2.3 Linear Approximation

(This is a portion of 2.3, not the whole section)

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Math 20C
Winter 2018
A **tangent plane** to a 3D surface $z = f(x, y)$ generalizes a **tangent line** to a 2D curve.

It’s a plane that just touches the surface at a point. It approximates the function when $(x, y)$ is near the starting point.

Find the tangent plane at point $P = (a, b, ?) = (a, b, f(a, b))$.
Find the tangent plane to surface $z = f(x, y)$ at $P = (a, b, f(a, b))$.

Holding $y = b$ constant gives a plane parallel to the $xz$ plane.

The intersection of the plane and surface is a 2D curve (red) with $z$ as a function of $x$.

The tangent line in that plane has slope $f_x(a, b)$:

$$y = b \quad \text{and} \quad z = f(a, b) + f_x(a, b) \cdot (x - a)$$
Find the tangent plane to surface $z = f(x, y)$ at $P = (a, b, f(a, b))$.

Holding $x = a$ constant gives a plane parallel to the $yz$ plane.

The intersection of the plane and surface is a 2D curve (green) with $z$ as a function of $y$.

The tangent line in that plane has slope $f_y(a, b)$:

$$x = a \quad \text{and} \quad z = f(a, b) + f_y(a, b) \cdot (y - b)$$
In 3D, the tangent plane to \( z = f(x, y) \) at \( P = (a, b, f(a, b)) \) is:

\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

- Intersecting with plane \( y = b \) gives \( z = f(a, b) + f_x(a, b)(x - a) \).
- Intersecting with plane \( x = a \) gives \( z = f(a, b) + f_y(a, b)(y - b) \).
- These are the tangent line formulas just described.

The tangent plane can also be written in vector format:

\[
z = f(a, b) + \nabla f(a, b) \cdot (x - a, y - b)
\]

where the \textit{gradient vector} is

\[
\nabla f = \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.
\]
Technicalities for the tangent plane to exist

Need \( f(x, y) \) and derivatives \( f_x(x, y) \) and \( f_y(x, y) \) to exist and be continuous at \((x, y) = (a, b)\), plus more technical conditions.

In these graphs, there is no tangent plane at the point at the top. On the right, there is no tangent plane at points along the creases.
Example: \( z = f(x, y) = x^2 + 4y^2 \)

Find the equation of the tangent plane at \((a, b) = (1, 2)\)

Need to fill in \( z \). At \((x, y) = (1, 2)\), \( z = 1^2 + 4(2^2) = 17 \).

Find the tangent plane at \((1, 2, 17)\).
Example: \( z = f(x, y) = x^2 + 4y^2 \)

Find the equation of the tangent plane at \((1, 2, 17)\)

\[
\begin{align*}
\text{Slopes} \\
 f(x, y) &= x^2 + 4y^2 \\
 f(x, y) &= 2x \\
 f(x, y) &= 8y \\
 f(1, 2) &= 1^2 + 4(2^2) = 17 \\
 f_x(1, 2) &= 2(1) = 2 \\
 f_y(1, 2) &= 8(2) = 16 \\
\end{align*}
\]

\[
\begin{align*}
\text{Tangent plane at } (a, b, f(a, b)) &= (1, 2, 17) \\
 z &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
 z &= f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \\
 z &= 17 + 2(x - 1) + 16(y - 2) \
\end{align*}
\]
Other ways to write tangent plane formula

\[ z = 17 + 2(x - 1) + 16(y - 2) \]

As a function

\[ L(x, y) = 17 + 2(x - 1) + 16(y - 2) \]

- \( f(x, y) \) is approximated by the tangent plane near the starting point:

\[
\begin{align*}
\underbrace{f(x, y)} & \approx \underbrace{L(x, y)} \quad \text{when } (x, y) \approx (1, 2) \\
z \text{ on surface} & \quad z \text{ on tangent plane}
\end{align*}
\]

- This is called *local linearity*. 

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Other ways to write tangent plane formula

\[ z = 17 + 2(x - 1) + 16(y - 2) \]

Surface: \( z = f(x, y) = x^2 + 4y^2 \)

Tangent plane: \( z = L(x, y) = 17 + 2(x - 1) + 16(y - 2) \)
Other ways to write tangent plane formula

\[ z = 17 + 2(x - 1) + 16(y - 2) \]

In terms of changes in \( x, y, z \)

\[ z - 17 = 2(x - 1) + 16(y - 2) \]

\[ \Delta z = 2 \Delta x + 16 \Delta y \]

where

\[ \Delta x = x - a \quad = x - 1 \]
\[ \Delta y = y - b \quad = y - 2 \]
\[ \Delta z = z - f(a, b) = z - 17 \]
Other ways to write tangent plane formula

\[ z = 17 + 2(x - 1) + 16(y - 2) \]

**Vector version**

\[ \nabla f(x, y) = \langle 2x, 8y \rangle \]

\[ \nabla f(1, 2) = \langle 2(1), 8(2) \rangle = \langle 2, 16 \rangle \]

\[ z = f(a, b) + \nabla f(a, b) \cdot \langle x - a, y - b \rangle = 17 + \nabla f(1, 2) \cdot \langle x - 1, y - 2 \rangle \]

\[ z = 17 + \langle 2, 16 \rangle \cdot \langle x - 1, y - 2 \rangle \]

**Vector version with changes in variables**

\[ \Delta z = \nabla f(a, b) \cdot \langle \Delta x, \Delta y \rangle \]

\[ \Delta z = \langle 2, 16 \rangle \cdot \langle \Delta x, \Delta y \rangle \]

where \( \Delta x = x - 1, \quad \Delta y = y - 2, \quad \Delta z = z - 17. \)
Example: Volume of a cylinder

Consider the volume of a cylinder of radius $r$ and height $h$:

$$V(r, h) = \pi r^2 h$$

Measurements:

$$r = 1 \pm 0.01 \text{ cm}$$
$$h = 2 \pm 0.04 \text{ cm}$$

Volume:

- Approximate $V$: $\pi \cdot 1^2 \cdot 2 = 2\pi \text{ cm}^3$
- Low estimate: $\pi \cdot (0.99)^2 \cdot (1.96) = 1.920996\pi \text{ cm}^3$
- High estimate: $\pi \cdot (1.01)^2 \cdot (2.04) = 2.081004\pi \text{ cm}^3$

The low and high estimates are about $(2 \pm 0.08)\pi \text{ cm}^3$. 
Example: Volume of a cylinder

The linear approximation to $V(r, h) = \pi r^2 h$ near point $(r, h) = (1, 2)$:

$$L(r + \Delta r, h + \Delta h) = V(r, h) + \frac{\partial V}{\partial r}(r, h) \Delta r + \frac{\partial V}{\partial h}(r, h) \Delta h$$

$$= \pi r^2 h + 2\pi rh \Delta r + \pi r^2 \Delta h$$

$$= \pi(1^2)(2) + 2\pi(1)(2) \Delta r + \pi(1)^2 \Delta h$$

$$= 2\pi + 4\pi \Delta r + \pi \Delta h$$

$L(1 + .01, 2 + .04) = 2\pi + 4\pi(.01) + \pi(.04) = 2.08\pi$

$L(1 - .01, 2 - .04) = 2\pi + 4\pi(-.01) + \pi(-.04) = 1.92\pi$
Example: Volume of a cylinder

Compare with the exact expansion of $V(r + \Delta r, h + \Delta h)$:

\[
V(r + \Delta r, h + \Delta h) = \pi (r + \Delta r)^2 (h + \Delta h)
\]

\[
= \pi (r^2 + 2r \Delta r + (\Delta r)^2)(h + \Delta h)
\]

0\textsuperscript{th} order (no $\Delta$’s) is $V(r, h)$:

\[
= \pi (r^2 h)
\]

1\textsuperscript{st} order/linear (1 $\Delta$):

\[
+ 2rh \Delta r + r^2 \Delta h
\]

2\textsuperscript{nd} order (2 $\Delta$’s):

\[
+ 2r(\Delta r)(\Delta h) + h(\Delta r)^2
\]

3\textsuperscript{rd} order (3 $\Delta$’s):

\[
+ (\Delta r)^2 (\Delta h)
\]

The linear approximation matches the 0\textsuperscript{th} plus 1\textsuperscript{st} order terms:

\[
L(r + \Delta r, h + \Delta h) = \pi r^2 h + 2\pi rh \Delta r + \pi r^2 h \Delta h
\]
Example: Volume of a cylinder

Plug in \( r = 1, \quad \Delta r = .01, \quad h = 2, \quad \Delta h = .04: \)

\[
V(r + \Delta r, h + \Delta h)
\]

0\(^{\text{th}}\) order: \( = \pi \left( r^2 h \right) = \pi \left( 1^2 \cdot 2 \right) \)

1\(^{\text{st}}\) order: \( + 2rh \Delta r + r^2 \Delta h \quad + 2(1)(2)(.01) + 1^2(.04) \)

2\(^{\text{nd}}\) order: \( + 2r(\Delta r)(\Delta h) + h(\Delta r)^2 \quad + 2(1)(.01)(.04) + 2(.01)^2 \)

3\(^{\text{rd}}\) order: \( + (\Delta r)^2(\Delta h) \quad + (.01)^2(.04) \)
Example: Volume of a cylinder

Plug in (a) \( r = 1, \quad \Delta r = .01, \quad h = 2, \quad \Delta h = .04, \) or (b) \( \Delta r = -.01 \) and \( \Delta h = -.04 \):

\[
V(r + \Delta r, h + \Delta h) = \pi \left( r^2 h + 2rh \Delta r + r^2 \Delta h \right) \pm .08\pi + .001\pi + .000004\pi
\]

Including the 0th and 1st order terms gives the linear approximation. Including all terms gives the exact value.
Linear approximation with more variables

- Four positive real numbers below 50 are rounded to one decimal place and multiplied together. Estimate the maximum error.

\[ u = f(w, x, y, z) = wxyz \quad f : \mathbb{R}^4 \to \mathbb{R} \]

- Rounding gives an error of up to ±0.05 in each variable.

- Estimated change in \( u \) due to changes in \( w, x, y, z \):

\[
\Delta u = \frac{\partial u}{\partial w} \Delta w + \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z = \nabla f \cdot \langle \Delta w, \Delta x, \Delta y, \Delta z \rangle \\
= xyz \Delta w + wyz \Delta x + wxz \Delta y + wxy \Delta z
\]

- Upper bound on error: \( w = x = y = z = 50 \), \( \Delta w = \Delta x = \Delta y = \Delta z = .05 \):

\[
\Delta u = 4(50)^3(.05) = 25000
\]

- The actual largest error is at \( w = x = y = z = 49.95 \) rounded up to 50:

\[
50^4 - (49.95)^4 \approx 24962.525
\]
Consider \( f(x, y) = \langle x^2y, e^{x^2}, y \rangle \) \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

- Break it into three functions:
  \[
  f_1(x, y) = x^2y \\
  f_2(x, y) = e^{x^2} \\
  f_3(x, y) = y
  \]

- The matrix of partial derivatives is

\[
Df(x, y) = \begin{bmatrix}
  \nabla f_1 \\
  \nabla f_2 \\
  \nabla f_3 \\
\end{bmatrix} = \begin{bmatrix}
  \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
  \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\
  \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \\
\end{bmatrix} = \begin{bmatrix}
  2xy & x^2 \\
  2xe^{x^2} & 0 \\
  0 & 1 \\
\end{bmatrix}
\]

\( Df(1, 2) = \begin{bmatrix}
  4 & 1 \\
  2e & 0 \\
  0 & 1 \\
\end{bmatrix} \)

- This is a “3 by 2 matrix” (3 \( \times \) 2):
  - 3 rows (one per output function)
  - 2 columns (one per input variable)

\( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) has an \( m \times n \) derivative matrix.