x, y) = \begin{cases} \frac{1}{4\pi} & \text{if } x^2 + y^2 \leq 4 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_X(x) = \begin{cases} \sqrt{4 - x^2}/(2\pi) & \text{if } -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_Y(y) = \begin{cases} \sqrt{4 - y^2}/(2\pi) & \text{if } -2 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \]

Check independence:

\[ f_X(x)f_Y(y) = \begin{cases} \sqrt{(4 - x^2)(4 - y^2)}/(4\pi^2) & \text{if } -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \]

This is different than \( f_{X,Y}(x, y) \). The formula is different, and it’s nonzero inside a square instead of inside a circle. So \( X, Y \) are dependent.
Expected values

Definition
For a function $g(X, Y)$ of continuous random variables, the expected value is

\[ E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f_{X,Y}(x, y) \, dA \]

This is similar to the definition in the discrete case, but using integrals instead of sums.

Compute $E(X)$ for the circle example

\[ E(X) = \int \int_{\text{left semicircle}} \frac{x}{4\pi} \, dA + \int \int_{\text{right semicircle}} \frac{x}{4\pi} \, dA = 0 \]

The two integrals are negatives of each other, so they sum to 0.
Compute $E(R)$ in the circle example

In polar coordinates, recall $R = \sqrt{X^2 + Y^2}$. Compute $E(R)$:

$$E(R) = E \left( \sqrt{X^2 + Y^2} \right) = \iint_C \sqrt{x^2 + y^2} \cdot \frac{1}{4\pi} \, dA$$

- This is easier in polar coordinates than in Cartesian coordinates. Switch to polar coordinates, and note that the integral separates:

$$E(R) = \iint_C \frac{r}{4\pi} \, dA = \int_0^{2\pi} \int_0^2 \frac{r}{4\pi} \cdot r \, dr \, d\theta = \frac{1}{4\pi} \left( \int_0^{2\pi} d\theta \right) \left( \int_0^2 r^2 \, dr \right)$$

- Evaluate the integrals:

$$\int_0^{2\pi} d\theta = \theta \bigg|_{\theta=0}^{\theta=2\pi} = 2\pi - 0 = 2\pi \quad \int_0^2 r^2 \, dr = \frac{r^3}{3} \bigg|_{r=0}^{r=2} = \frac{2^3 - 0^3}{3} = \frac{8}{3}$$

- Plug in their values:

$$E(R) = \frac{1}{4\pi} \left( 2\pi \right) \left( \frac{8}{3} \right) = \frac{16\pi}{12\pi} = \frac{4}{3}$$
The variance formula is the same for continuous as for discrete:

\[ \text{Var}(X) = E((X - \mu)^2) = E(X^2) - (E(X))^2 \]

However, expected value is computed using an integral instead of a sum.

Compute \( \text{Var}(R) \) and \( \text{SD}(R) \):

\[
E(R^2) = \iint_C \frac{r^2}{4\pi} \, dA = \int_0^{2\pi} \int_0^2 \frac{r^2}{4\pi} \cdot r \, dr \, d\theta = \frac{1}{4\pi} \left( \int_0^{2\pi} d\theta \right) \left( \int_0^2 r^3 \, dr \right)
\]

\[
\int_0^{2\pi} d\theta = 2\pi \quad \int_0^2 r^3 \, dr = \frac{r^4}{4} \bigg|_{r=0}^{r=2} = \frac{2^4 - 0^4}{4} = 4
\]

\[
E(R^2) = \frac{1}{4\pi} (2\pi)(4) = 2
\]

\[
\text{Var}(R) = E(R^2) - (E(R))^2 = 2 - \left( \frac{4}{3} \right)^2 = \frac{2}{9} \quad \text{SD}(R) = \sqrt{\frac{2}{9}}
\]
## Mass density in physics vs. continuous pdf

<table>
<thead>
<tr>
<th>Physics</th>
<th>Probability</th>
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<tbody>
<tr>
<td><strong>Mass density</strong></td>
<td><strong>Probability density function</strong></td>
</tr>
<tr>
<td>$\rho(x, y) \geq 0$</td>
<td>$f_{X,Y}(x, y) \geq 0$</td>
</tr>
<tr>
<td><strong>Mass of shape $D \subset \mathbb{R}^2$:</strong></td>
<td><strong>Probability of event $D \subset \mathbb{R}^2$:</strong></td>
</tr>
<tr>
<td>$M = \int\int_D \rho(x, y) , dA \geq 0$</td>
<td>$P(D) = \int\int_D f_{X,Y}(x, y) , dA$ and $P(\mathbb{R}^2) = 1$</td>
</tr>
<tr>
<td><strong>Center of mass $(\bar{x}, \bar{y})$</strong></td>
<td><strong>Expected value</strong></td>
</tr>
<tr>
<td>$\bar{x} = \frac{\int\int_D x \cdot \rho(x, y) , dA}{\int\int_D \rho(x, y) , dA}$</td>
<td>$E(X) = \int\int_{\mathbb{R}^2} x \cdot f_{X,Y}(x, y) , dA = \text{numerator of } \bar{x}$.</td>
</tr>
<tr>
<td></td>
<td>The denominator of $\bar{x}$ is 1, since $M = 1$.</td>
</tr>
<tr>
<td>$\bar{y}$ formula is similar</td>
<td>$E(Y)$ formula is similar</td>
</tr>
</tbody>
</table>

- When the total mass is 1, we have $(\bar{x}, \bar{y}) = (E(X), E(Y))$.
- We used $\mathbb{R}^2$. For $\mathbb{R}^n$, use $(x_1, \ldots, x_n)$ instead of $(x, y)$.
**Question:** Determine the formula of the probability density if it is proportional to \( x + 4y \) inside the rectangle and is 0 outside.

- We have \( f_{X,Y}(x, y) = c(x + 4y) \) inside the rectangle and 0 outside, for some constant \( c \).
- Find \( c \) so that the total probability is 1:

\[
P = \int_{0}^{2} \int_{0}^{1} c(x + 4y) \, dy \, dx = 1
\]

- The inside integral is

\[
c(xy + 2y^2) \bigg|_{y=0}^{y=1} = c(x(1 - 0) + 2(1^2 - 0^2)) = c(x + 2)
\]

- Plug that back in: \( P = \int_{0}^{2} c(x + 2) \, dx \)
Determining the constant

Continue evaluating:

\[ P = \int_{0}^{2} c(x + 2) \, dx = c \left( \frac{x^2}{2} + 2x \right){\bigg|}_{x=0}^{2} \]

\[ = c \left( \frac{2^2-0^2}{2} + 2(2 - 0) \right) = c \cdot (2 + 4) = 6c \]

To get \( P = 1 \), solve \( 6c = 1 \), so \( c = 1/6 \).

Plug this value of \( c \) into the formula \( f_{X,Y}(x, y) = c(x + 4y) \).

Thus, the pdf is

\[ f_{X,Y}(x, y) = \begin{cases} 
\frac{x+4y}{6} & \text{inside rectangle: } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\
0 & \text{outside rectangle}
\end{cases} \]
For $0 \leq x \leq 2$:

$$f_X(x) = \int_0^1 \frac{x + 4y}{6} \, dy = \frac{xy + 2y^2}{6} \bigg|_{y=0}^{1} = \frac{x(1 - 0) + 2(1^2 - 0^2)}{6} = \frac{x + 2}{6}$$

Otherwise, $f_X(x) = 0$. 
Marginal densities for rectangle example

For $0 \leq y \leq 1$:

$$f_Y(y) = \int_0^2 \frac{x + 4y}{6} \, dx = \left( \frac{x^2 + 4xy}{12} \right) \bigg|_{x=0}^2$$

$$= \left( \frac{2^2 - 0^2}{12} + \frac{4(2 - 0)y}{6} \right)$$

$$= \frac{4}{12} + \frac{8y}{6} = \frac{4y + 1}{3}$$

Otherwise, $f_Y(y) = 0$. 
Independence in rectangle example

- Recall we computed:

\[ f_{X,Y}(x, y) = \begin{cases} \frac{x+4y}{6} & \text{inside rectangle: } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{outside rectangle} \end{cases} \]

- Check independence:

\[ f_X(x) = \begin{cases} \frac{x+2}{6} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_Y(y) = \begin{cases} \frac{4y+1}{3} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_X(x) \cdot f_Y(y) = \begin{cases} \frac{(x+2)(4y+1)}{18} & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \neq f_{X,Y}(x, y) \] so \( X \) and \( Y \) are dependent.
The **joint cumulative distribution function (cdf)** for two random variables $X, Y$ is

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

For multiple random variables, the formula is similar.

As an integral:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v) \, dv \, du = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v) \, du \, dv.$$

Since we used $(x, y)$ in the limits of the integral, the integration variables had to be renamed; here, we used $(u, v)$ instead. Alternatively, some people prefer to do it the other way around:

$$F_{X,Y}(u, v) = P(X \leq u, Y \leq v) = \int_{-\infty}^{v} \int_{-\infty}^{u} f_{X,Y}(x, y) \, dx \, dy.$$
Joint cdf in rectangle example

\[ f_{X,Y}(x,y) = \begin{cases} 
\frac{x+4y}{6} & \text{inside rectangle:} \\
0 & \text{outside rectangle}
\end{cases} \]

\[ 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \]

Consider \((x,y)\) inside the rectangle:

\[ F_{X,Y}(x,y) = \int_0^x \int_0^y \frac{u + 4v}{6} \, dv \, du \]

Inside integral:

\[
\int_0^y \frac{u + 4v}{6} \, dv = \frac{uv + 2v^2}{6} \bigg|_{v=0}^{v=y} = \frac{u(y - 0) + 2(y^2 - 0^2)}{6} = \frac{uy + 2y^2}{6}
\]

Outside integral:

\[
F_{X,Y}(x,y) = \int_0^x \frac{uy + 2y^2}{6} \, du = \left( \frac{u^2y}{12} + \frac{uy^2}{3} \right) \bigg|_{u=0}^{u=x} = \frac{x^2y}{12} + \frac{xy^2}{3}
\]
Differentiating the cdf

- Evaluate

\[
\frac{\partial^2}{\partial x \, \partial y} F_{X,Y}(x, y) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} F_{X,Y}(x, y) \right)
\]

- Inside derivative:

\[
\frac{\partial}{\partial y} F_{X,Y}(x, y) = \frac{\partial}{\partial y} \left( \frac{x^2 y}{12} + \frac{xy^2}{3} \right) = \frac{x^2}{12} + \frac{2xy}{3}
\]

- Outside derivative:

\[
\frac{\partial^2}{\partial x \, \partial y} F_{X,Y}(x, y) = \frac{\partial}{\partial x} \left( \frac{x^2}{12} + \frac{2xy}{3} \right) = \frac{x}{6} + \frac{2y}{3} = \frac{x + 4y}{6} = f_{X,Y}(x, y)
\]

- In general, the cdf is the double integral of the pdf with respect to \( x \) and \( y \), and inversely, the pdf is the double derivative of the cdf with respect to \( x \) and \( y \).
The pdf $f_{X,Y}(x, y) = (x + 4y)/6$ is nonzero only in the blue rectangle.

The region $\leq (x, y)$ can intercept the rectangle in different ways, depending on where $(x, y)$ is in relation to the rectangle.

If $x < 0$ or $y < 0$, then the pdf is 0 in the whole integration region, so $F_{X,Y}(x, y) = 0$.

This satisfies $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = 0$. 

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Remaining cases: In this example, when \((x, y)\) is right of and/or above the rectangle, the intercepted region becomes the cdf of another point:

**Right:** If \(x > 2\) and \(0 \leq y \leq 1\)

\[
F_{X,Y}(x, y) = F_{X,Y}(2, y) = \frac{4y}{12} + \frac{2y^2}{3} = \frac{y + 2y^2}{3}
\]

**Above:** If \(y > 1\) and \(0 \leq x \leq 2\)

\[
F_{X,Y}(x, y) = F_{X,Y}(x, 1) = \frac{x^2}{12} + \frac{x}{3} = \frac{x^2 + 4x}{12}
\]

**Above and right:** If \(x > 2\) and \(y > 1\)

\[
F_{X,Y}(x, y) = F_{X,Y}(2, 1) = \frac{2^2 \cdot 1}{12} + \frac{2 \cdot 1}{3} = 1
\]

In all of these, \(f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = 0\).
Evaluate $P(Y > 3X)$ in the rectangle example:

Compute $\int\int_D f_{X,Y}(x,y) \, dA = \int\int_D \frac{x+4y}{6} \, dA$ over the shaded triangle, $D$.

Can use $x$-slices or $y$-slices. Both give the same final answer. $x$-slices are left as an exercise for you. The $y$-slices are:

One $y$-slice for each $0 \leq y \leq 1$. It runs over $0 \leq x \leq y/3$.

The integral is

$$P(Y > 3X) = \int_0^1 \int_0^{y/3} \frac{x + 4y}{6} \, dx \, dy$$
Probability of an event

\[ P(Y > 3X) = \int_0^1 \int_0^{y/3} \frac{x + 4y}{6} \, dx \, dy \]

- **Inside integral:**

\[ \int_0^{y/3} \frac{x + 4y}{6} \, dx = \left( \frac{x^2}{12} + \frac{4xy}{6} \right) \bigg|_{x=0}^{x=y/3} \]

\[ = \frac{(y/3)^2 - 0^2}{12} + \frac{4(y/3)y}{6} = \frac{y^2}{108} + \frac{4y^2}{18} = \frac{25y^2}{108} \]

- **Outside integral:**

\[ P(Y > 3X) = \int_0^1 \frac{25y^2}{108} \, dy = \frac{25y^3}{324} \bigg|_0^1 = \frac{25(1^3 - 0^3)}{324} = \frac{25}{324} \]
Conditional probability example #1

\[ P(A|B) \text{ where } A \text{ and } B \text{ have the same dimension} \]

Evaluate \( P(Y > \frac{1}{2} \mid X < 1) \) in the rectangle example:

- This is \( 1 - P(Y \leq \frac{1}{2} \mid X < 1) \). We have:
  \[
  P(Y \leq \frac{1}{2} \mid X < 1) = \frac{P(Y \leq \frac{1}{2} \text{ and } X < 1)}{P(X < 1)} = \frac{F_{X,Y}(1, 1/2)}{F_X(1)}
  \]

- Recall that inside the rectangle, we have
  \[
  F_{X,Y}(x, y) = \frac{x^2y}{12} + \frac{xy^2}{3}
  \]
  and we used tricks to evaluate it outside the rectangle.

- \[
  F_{X,Y}(1, 1/2) = \frac{(1^2)(1/2)}{12} + \frac{(1)(1/2)^2}{3} = \frac{1}{24} + \frac{1}{12} = \frac{3}{24} = \frac{1}{8}
  \]
- \[
  F_X(1) = F_{X,Y}(1, \infty) = F_{X,Y}(1, 1) = \frac{(1^2)(1)}{12} + \frac{(1)(1^2)}{3} = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}
  \]
- Plug these into the above formulas to get
  \[
  P(Y > \frac{1}{2} \mid X < 1) = 1 - P(Y \leq \frac{1}{2} \mid X < 1) = 1 - \frac{F_{X,Y}(1,1/2)}{F_X(1)}
  \]
  \[
  = 1 - \frac{1/8}{5/12} = 1 - \frac{3}{10} = \frac{7}{10}
  \]
Conditional probability example #2

\(P(A|B)\) where \(A\) and \(B\) have different dimensions

Evaluate \(P(Y > \frac{1}{2} \mid X = 1)\) in the rectangle example:

- \(\frac{P(Y > \frac{1}{2} \text{ and } X=1)}{P(X=1)} = \frac{0}{0}\) does not work.

- Instead, define the conditional probability density at \(X = x\):

\[
f_Y(y \mid X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)}
\]

For a given value of \(x\), this is a function of varying \(y\).
It’s proportional to \(f_{X,Y}(x, y)\) but is renormalized so that the total probability as \(y\) varies in the strip \(X = x\) is 1:

\[
\int_{-\infty}^{\infty} f_Y(y \mid X = x) \, dy = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x, y)}{f_X(x)} \, dy = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy}{f_X(x)} = \frac{f_X(x)}{f_X(x)} = 1
\]
Conditional probability example #2

\[ P(A \mid B) \text{ where } A \text{ and } B \text{ have different dimensions} \]

Evaluate \( P(Y > \frac{1}{2} \mid X = 1) \) in the rectangle example:

- The conditional probability density at \( X = x \) is

\[
f_Y(y \mid X = x) = \frac{f_{X,Y}(x,y)}{f_X(x)}
\]

- In the rectangle example, for \( x \) and \( y \) within the rectangle:

\[
f_Y(y \mid X = x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x + 4y)/6}{(x + 2)/6} = \frac{x + 4y}{x + 2}
\]

\[
f_Y(y \mid X = 1) = \frac{1 + 4y}{3}
\]

\[
P(Y > \frac{1}{2} \mid X = 1) = \int_{1/2}^{\infty} f_Y(y \mid X = 1) \, dy = \int_{1/2}^{1} \frac{1 + 4y}{3} \, dy = \frac{y + 2y^2}{3} \bigg|_{y=1/2}^{y=1} = \frac{1 - (1/2)}{3} + \frac{2(1^2 - (1/2)^2)}{3} = \frac{1}{3} + \frac{1}{3} + \frac{2(3/4)}{3} = \frac{2}{3}
\]