MATH 20C: AREA OF ONE BAND OF A SPIRAL

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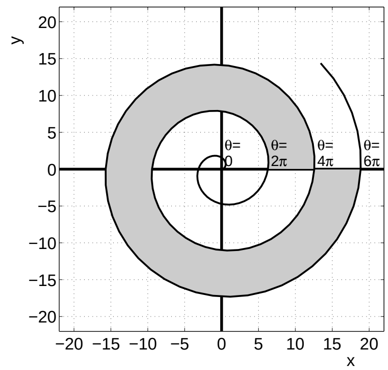
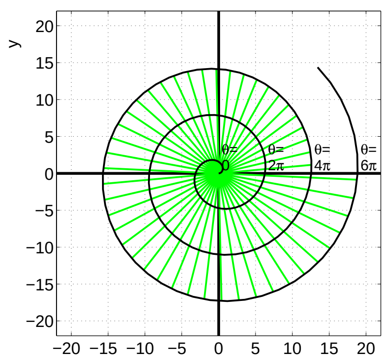
Problem: Find the area of one band of the spiral $r = \theta$ as shown here (call the region B):

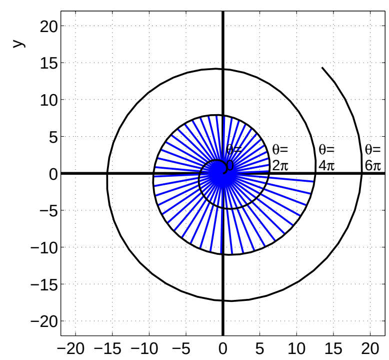


Solution 1:

This is the area between two curves that are parameterized differently: the polar curve on the outside border of the region is $r_{out} = \theta$, $4\pi \leq \theta \leq 6\pi$ (the area enclosed by it is shown in green), and the polar curve bounding the inside border of the region is $r_{in} = \theta$, $2\pi \leq \theta \leq 4\pi$ (enclosed area shown in blue):



$r_{out} = \theta, \quad 4\pi \leq \theta \leq 6\pi$



$r_{in} = \theta, \quad 2\pi \leq \theta \leq 4\pi$

The area of region B is the the difference of the areas bounded by those two curves.

The area inside r_{out} (green) is

$$A_{out} = \frac{1}{2}\int_{4\pi}^{6\pi} \theta^2 \, d\theta = \frac{1}{6}\theta^3\Big|_{4\pi}^{6\pi} = \frac{1}{6}\left((6\pi)^3 - (4\pi)^3\right) = \frac{1}{6}\left((216-64)\pi^3\right) = \frac{152}{6}\pi^3 = \frac{76}{3}\pi^3$$

The area inside r_{in} (blue) is

$$A_{in} = \frac{1}{2}\int_{2\pi}^{4\pi} \theta^2 \, d\theta = \frac{1}{6}\theta^3\Big|_{2\pi}^{4\pi} = \frac{1}{6}\left((4\pi)^3 - (2\pi)^3\right) = \frac{1}{6}\left((64-8)\pi^3\right) = \frac{56}{6}\pi^3 = \frac{28}{3}\pi^3$$

The difference between these areas is the area of B:

$$A = A_{out} - A_{in} = \frac{76}{3}\pi^3 - \frac{28}{3}\pi^3 = \frac{48}{3}\pi^3 = \boxed{16\pi^3}$$

(turn page)

Solution 2:

Express the outside and inside curves in terms of the same parameter θ, $4\pi \le \theta \le 6\pi$, instead of treating the two curves separately:

The green area is the area inside $r_{out} = \theta, \quad 4\pi \le \theta \le 6\pi$.

The blue area is the area inside $r_{in} = \theta - 2\pi, \quad 4\pi \le \theta \le 6\pi$.

The area in-between r_{in} and r_{out} is

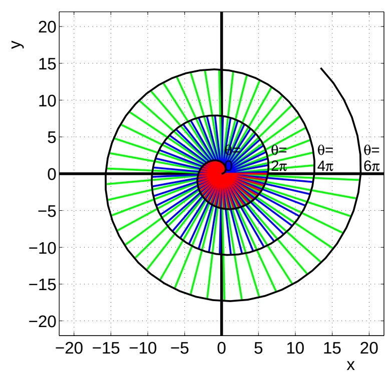
$$\begin{aligned} A &= \frac{1}{2}\int_{4\pi}^{6\pi} \left(r_{out}^2 - r_{in}^2\right)\, d\theta \;=\; \frac{1}{2}\int_{4\pi}^{6\pi} \left(\theta^2 - (\theta-2\pi)^2\right)\, d\theta \\ &= \frac{1}{2}\int_{4\pi}^{6\pi} \left(\theta^2 - (\theta^2 - 4\pi\theta + 4\pi^2)\right)\, d\theta \;=\; \frac{1}{2}\int_{4\pi}^{6\pi} \left(4\pi\theta - 4\pi^2\right)\, d\theta \;=\; \frac{1}{2}\left(2\pi\theta^2 - 4\pi^2\theta\right)\Big|_{4\pi}^{6\pi} \\ &= \frac{1}{2}\left(2\pi\left((6\pi)^2 - (4\pi)^2\right) - 4\pi^2(6\pi - 4\pi)\right) \;=\; \boxed{16\pi^3} \end{aligned}$$

Questions about counting the area multiple times:

It's important to put the proper bounds on θ for each curve. If we were to use $0 \le \theta \le 6\pi$

$$A = \frac{1}{2}\int_0^{6\pi} \theta^2\, d\theta = \frac{1}{6}\theta^3\Big|_0^{6\pi} = \frac{1}{6}\left((6\pi)^3 - 0\right) = \frac{1}{6}\left(216\pi^3\right) = 36\pi^3$$

instead of breaking it up as described above, then portions of the spiral area would be traced out up to three times!

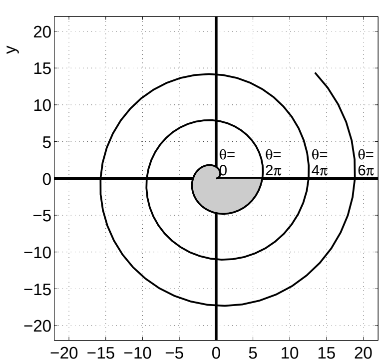


In this picture, the area inside the spiral $r = \theta$, $0 \le \theta \le 2\pi$, is swept out in red stripes.

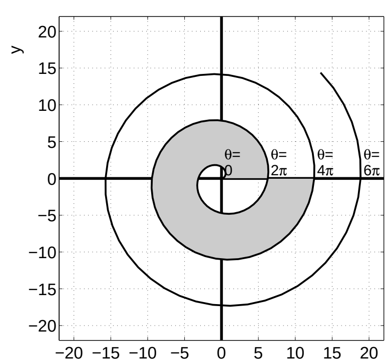
The area inside $r = \theta$, $2\pi \le \theta \le 4\pi$ is swept out in blue stripes. Notice that it includes the region previously covered in red.

The area inside $r = \theta$, $4\pi \le \theta \le 6\pi$ is swept out in green stripes. Notice that it includes the regions previously covered in red or blue.

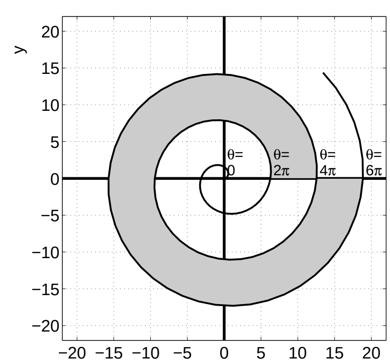
Thus, the "area" $36\pi^3$ computed by that integral is actually



three times the area of the first band of the spiral



plus two times the area of the second band



plus the area of the third band.