2.3 Partial Derivatives, Linear Approximation

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Math 20C
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Partial derivatives

$$f(x, y, z) = \sin(x^2 + 4xy + 3z)$$

- The *partial derivative of f with respect to x* means
  - Treat x as a variable.
  - Treat the other variables (y and z) as constants.
  - Differentiate as a function of x.
- Result: $$\frac{\partial f}{\partial x} = \cos(x^2 + 4xy + 3z) \cdot (2x + 4y)$$

Notation

Partial derivatives	One variable derivative
∂: partial derivative symbol	d
$\frac{\partial f}{\partial x}$	$\frac{df}{dx}$
$\frac{\partial}{\partial x} f$	$\frac{d}{dx} f$
f_x or $f_x(x, y, z)$	$f'(x)$

Partial derivatives

$$f(x, y, z) = \sin(x^2 + 4xy + 3z)$$

- The *partial derivative of f with respect to y* means
  - Treat y as a variable.
  - Treat the other variables (x and z) as constants.
  - Differentiate as a function of y.
- Result:

$$\frac{\partial f}{\partial y} = 4x \cos(x^2 + 4xy + 3z)$$

Partial derivatives

$$f(x, y, z) = \sin(x^2 + 4xy + 3z)$$

- The *partial derivative of f with respect to z* means
  - Treat z as a variable.
  - Treat the other variables (x and y) as constants.
  - Differentiate as a function of z.
- Result:

$$\frac{\partial f}{\partial z} = 3\cos(x^2 + 4xy + 3z)$$

Partial derivative at a point

One variable

- $f'(10)$: Evaluate function $f'(x)$ first, and then plug in value $x = 10$.
- $f(x) = x^3 \qquad f'(x) = 3x^2 \qquad f'(10) = 3(10)^2 = 300$

Multiple variables

$$f(x,y) = x^4 y$$

- $f_x(1,2)$: Compute derivative as function: $f_x(x,y) = 4x^3 y$
  and then plug in $(x,y) = (1,2)$: $f_x(1,2) = 4(1^3)(2) = 8$
- Several notations for this:
$$f_x(1,2) \quad = \quad \frac{\partial f}{\partial x}(1,2) \quad = \quad \left.\frac{\partial f}{\partial x}\right|_{x=1,y=2} \quad = \quad \left.\frac{\partial f}{\partial x}\right|_{(1,2)}$$
- $f_y(x,y) = x^4$ and $f_y(1,2) = 1^4 = 1$
- For $z = x^4 y$: $\qquad \frac{\partial z}{\partial x} = 4x^3 y \qquad \frac{\partial z}{\partial y} = x^4$

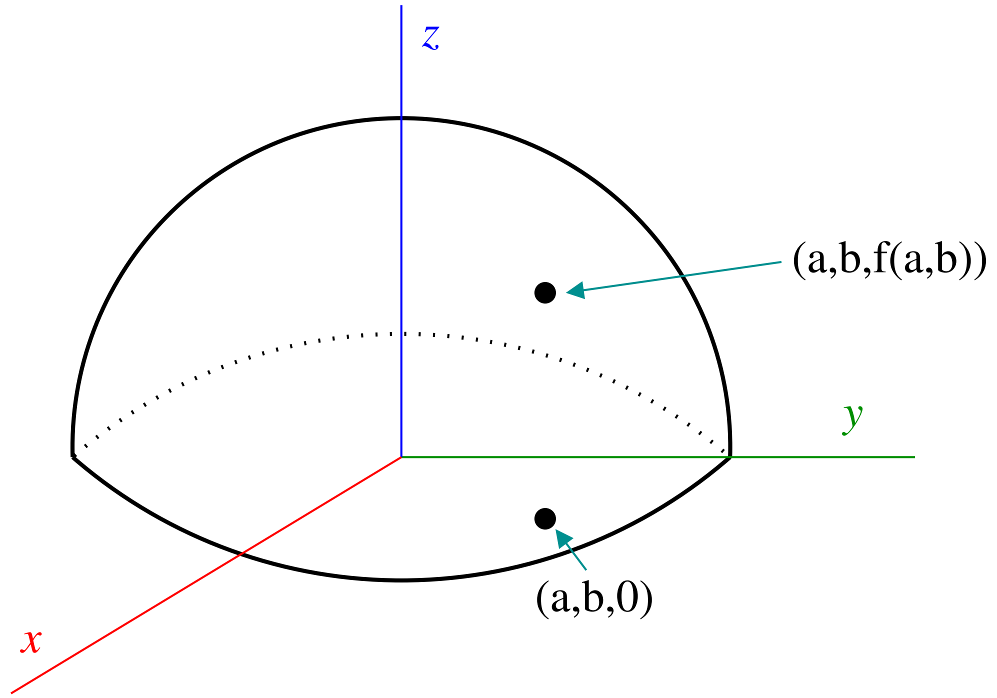
$z = x^y$

- For $z = x^y$, what are $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$?
- $\dfrac{d}{dx}(x^3) = 3x^2$ $\qquad$ $\dfrac{\partial z}{\partial x} = y \cdot x^{y-1}$
- $\dfrac{d}{dy}(3^y) = 3^y \ln 3$ $\qquad$ $\dfrac{\partial z}{\partial y} = x^y \ln(x)$

Gradient

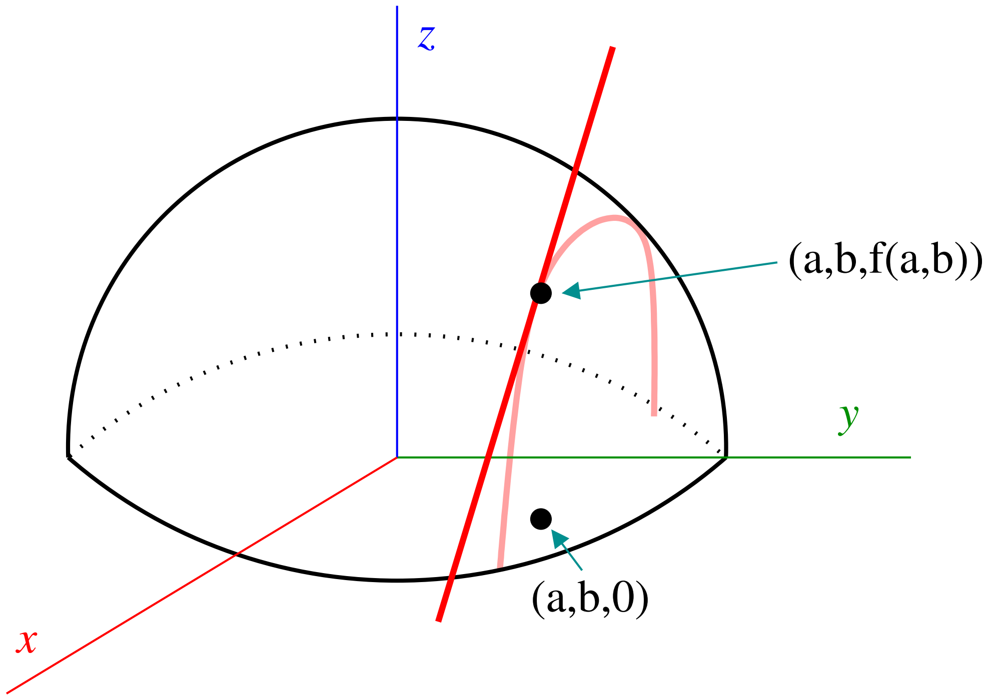
- The *gradient* of $f(x, y)$ is
$$\nabla f = \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$
- For $f(x, y) = x^2 y^4$, we get $\nabla f = \langle 2xy^4, 4x^2y^3 \rangle$.
- At point $(x, y) = (1, 10)$:
$$\nabla f(1, 10) = \langle 2 \cdot 1 \cdot 10^4, 4 \cdot 1^2 \cdot 10^3 \rangle = \langle 20000, 4000 \rangle$$
- For a function of three variables:
$$\nabla f = \nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$
This generalizes to any number of variables.
- Symbol "∇" is called *Nabla*.
It's an upside down Greek letter Delta, Δ.

Formal definition of partial derivative



- Graph the surface $z = f(x, y)$.
- Consider point $P = (a, b, ?)$ on surface.
- $z = f(x, y) = f(a, b)$, so the point on the surface is $P = (a, b, f(a, b))$.

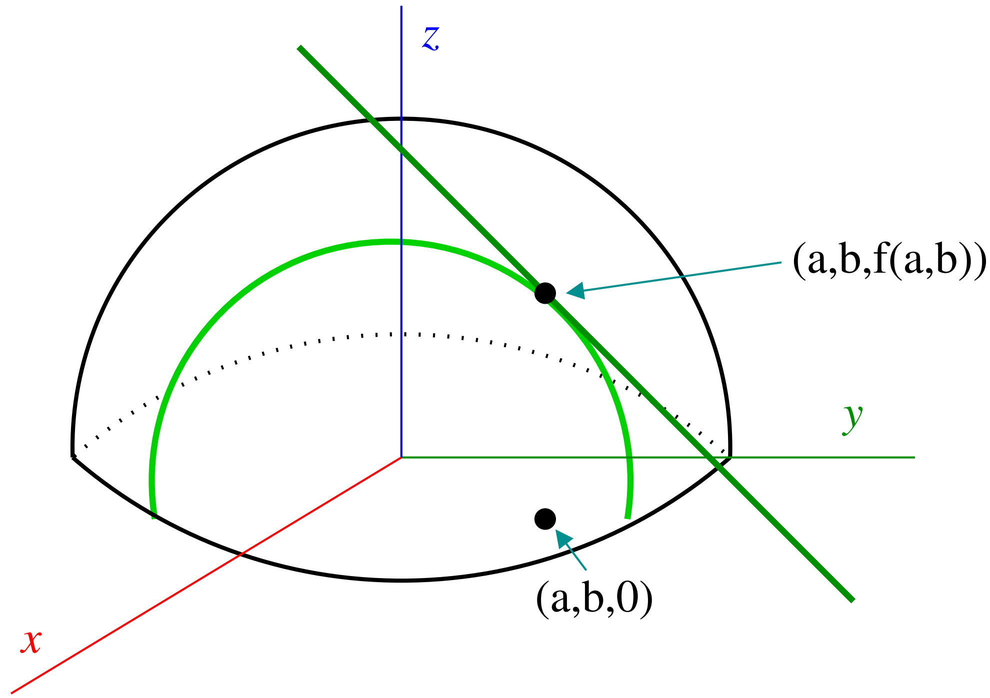
Formal definition of partial derivative



- $\frac{\partial f}{\partial x}$: Compute derivative treating x as a variable and y as a constant.
- $y = b =$ constant is a plane parallel to the xz plane ($y = 0$).
- The graph of $z = f(x,b)$ with x varying and $y = b =$ constant gives the red curve on the surface.
- The tangent line in that plane has slope $f_x(a,b)$:

$$y = b \quad \text{and} \quad z = f(a,b) + f_x(a,b) \cdot (x - a)$$

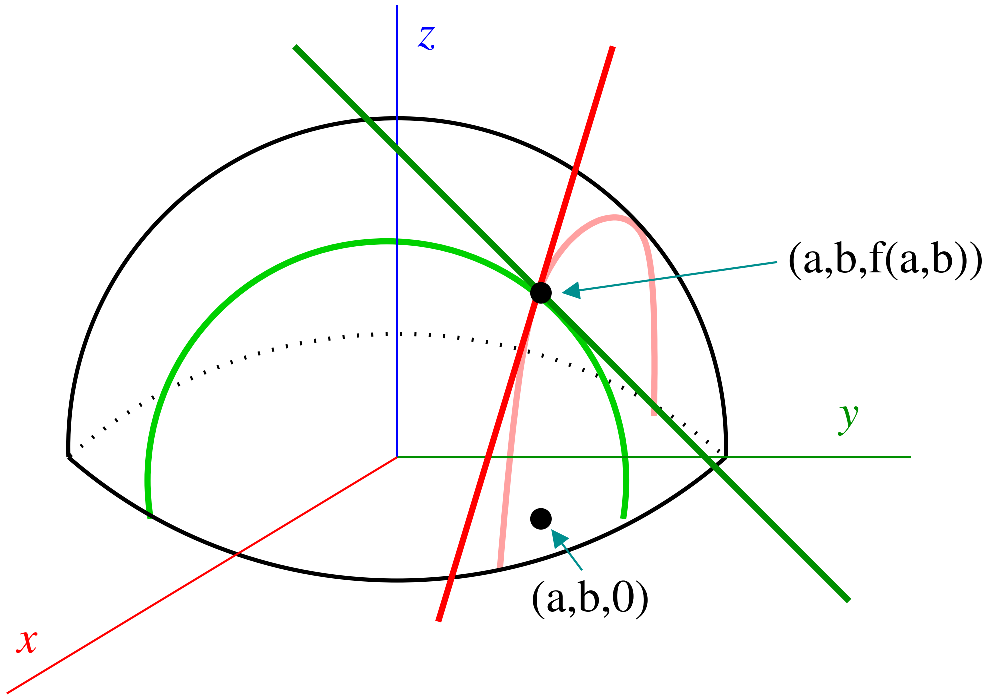
Formal definition of partial derivative



- $\frac{\partial f}{\partial y}$: Compute derivative treating y as a variable and x as a constant.
- $x = a =$ constant is a plane parallel to the yz plane ($x = 0$).
- The graph of $z = f(a, y)$ with y varying and $x = a =$ constant gives the green curve on the surface.
- The tangent line in that plane has slope $f_y(a, b)$:

$$x = a \quad \text{and} \quad z = f(a, b) + f_y(a, b) \cdot (y - b)$$

Formal definition of partial derivative



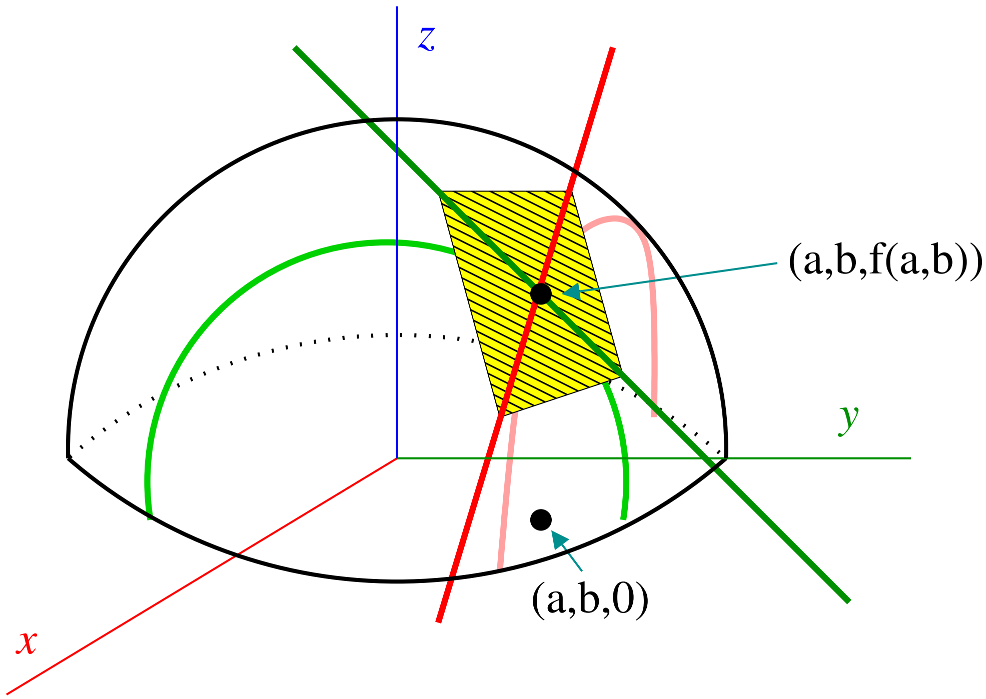
$$f_x(a,b) = \text{rate of change of } f \text{ w.r.t. } x \text{ at } (a,b)$$

$$= \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x} \qquad = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x-a}$$

$$f_y(a,b) = \text{rate of change of } f \text{ w.r.t. } y \text{ at } (a,b)$$

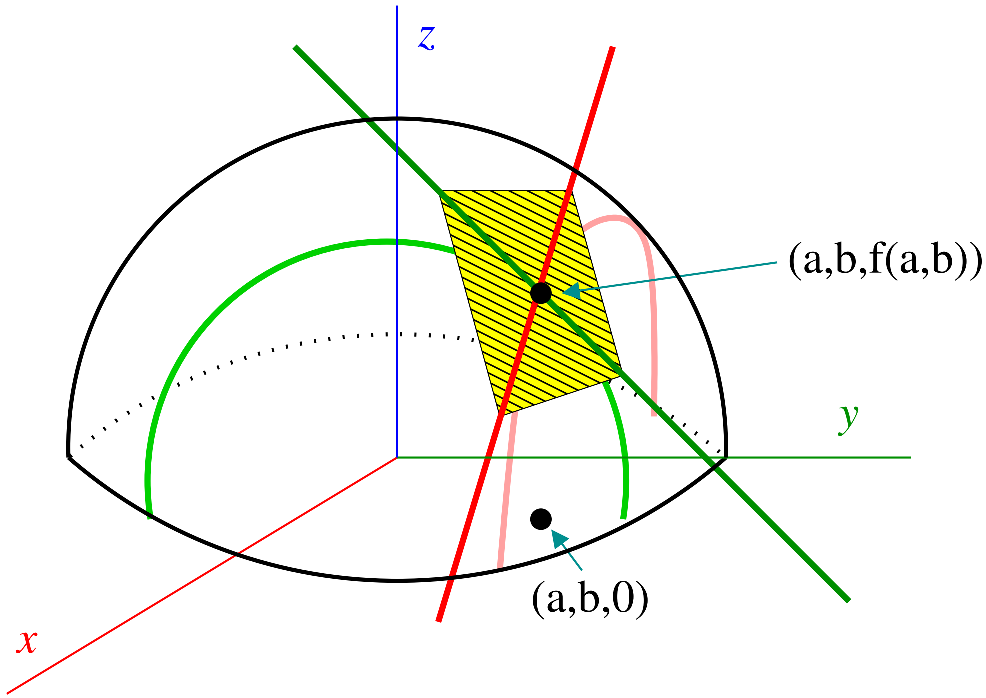
$$= \lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} \qquad = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b}$$

Tangent plane



- A *tangent plane* to a 3D surface $z = f(x, y)$ generalizes a *tangent line* to a 2D curve.
- It's a plane that just touches the surface at a given point.
  It approximates the function when (x, y) is near the starting point.

Tangent plane

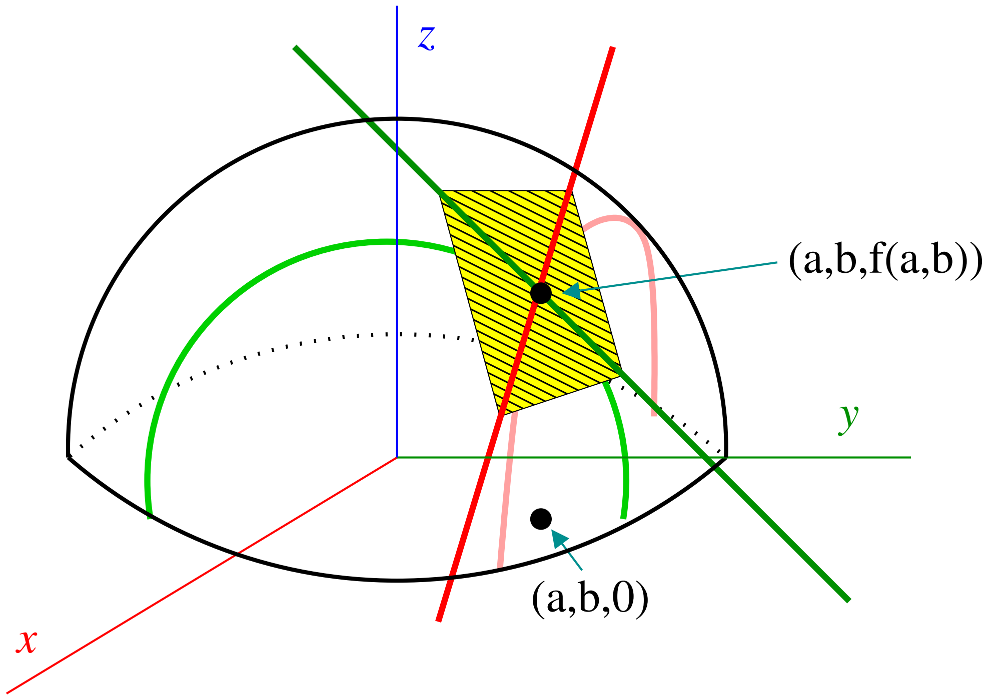


- The tangent plane at point P contains both tangent lines.
- The formula of the tangent plane is:

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

- Holding $x = a$ constant gives the green tangent line, and holding $y = b$ constant gives the red tangent line.

Tangent plane — Vector formula



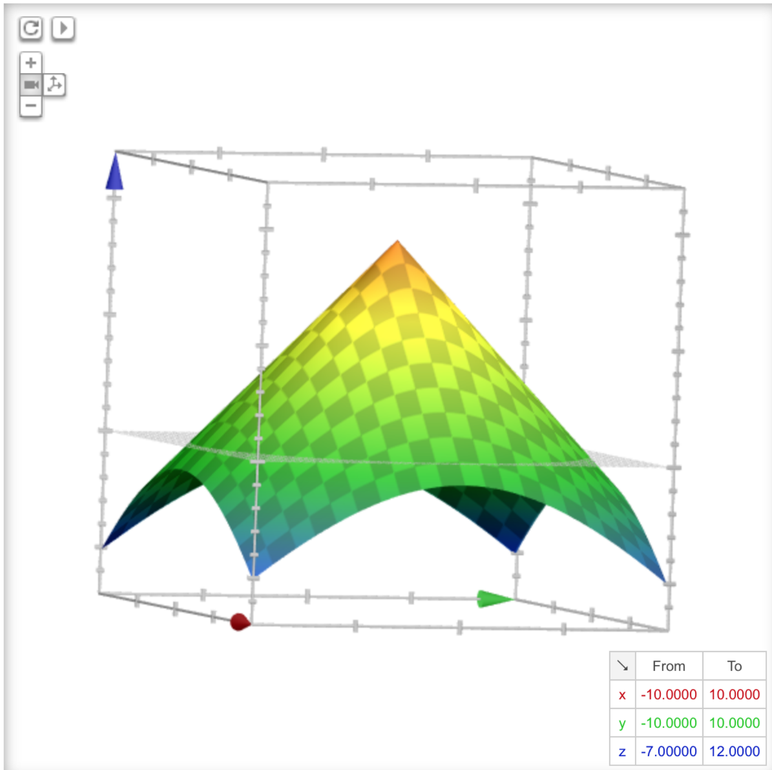
$$
\begin{aligned}
z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\
&= f(a,b) + \big\langle f_x(a,b), f_y(a,b) \big\rangle \cdot \langle x-a, y-b \rangle
\end{aligned}
$$

which gives an alternate formula

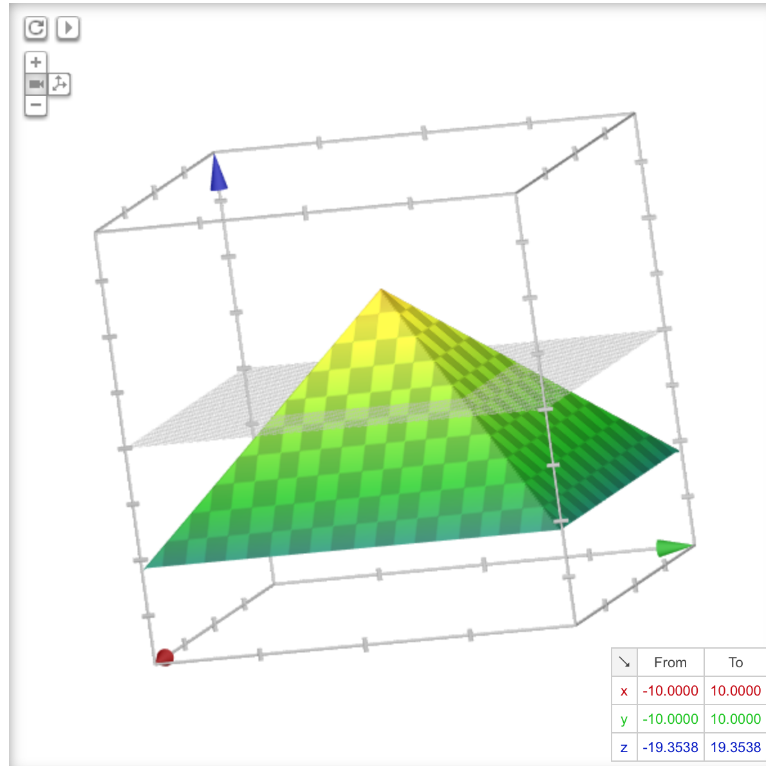
$$
\boxed{\boldsymbol{z = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle}}
$$

Technicalities for the tangent plane to exist

Graph for 9-sqrt(x^2+y^2)



Graph for 9-abs(x+y)-abs(x-y)

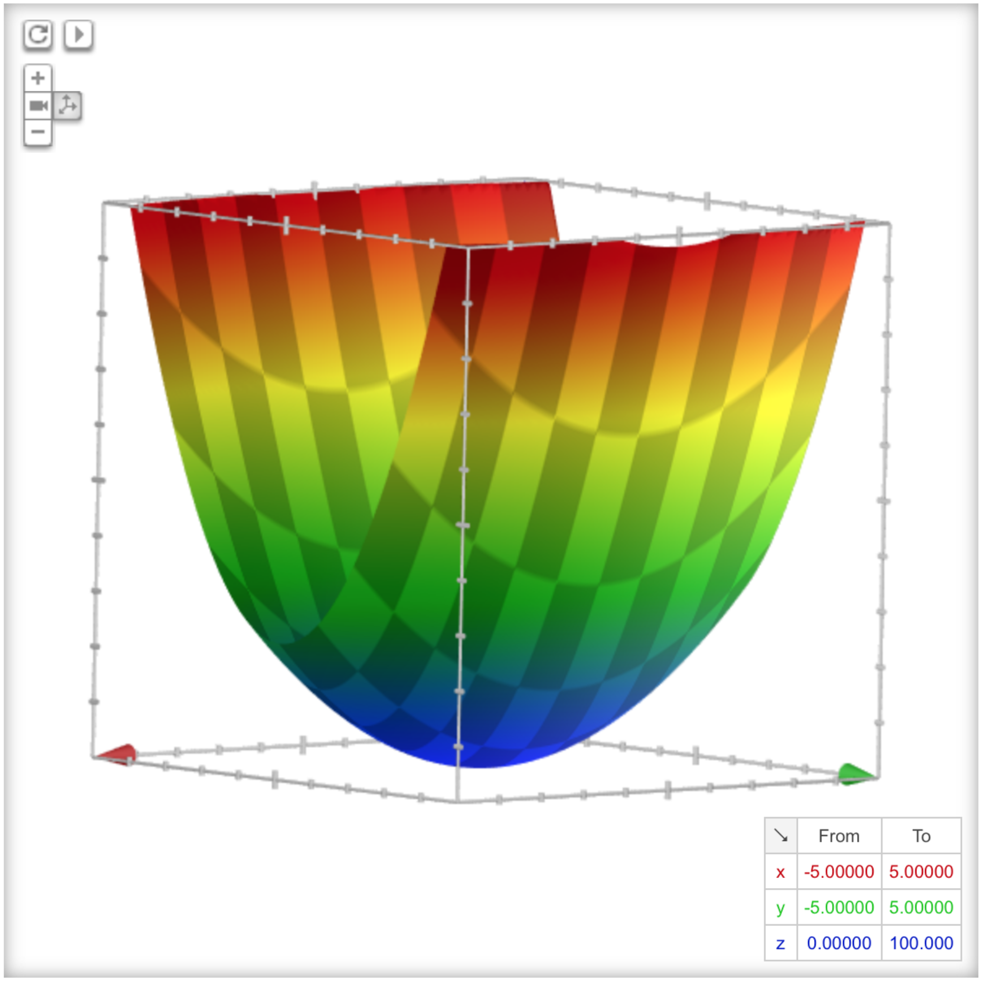


- **Left graph:** no tangent plane at the top point.
  Right graph: no tangent plane at any point along the creases.
- Need $f(x,y)$ and derivatives $f_x(x,y)$ and $f_y(x,y)$ to exist and be continuous at $(x,y)=(a,b)$, plus more technical conditions.

Example: $z = f(x,y) = x^2 + 4y^2$

Find the equation of the tangent plane at $(a,b) = (1,2)$

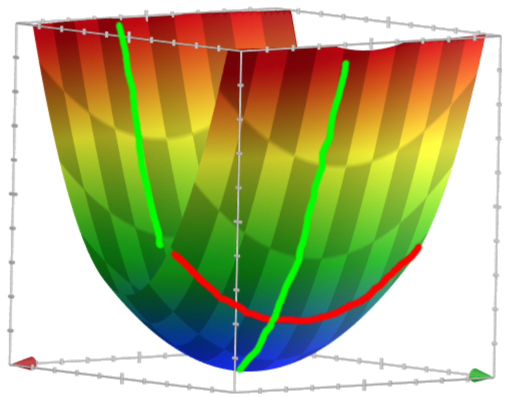
Graph for x^2+4*y^2



- Need to fill in z. At $(x,y) = (1,2)$, $z = 1^2 + 4(2^2) = 17$.
- Find the tangent plane at $(1,2,17)$.

Example: $z = f(x,y) = x^2 + 4y^2$

Find the equation of the tangent plane at $(1,2,17)$



Slopes

$f(x,y) = x^2 + 4y^2$ $\quad f_x(x,y) = 2x$ $\quad f_y(x,y) = 8y$

$f(1,2) = 1^2 + 4(2^2) = 17$ $\quad f_x(1,2) = 2(1) = 2$ $\quad f_y(1,2) = 8(2) = 16$

Tangent plane at $(a,b,f(a,b)) = (1,2,17)$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

or $$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$\boxed{\mathbf{z = 17 + 2(x-1) + 16(y-2)}}$$

Other ways to write tangent plane formula

$$z = 17 + 2(x-1) + 16(y-2)$$

As a function

$$\boxed{L(x,y) = 17 + 2\,(x-1) + 16\,(y-2)}$$

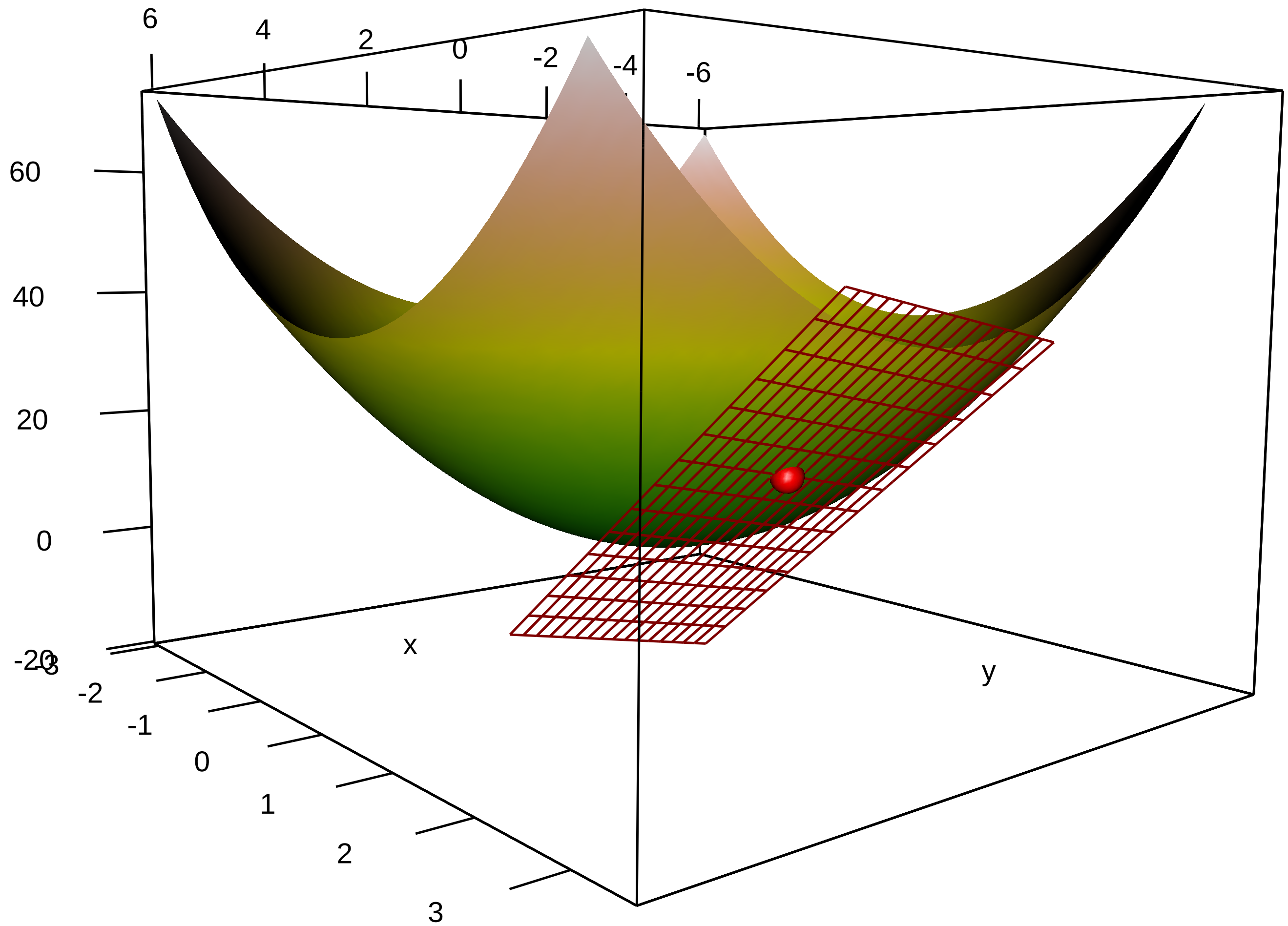
- $f(x,y)$ is approximated by the tangent plane near the starting point:

$$\underbrace{f(x,y)}_{z \text{ on surface}} \approx \underbrace{L(x,y)}_{z \text{ on tangent plane}} \qquad \text{when } (x,y) \approx (1,2)$$

- This is called *local linearity*.

Other ways to write tangent plane formula

$$z = 17 + 2(x-1) + 16(y-2)$$



Surface: $z = f(x, y) = x^2 + 4y^2$

Tangent plane: $z = L(x, y) = 17 + 2(x-1) + 16(y-2)$

Other ways to write tangent plane formula
$z = 17 + 2(x - 1) + 16(y - 2)$

In terms of changes in x, y, z

$$z - 17 = 2(x-1) + 16(y-2)$$

$$\boxed{\mathbf{\Delta z = 2\,\Delta x + 16\,\Delta y}}$$

where
$$\begin{aligned}
\Delta x &= x - a &&= x - 1\\
\Delta y &= y - b &&= y - 2\\
\Delta z &= z - f(a,b) &&= z - 17
\end{aligned}$$

General formula

$$\begin{aligned}
\Delta z &= f_x(a,b)\Delta x + f_y(a,b)\Delta y\\
&= \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y
\end{aligned}$$

Other ways to write tangent plane formula $z = 17 + 2(x-1) + 16(y-2)$

Vector version

- $f(x,y) = x^2 + 4y^2$ has $\nabla f(x,y) = \langle 2x, 8y \rangle$
- $\nabla f(1,2) = \langle 2(1), 8(2) \rangle = \langle 2, 16 \rangle$

$$z = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$$

$$z = 17 + \nabla f(1,2) \cdot \langle x-1, y-2 \rangle$$

$$\boxed{\mathbf{z = 17 + \langle 2, 16 \rangle \cdot \langle x-1, y-2 \rangle}}$$

Vector version with changes in variables

$$\Delta z = \nabla f(a,b) \cdot \langle \Delta x, \Delta y \rangle$$

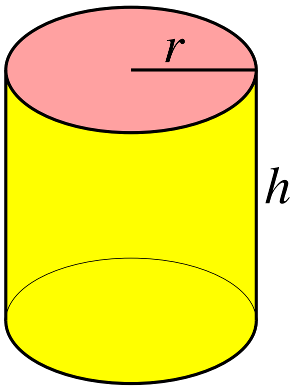
$$\boxed{\mathbf{\Delta z = \langle 2, 16 \rangle \cdot \langle \Delta x, \Delta y \rangle}}$$

where $\quad \Delta x = x - 1, \quad \Delta y = y - 2, \quad \Delta z = z - 17.$

Example: Volume of a cylinder

- Consider the volume of a cylinder of radius r and height h:

$$V(r,h) = \pi r^2 h$$



- Measurements:

$$r = 1 \pm .01 \text{ cm}$$
$$h = 2 \pm .04 \text{ cm}$$

- Volume:

$$\begin{array}{lll} \text{Approximate } V\text{:} & \pi \cdot 1^2 \cdot 2 & = 2\pi \ \text{ cm}^3 \\ \text{Low estimate:} & \pi \cdot (0.99)^2 \cdot (1.96) & = 1.920996\,\pi \ \text{ cm}^3 \\ \text{High estimate:} & \pi \cdot (1.01)^2 \cdot (2.04) & = 2.081004\,\pi \ \text{ cm}^3 \end{array}$$

- The low and high estimates are about $(2 \pm .08)\pi$ cm^3.

Example: Volume of a cylinder

The linear approximation to $V(r,h) = \pi r^2 h$ near point $(r,h) = (1,2)$:

$$\begin{aligned}
L(r+\Delta r, h+\Delta h) &= V(r,h) + \frac{\partial V}{\partial r}(r,h)\,\Delta r + \frac{\partial V}{\partial h}(r,h)\,\Delta h \\
&= \pi r^2 h + 2\pi r h\,\Delta r + \pi r^2\,\Delta h \\
&= \pi(1^2)(2) + 2\pi(1)(2)\,\Delta r + \pi(1)^2\,\Delta h \\
&= 2\pi + 4\pi\,\Delta r + \pi\,\Delta h
\end{aligned}$$

$$L(1+.01, 2+.04) = 2\pi + 4\pi(.01) + \pi(.04) = 2.08\pi$$
$$L(1-.01, 2-.04) = 2\pi + 4\pi(-.01) + \pi(-.04) = 1.92\pi$$

Example: Volume of a cylinder

Compare with the exact expansion of $V(r+\Delta r, h+\Delta h)$:

$$
\begin{aligned}
V(r+\Delta r, h+\Delta h) &= \pi(r+\Delta r)^2(h+\Delta h) \\
&= \pi(r^2+2r\,\Delta r+(\Delta r)^2)(h+\Delta h) \\
\text{0}^{\text{th}}\text{ order (no }\Delta\text{'s) is }V(r,h)\text{:} \quad &= \pi\Big(r^2 h \\
\text{1}^{\text{st}}\text{ order/linear (1 }\Delta\text{):} \quad & \qquad + 2rh\,\Delta r + r^2\,\Delta h \\
\text{2}^{\text{nd}}\text{ order (2 }\Delta\text{'s):} \quad & \qquad + 2r(\Delta r)(\Delta h) + h(\Delta r)^2 \\
\text{3}^{\text{rd}}\text{ order (3 }\Delta\text{'s):} \quad & \qquad + (\Delta r)^2(\Delta h)\Big)
\end{aligned}
$$

The linear approximation matches the 0$^{\text{th}}$ plus 1$^{\text{st}}$ order terms:

$$L(r+\Delta r, h+\Delta h) = \pi r^2 h + 2\pi rh\,\Delta r + \pi r^2 h\,\Delta h$$

Example: Volume of a cylinder

Plug in $r = 1, \quad \Delta r = .01, \quad h = 2, \quad \Delta h = .04$:

$$
\begin{array}{llll}
 & V(r+\Delta r, h+\Delta h) & & \\
0^{\text{th}} \text{ order:} & = \pi\Big(r^2 h & & = \pi\Big(1^2 \cdot 2 \\
1^{\text{st}} \text{ order:} & \quad + 2rh\,\Delta r + r^2\,\Delta h & & \quad + 2(1)(2)\,(.01) + 1^2\,(.04) \\
2^{\text{nd}} \text{ order:} & \quad + 2r(\Delta r)(\Delta h) + h(\Delta r)^2 & & \quad + 2(1)(.01)(.04) + 2(.01)^2 \\
3^{\text{rd}} \text{ order:} & \quad + (\Delta r)^2(\Delta h)\Big) & & \quad + (.01)^2(.04)\Big)
\end{array}
$$

Example: Volume of a cylinder

Plug in (a) $r = 1, \quad \Delta r = .01, \quad h = 2, \quad \Delta h = .04,$ or
(b) $\Delta r = -.01$ and $\Delta h = -.04$:

$$
\begin{array}{rlr}
 & V(r+\Delta r, h+\Delta h) & \\
0^{\text{th}} \text{ order:} & = \pi\Big(r^2 h & = 2\pi \\
1^{\text{st}} \text{ order:} & \quad + 2rh\,\Delta r + r^2\,\Delta h & \pm .08\pi \\
2^{\text{nd}} \text{ order:} & \quad + 2r(\Delta r)(\Delta h) + h(\Delta r)^2 & + .001\pi \\
3^{\text{rd}} \text{ order:} & \quad + (\Delta r)^2(\Delta h)\Big) & \pm .000004\pi \\
\hline
 & & \text{(a) } 2.081004\pi \\
 & & \text{(b) } 1.920996\pi
\end{array}
$$

Including the 0^{th} and 1^{st} order terms gives the linear approximation.
Including all terms gives the exact value.

Linear approximation with more variables

- Four positive real numbers below 50 are rounded to one decimal place and multiplied together. Estimate the maximum error.

$$u = f(w,x,y,z) = wxyz \qquad f : \mathbb{R}^4 \to \mathbb{R}$$

- Rounding gives an error of up to ± 0.05 in each variable.
- Estimated change in u due to changes in w, x, y, z:

$$\Delta u = \frac{\partial u}{\partial w}\Delta w + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z = \nabla f \cdot \langle \Delta w, \Delta x, \Delta y, \Delta z \rangle$$

$$= xyz\,\Delta w + wyz\,\Delta x + wxz\,\Delta y + wxy\,\Delta z$$

- Upper bound on error: $\quad w{=}x{=}y{=}z{=}50, \quad \Delta w{=}\Delta x{=}\Delta y{=}\Delta z{=}.05$:

$$\Delta u = 4(50)^3(.05) = 25000$$

- The actual largest error is at $w{=}x{=}y{=}z{=}49.95$ rounded up to 50:

$$50^4 - (49.95)^4 \approx 24962.525$$

Derivative matrix

Consider $f(x,y) = \langle x^2y, e^{x^2}, y\rangle$ $\qquad f:\mathbb{R}^2 \to \mathbb{R}^3$

- Break it into three functions:

$$f_1(x,y) = x^2y \qquad f_2(x,y) = e^{x^2} \qquad f_3(x,y) = y$$

- The *matrix of partial derivatives* is

$$Df(x,y) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 2xe^{x^2} & 0 \\ 0 & 1 \end{bmatrix} \qquad Df(1,2) = \begin{bmatrix} 4 & 1 \\ 2e & 0 \\ 0 & 1 \end{bmatrix}$$

- This is a "3 by 2 matrix" (3×2):
  3 rows (one per output function)
  2 columns (one per input variable)
- $f:\mathbb{R}^n \to \mathbb{R}^m$ has an $m \times n$ derivative matrix.