3.3 Optimizing Functions of Several Variables
3.4 Lagrange Multipliers

Prof. Tesler

Math 20C
Fall 2016
Optimizing $y = f(x)$

- In Math 20A, we found the minimum and maximum of $y = f(x)$ by using derivatives.

- **First derivative:**
  Solve for points where $f'(x) = 0$.
  Each such point is called a *critical point*.

- **Second derivative:**
  For each critical point $x = a$, check the sign of $f''(a)$:
  - $f''(a) > 0$: The value $y = f(a)$ is a local minimum.
  - $f''(a) < 0$: The value $y = f(a)$ is a local maximum.
  - $f''(a) = 0$: The test is inconclusive.

- Also may need to check points where $f(x)$ is defined but the derivatives aren’t, as well as boundary points.

- We will generalize this to functions $z = f(x, y)$. 
Local extrema (= maxima or minima)

Consider a function \( z = f(x, y) \).

The point \((x, y) = (a, b)\) is a

- **local maximum** when \( f(x, y) \leq f(a, b) \) for all \((x, y)\) in a small disk (filled-in circle) around \((a, b)\);
- **global maximum** (a.k.a. absolute maximum) when \( f(x, y) \leq f(a, b) \) for all \((x, y)\);
- **local minimum** and **global minimum** are similar with \( f(x, y) \geq f(a, b) \).

- \(A, C, E\) are local maxima (plural of maximum)
- \(E\) is the global maximum
- \(D, G\) are local minima
- \(G\) is the global minimum

- \(B\) is maximum in the red cross-section but minimum in the purple cross-section!
  It’s called a **saddle point**.
Classify each point $P, Q, R, S$ as local maximum or minimum, saddle point, or none.

- Isolated max/min usually have small closed curves around them. Values decrease towards $P$, so $P$ is a local minimum. Values increase towards $Q$, so $Q$ is a local maximum.
The crossing contours have the same value, 1. (If they have different values, the function is undefined at that point.)

Here, the crossing contours give four regions around \( R \).

The function has
- a local min. at \( R \) on lines with positive slope (goes from \( >1 \) to \( 1 \) to \( >1 \))
- a local max. at \( R \) on lines with neg. slope (goes from \( <1 \) to \( 1 \) to \( <1 \)).

Thus, \( R \) is a saddle point.
$S$ is a regular point.
Its level curve $\approx 8$ is implied but not shown.
The values are bigger on one side and smaller on the other.

$P$: local min  $Q$: local max  $R$: saddle point  $S$: none
Contours of $z = y/x$ are diagonal lines: $z = c$ along $y = cx$.

Contours cross at $(0, 0)$ and have different values there.

Function $z = y/x$ is undefined at $(0, 0)$. 
Contour map of $z = \sin(y)$

Minimum and maximum form curves, not just isolated points

- Contours of $z = f(x, y) = \sin(y)$ are horizontal lines $y = \arcsin(z)$
- Maximum at $y = (2k + \frac{1}{2})\pi$ for all integers $k$
- Minimum at $y = (2k - \frac{1}{2})\pi$
- These are curves, not isolated points enclosed in contours.
Finding the minimum/maximum values of $z = f(x, y)$

- The tangent plane is horizontal at a local minimum or maximum:
  $$f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) - z = 0.$$  
  The normal vector $\langle f_x(a, b), f_y(a, b), -1 \rangle \parallel z$-axis   
  when $f_x(a, b) = f_y(a, b) = 0$, or $\nabla f(a, b) = \vec{0}$.

- At points where $\nabla f \neq \vec{0}$, we can make $f(x, y)$
  - larger by moving in the direction of $\nabla f$;
  - smaller by moving in the direction of $-\nabla f$.

- $(a, b)$ is a critical point if $\nabla f(a, b)$ is $\vec{0}$ or is undefined.
  These are candidates for being maximums or minimums.

- Critical points found in the same way for $f(x, y, z, \ldots)$. 

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3.3–3.4 Optimization

Math 20C / Fall 2016 9 / 54
Critical points

- Let $f(x, y) = x^2 - 2x + y^2 - 4y + 15$
  \[ \nabla f = (2x - 2, 2y - 4) \]

- $\nabla f = \vec{0}$ at $x = 1, y = 2$, so $(1, 2)$ is a critical point.

- Use $\begin{align*}
(x - 1)^2 &= x^2 - 2x + 1 \\
(y - 2)^2 &= y^2 - 4y + 4
\end{align*}$
  \[ f(x, y) = (x - 1)^2 + (y - 2)^2 + 10 \]

  We “completed the squares”: $x^2 - ax = (x - \frac{a}{2})^2 - (\frac{a}{2})^2$

- $f(x, y) \geq 10$ everywhere, with global minimum 10 at $(x, y) = (1, 2)$. 
Compute all points where $\nabla f(a, b) = \vec{0}$, and classify each as follows:

- Compute the **discriminant**

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

Determinant of “Hessian matrix” at $(x, y) = (a, b)$

- If $D > 0$ and $f_{xx} > 0$ then $z = f(a, b)$ is a local minimum;
- If $D > 0$ and $f_{xx} < 0$ then $z = f(a, b)$ is a local maximum;
- If $D < 0$ then $f$ has a saddle point at $(a, b)$;
- If $D = 0$ then it’s inconclusive;
- min, max, saddle, or none of these, are all possible.
**Example:**

\[ f(x, y) = x^2 - y^2 \]

Find the critical points of \( f(x, y) = x^2 - y^2 \) and classify them using the second derivatives test.

\[ \nabla f = \langle 2x, -2y \rangle = \vec{0} \text{ at } (x, y) = (0, 0). \]

The \( x = 0 \) cross-section is \( f(0, y) = -y^2 \leq 0 \).

The \( y = 0 \) cross-section is \( f(x, 0) = x^2 \geq 0 \).

It is neither a minimum nor a maximum.

\( f_{xx}(x, y) = 2 \) and \( f_{xx}(0, 0) = 2 \)

\( f_{yy}(x, y) = -2 \) and \( f_{yy}(0, 0) = -2 \)

\( f_{xy}(x, y) = 0 \) and \( f_{xy}(0, 0) = 0 \)

\[
D = f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2 \\
= (2)(-2) - 0^2 = -4 < 0
\]

so \( (0, 0) \) is a saddle point.
Example: 
\[ f(x, y) = x^2 - y^2 \]

- \( \nabla f = \langle 2x, -2y \rangle \) points in the direction of greatest increase of the function.
- The function increases as we move towards the \( x \)-axis and away from the \( y \)-axis. At the origin, it increases or decreases depending on the direction of approach.
Example: \[ f(x, y) = 8y^3 + 12x^2 - 24xy \]

Find the critical points of \( f(x, y) \) and classify them using the second derivatives test.

- Solve for first derivatives equal to 0:
  
  \[
  f_x = 24x - 24y = 0 \quad \text{gives} \quad x = y \\
  f_y = 24y^2 - 24x = 0 \quad \text{gives} \quad 24y^2 - 24y = 24y(y - 1) = 0 \\
  \text{so} \quad y = 0 \quad \text{or} \quad y = 1 \\
  \text{so} \quad x = y \\
  \]

- Critical points: \((0, 0)\) and \((1, 1)\)

- Second derivative test: \[ (D = f_{xx}f_{yy} - (f_{xy})^2) \]

<table>
<thead>
<tr>
<th>Crit pt</th>
<th>( f )</th>
<th>( f_{xx} = 24 )</th>
<th>( f_{yy} = 48y )</th>
<th>( f_{xy} = -24 )</th>
<th>( D )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>0</td>
<td>24</td>
<td>0</td>
<td>-24</td>
<td>-576</td>
<td>( D &lt; 0 ) saddle</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>-4</td>
<td>24</td>
<td>48</td>
<td>-24</td>
<td>576</td>
<td>( D &gt; 0 ) and ( f_{xx} &gt; 0 ) local minimum</td>
</tr>
</tbody>
</table>

- No absolute min or max: \( f(0, y) = 8y^3 \) ranges over \(( -\infty, \infty )\)
Example: \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \)

Find the critical points of \( f(x, y) \) and classify them using the second derivatives test.

- Solve for first derivatives equal to 0:
  
  \[
  f_x = 3x^2 - 3 = 0 \quad \text{gives} \quad x = \pm 1
  \]
  
  \[
  f_y = 3y^2 - 6y = 3y(y - 2) = 0 \quad \text{gives} \quad y = 0 \text{ or } y = 2
  \]

- Critical points: \((-1, 0), (1, 0), (−1, 2), (1, 2)\)
- Second derivative test: \( D = f_{xx}f_{yy} - (f_{xy})^2 \)

<table>
<thead>
<tr>
<th>Crit pt</th>
<th>( f )</th>
<th>( f_{xx} = 6x )</th>
<th>( f_{yy} = 6y - 6 )</th>
<th>( f_{xy} = 0 )</th>
<th>( D )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1, 0))</td>
<td>3</td>
<td>−6</td>
<td>−6</td>
<td>0</td>
<td>36</td>
<td>( D &gt; 0 ) and ( f_{xx} &lt; 0 ): local max</td>
</tr>
<tr>
<td>((1, 0))</td>
<td>−1</td>
<td>6</td>
<td>−6</td>
<td>0</td>
<td>−36</td>
<td>( D &lt; 0 ): saddle</td>
</tr>
<tr>
<td>((-1, 2))</td>
<td>−1</td>
<td>−6</td>
<td>6</td>
<td>0</td>
<td>−36</td>
<td>( D &lt; 0 ): saddle</td>
</tr>
<tr>
<td>((1, 2))</td>
<td>−5</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>36</td>
<td>( D &gt; 0 ) and ( f_{xx} &gt; 0 ): local min</td>
</tr>
</tbody>
</table>
Consider a region $A \subset \mathbb{R}^n$.

A point is a **boundary point** of $A$ if every disk (blue) around that point contains some points in $A$ and some points not in $A$.

A point is an **interior point** of $A$ if there is a small enough disk (pink) around it fully contained in $A$.

In both $A$ and $B$, the boundary points are the same: the perimeter of the hexagon.

$\partial A$ denotes the set of boundary points of $A$. 
Extreme Value Theorem

A region is **bounded** if it fits in a disk of finite radius.

A region is **closed** if it contains all its boundary points and **open** if every point in it is an interior point.

Open and closed are *not* opposites: e.g., $\mathbb{R}^2$ is open and closed!

The third example above is neither open nor closed.

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Extreme Value Theorem

If $f(x, y)$ is continuous on a closed and bounded region, then it has a global maximum and a global minimum within that region.

To find these, consider the local minima/maxima of $f(x, y)$ that are within the region, and also analyze the boundary of the region.
Example: $f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ in a triangle

Find the global minimum and maximum of $f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ in the triangle with vertices $(0, 0), (0, 3), (3, 3)$.

**Critical points inside the region**

- $f(1, 2) = -5$ is a local minimum and is inside the triangle.
- Ignore the other critical points since they are outside the triangle.
- Ignore the saddle points.
Example: \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in a triangle

Find the global minimum and maximum of \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in the triangle with vertices \((0, 0), (0, 3), (3, 3)\).

**Extrema on left edge:** \( x = 0 \) and \( 0 \leq y \leq 3 \)

- Set
  \[
  g(y) = f(0, y) = y^3 - 3y^2 + 1 \quad \text{for} \quad 0 \leq y \leq 3
  \]
  \[
  g'(y) = 3y^2 - 6y = 3y(y - 2)
  \]
  \[
  g'(y) = 0 \quad \text{at} \quad y = 0 \text{ or } 2.
  \]
- We consider \( y = 0 \) and 2 by that test.
- We also consider boundaries \( y = 0 \) and 3.
- Candidates:
  \[
  f(0, 0) = 1
  \]
  \[
  f(0, 2) = -3
  \]
  \[
  f(0, 3) = 1
  \]
- We could use the second derivatives test for one variable, but we’ll do it another way.
Example: \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in a triangle

Find the global minimum and maximum of \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in the triangle with vertices \((0, 0), (0, 3), (3, 3)\).

### Extrema on top edge: \( y = 3 \) and \( 0 \leq x \leq 3 \)

- Set
  \[
  h(x) = f(x, 3) = x^3 + 27 - 3x - 27 + 1 = x^3 - 3x + 1 \quad \text{for} \quad 0 \leq x \leq 3
  \]
  \[
  h'(x) = 3x^2 - 3
  \]
  \[
  h'(x) = 0 \quad \text{at} \quad x = \pm 1 \quad (\text{but} -1 \text{ is out of range})
  \]
- Also consider the boundaries \( x = 0 \) and \( 3 \).
- Candidates: \( f(0, 3) = 1 \)
  \( f(1, 3) = -1 \)
  \( f(3, 3) = 19 \)
Example: \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in a triangle

Find the global minimum and maximum of \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in the triangle with vertices \((0, 0), (0, 3), (3, 3)\).

**Diagonal edge: \( y = x \) for \( 0 \leq x \leq 3 \)**

- For \( 0 \leq x \leq 3 \), set
  \[
  p(x) = f(x, x) = 2x^3 - 3x - 3x^2 + 1 = 2x^3 - 3x^2 - 3x + 1
  \]
  \[
  p'(x) = 6x^2 - 6x - 3
  \]
  \[
  p'(x) = 0\quad\text{at}\quad x = \frac{1 \pm \sqrt{3}}{2} \approx -0.366, 1.366
  \]
  (but \( \frac{1 - \sqrt{3}}{2} \) is out of range)

- Also consider the boundaries \( x = 0 \) and \( 3 \).

- **Candidates**: \( f(0, 0) = 1 \)
  \[
  f(3, 3) = 19
  \]
  \[
  f \left( \frac{1 + \sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2} \right) = -1 - \frac{3 \sqrt{3}}{2} \approx -3.598
  \]
Example: \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in a triangle

Find the global minimum and maximum of \( f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1 \) in the triangle with vertices \((0, 0), (0, 3), (3, 3)\).

Compare all candidate points

\[ f(1, 2) = -5: \]
The global minimum is \(-5\). It occurs at \((x, y) = (1, 2)\).

\[ f(3, 3) = 19: \]
The global maximum is 19. It occurs at \((x, y) = (3, 3)\).
Example: \[ f(x, y) = xy(1 - x - y) \]

Find the critical points of \( f(x, y) \) and classify them.

- Solve for first derivatives equal to 0:
  \[
  f' = xy - x^2y - xy^2
  \]
  \[
  f_x = y - 2xy - y^2 = y(1 - 2x - y) \quad \text{gives} \quad y = 0 \text{ or } 1 - 2x - y = 0
  \]
  \[
  f_y = x - x^2 - 2xy = x(1 - x - 2y) \quad \text{gives} \quad x = 0 \text{ or } 1 - x - 2y = 0
  \]

- Two solutions of \( f_x = 0 \) and two of \( f_y = 0 \) gives \( 2 \cdot 2 = 4 \) combinations:
  - \( y = 0 \) and \( x = 0 \) gives \( (x, y) = (0, 0) \).
  - \( y = 0 \) and \( 1 - x - 2y = 0 \) gives \( (x, y) = (1, 0) \).
  - \( 1 - 2x - y = 0 \) and \( x = 0 \) gives \( (x, y) = (0, 1) \).
  - \( 1 - 2x - y = 0 \) and \( 1 - x - 2y = 0 \):
    
    The 1\textsuperscript{st} equation gives \( y = 1 - 2x \). Plug that into the 2\textsuperscript{nd} equation:
    
    \[ 0 = 1 - x - 2y = 1 - x - 2(1 - 2x) = 1 - x - 2 + 4x = 3x - 1 \]
    
    so \( x = \frac{1}{3} \) and \( y = 1 - 2x = 1 - 2\left( \frac{1}{3} \right) = \frac{1}{3} \) gives \( (x, y) = \left( \frac{1}{3}, \frac{1}{3} \right) \).
Example: \[ f(x, y) = xy(1 - x - y) \]

Classify the critical points using the second derivatives test.

- Derivatives:
  \[
  f = xy - x^2y - xy^2 \quad f_x = y - 2xy - y^2 \quad f_y = x - x^2 - 2xy \\
  f_{xx} = -2y \quad f_{yy} = -2x \\
  f_{xy} = f_{yx} = 1 - 2x - 2y
  \]

- Second derivative test: \[ (D = f_{xx}f_{yy} - (f_{xy})^2) \]

<table>
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<tr>
<th>Crit pt</th>
<th>( f )</th>
<th>( f_{xx} )</th>
<th>( f_{yy} )</th>
<th>( f_{xy} )</th>
<th>( D )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>−1</td>
<td>( D &lt; 0 ): saddle</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>0</td>
<td>0</td>
<td>−2</td>
<td>−1</td>
<td>−1</td>
<td>( D &lt; 0 ): saddle</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>0</td>
<td>−2</td>
<td>0</td>
<td>−1</td>
<td>−1</td>
<td>( D &lt; 0 ): saddle</td>
</tr>
<tr>
<td>(1/3, 1/3)</td>
<td>1/27</td>
<td>−2/3</td>
<td>−2/3</td>
<td>−1/3</td>
<td>1/3</td>
<td>( D &gt; 0 ) and ( f_{xx} &lt; 0 ): local maximum</td>
</tr>
</tbody>
</table>
Extrema of \( f(x, y) = |xy| \): \( \nabla f \) isn’t defined everywhere

Extrema of \( f(x, y) = |xy| \) on rectangle
\(-1 \leq x \leq 1, -2 \leq y \leq 2 \)

- 1st & 3rd quadrants: \( f(x, y) = xy \) and \( \nabla f = \langle y, x \rangle \).
- 2nd & 4th quadrants: \( f(x, y) = -xy \) and \( \nabla f = -\langle y, x \rangle \).
- Away from the axes, \( \nabla f \neq 0 \).
- On the axes, \( \nabla f \) is undefined.
  - \( f(x, 0) = f(0, y) = 0 \) on the axes.
    All points on the axes are tied for global minimum.
- On the perimeter, \( f(\pm 1, y) = |y| \) and \( f(x, \pm 2) = 2|x| \):
  - Minimum \( f = 0 \) at \((\pm 1, 0)\) and \((0, \pm 2)\).
  - Maximum \( f = 2 \) at \((1, 2), (1, -2), (-1, 2), (-1, -2)\).
- The global maximum is \( f = 2 \) at
  \((1, 2), (1, -2), (-1, 2), (-1, -2)\).
Extrema of $f(x, y) = |xy|$: $\nabla f$ isn’t defined everywhere.

Extrema of $f(x, y) = |xy|$ on open rectangle $-1 < x < 1$, $-2 < y < 2$

- Global minimum is still $f = 0$ on axes.
- No global maximum. While $f(x, y)$ gets arbitrarily close to 2, it never reaches 2 since those corners are not in the open rectangle.
The **Hessian matrix** of $f(x, y, z)$ is

$$
\begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\
\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\
\frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2}
\end{bmatrix}
$$

For $f : \mathbb{R}^n \to \mathbb{R}$, it’s an $n \times n$ matrix of 2\textsuperscript{nd} partial derivatives.

For each point with $\nabla f = \vec{0}$, compute the determinants of the upper left $1 \times 1$, $2 \times 2$, $3 \times 3$, \ldots, $n \times n$ submatrices.

- If the $n \times n$ determinant is zero, the test is inconclusive.
- If the determinants are all positive, it’s a local minimum.
- If signs of determinants alternate $-, +, -, \ldots$, it’s a local maximum.
- Otherwise, it’s a saddle point.

We did $2 \times 2$ and $3 \times 3$ determinants. For $1 \times 1$, $\det[x] = x$.

$n \times n$ determinants are covered in Linear Algebra (Math 18).
Optional example: \( f(x, y, z) = x^2 + y^2 + z^2 + 2xyz + 10 \)

- Solve \( \nabla f = \vec{0} \):
  \[
  \nabla f = \langle 2x + 2yz, 2y + 2xz, 2z + 2xy \rangle = \vec{0}
  \]
  \[
  x = -yz, \quad y = -xz, \quad z = -xy.
  \]

- There are five solutions \((x, y, z)\) of \( \nabla f = \vec{0} \) (work not shown):
  \((0, 0, 0), (1, 1, -1), (-1, 1, 1), (1, -1, 1), (-1, -1, -1)\).

- Hessian
  \[
  \begin{bmatrix}
  2 & 2z & 2y \\
  2z & 2 & 2x \\
  2y & 2x & 2
  \end{bmatrix}
  \]
  At \((0, 0, 0)\):
  \[
  \begin{bmatrix}
  2 & 0 & 0 \\
  0 & 2 & 0 \\
  0 & 0 & 2
  \end{bmatrix}
  \]
  \[
  \det \begin{bmatrix} 2 \end{bmatrix} = 2 \quad \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 \quad \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 8
  \]

- All positive, so \( f(0, 0, 0) = 10 \) is a local minimum.
Optional example: \( f(x, y, z) = x^2 + y^2 + z^2 + 2xyz + 10 \)

Hessian = \[
\begin{bmatrix}
2 & 2z & 2y \\
2z & 2 & 2x \\
2y & 2x & 2
\end{bmatrix}
\]

At \((1, 1, -1)\):
\[
\begin{bmatrix}
2 & -2 & 2 \\
-2 & 2 & 2 \\
2 & 2 & 2
\end{bmatrix}
\]

\[\det[2] = 2 \quad \det\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 0 \quad \det\begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = -32\]

Signs +, 0, −, so saddle point.

Critical points \((-1, 1, 1), (1, -1, 1), (-1, -1, -1)\) give the same determinants \(2, 0, -32\) as this case, so they’re also saddle points.
A hiker hikes on a mountain \( z = f(x, y) = \sqrt{1 - x^2 - y^2} \). Plot their trail on a topographic map: \( x^2 + 4y^2 = 1 \) (red ellipse).

What is the minimum and maximum height reached, and where?

On the ellipse, \( y^2 = (1 - x^2)/4 \) and \(-1 \leq x \leq 1\), so

\[
  z = \sqrt{1 - x^2 - (1 - x^2)/4} = \sqrt{\frac{3}{4}(1 - x^2)}
\]

**Minimum at \( x = \pm 1 \)**

- \( y^2 = (1 - (\pm 1)^2)/4 = 0 \) so \( y = 0 \)
- \( z = \sqrt{(3/4)(1 - (\pm 1)^2)} = 0 \)
- Min: \( z = 0 \) at \((x, y) = (\pm 1, 0)\)

**Maximum at \( x = 0 \)**

- \( y^2 = (1 - 0^2)/4 = 1/4 \) so \( y = \pm \frac{1}{2} \)
- \( z = \sqrt{\frac{3}{4}(1 - 0^2)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \)
- Max: \( z = \frac{\sqrt{3}}{2} \) at \((x, y) = (0, \pm \frac{1}{2})\)
3.4. Lagrange Multipliers

General problem

Find the minimum and maximum of \( f(x, y, z, \ldots) \)
subject to the constraint
\[
g(x, y, z, \ldots) = c \ \text{(constant)}
\]

This problem

Find the minimum and maximum of \( f(x, y) = \sqrt{1 - x^2 - y^2} \)
subject to the constraint
\[
g(x, y) = x^2 + 4y^2 = 1
\]

Approaches

- Use the constraint \( g \) to solve for one variable in terms of the other(s), then plug into \( f \) and find its extrema.
- New method: Lagrange Multipliers
On the contour map, when the trail \((g(x,y) = c, \text{ in red})\) crosses a contour of \(f(x,y)\), \(f\) is lower on one side and higher on the other.

The min/max of \(f(x,y)\) on the trail occurs when the trail is tangent to a contour of \(f(x,y)\)! The trail goes up to a max and then back down, staying on the same side of the contour of \(f\).

Recall \(\nabla f \perp \) contours of \(f\) \(\nabla g \perp \) contours of \(g\)

So contours of \(f\) and \(g\) are tangent when \(\nabla f \parallel \nabla g\), or \(\nabla f = \lambda \nabla g\) for some scalar \(\lambda\) (called a \textit{Lagrange Multiplier}).
Lagrange Multipliers for the ellipse path

- Find the minimum and maximum of \( z = \sqrt{1 - x^2 - y^2} \) 
  subject to the constraint \( x^2 + 4y^2 = 1 \).
- This is equivalent to finding the extrema of \( z^2 = 1 - x^2 - y^2 \).
- Set \( f(x, y) = 1 - x^2 - y^2 \) and \( g(x, y) = x^2 + 4y^2 \) (constraint: \( = 1 \)).
- \[ \nabla f = \langle -2x, -2y \rangle \quad \nabla g = \langle 2x, 8y \rangle \]
- Solve \( \nabla f = \lambda \nabla g \) and \( g(x, y) = c \) for \( x, y, \lambda \):
  \[
  \begin{align*}
  -2x &= 2\lambda x \\
  -2y &= 8\lambda y \\
  2x(1 + \lambda) &= 0 \\
  y(2 + 8\lambda) &= 0 \\
  x = 0 \text{ or } \lambda &= -1 \\
  y = 0 \text{ or } \lambda &= -1/4
  \end{align*}
  \]
- Solutions:
  - \( x = 0 \) gives \( y = \pm \sqrt{1 - 0^2/2} = \pm \frac{1}{2}, \quad \lambda = -2/8 = -1/4, \)
  \[ z = \sqrt{1 - 0^2 - (1/2)^2} = \sqrt{3}/2. \]
  - \( \lambda = -1 \) gives \( y = 0, \quad x = \pm \sqrt{1 - 4(0)^2} = \pm 1, \)
  \[ z = \sqrt{1 - (\pm 1)^2 - 0^2} = 0. \]
Lagrange Multipliers for the ellipse path

- \( \sqrt{1 - x^2 - y^2} \) is continuous along the closed path \( x^2 + 4y^2 = 1 \), so
  - \( z = \frac{\sqrt{3}}{2} \) at \( (x, y) = (0, \pm \frac{1}{2}) \) are absolute maxima
  - \( z = 0 \) at \( (x, y) = (\pm 1, 0) \) are absolute minima

- \( \lambda \) is a tool to solve for the extremal points; its value isn’t important.
Find the extrema of \( z = \sqrt{1 - x^2 - y^2} \) subject to the constraint \( x^2 + 4y^2 \leq 1 \).

- Analyze interior points and boundary points separately. Then select the minimum and maximum out of all candidates.

- In \( x^2 + 4y^2 < 1 \) (yellow interior), use critical points to show the maximum is \( f(0, 0) = 1 \).

- On boundary \( x^2 + 4y^2 = 1 \) (red ellipse), use Lagrange Multipliers. minimum \( f(\pm 1, 0) = 0 \), maximum \( f(0, \pm \frac{1}{2}) = \frac{\sqrt{3}}{2} \approx 0.866 \).

- Comparing candidates (red spots) gives absolute minimum \( f(\pm 1, 0) = 0 \), absolute maximum \( f(0, 0) = 1 \).
An open rectangular box (5 sides but no top) has volume \( 500 \, \text{cm}^3 \). What dimensions give the minimum surface area, and what is that area? 

Physical intuition says there is some minimum amount of material needed in order to hold a given volume. We will solve for this.

There’s no maximum, though:
- e.g., let \( x = y, \, z = \frac{500}{xy} = \frac{500}{x^2} \), and let \( x \to \infty \). Then \( A \to \infty \).
Example: Rectangular box
Method 1: Critical points

An open rectangular box (5 sides but no top) has volume 500 cm$^3$. What dimensions give the minimum surface area, and what is that area?

Dimensions $x, y, z > 0$
Volume $V = xyz = 500$
Area $A = xy + 2xz + 2yz$

- The volume equation gives $z = \frac{500}{xy}$
- Plug that into the area equation:

$$A = xy + 2x \cdot \frac{500}{xy} + 2y \cdot \frac{500}{xy} = xy + \frac{1000}{y} + \frac{1000}{x}$$
Example: Rectangular box

Method 1: Critical points

\[ A = xy + \frac{1000}{y} + \frac{1000}{x} \]

- Find first derivatives:
  \[ A_x = y - \frac{1000}{x^2} \quad A_y = x - \frac{1000}{y^2} \]

- Solve \( A_x = A_y = 0 \): Plug \( y = \frac{1000}{x^2} \) into \( x = \frac{1000}{y^2} \) to get
  \[ x = \frac{1000}{(1000/x^2)^2} = \frac{x^4}{1000} \quad x^4 - 1000x = 0 \quad x(x^3 - 1000) = 0 \]
  so \( x = 0 \) or \( x = 10 \) (and two complex solutions)

- \( x = 0 \) violates \( V = xyz = 500 \).
  Also, we need \( x > 0 \) for a real box.

- \( x = 10 \) gives \( y = \frac{1000}{x^2} = \frac{1000}{10^2} = 10 \) and \( z = \frac{500}{xy} = \frac{500}{(10)(10)} = 5 \)
Example: Rectangular box
Method 1: Critical points

\[ A = xy + \frac{1000}{y} + \frac{1000}{x} \]

- Check if \( x = y = 10 \) is a critical point:
  \[ A_x = y - \frac{1000}{x^2} = 10 - \frac{1000}{10^2} = 10 - 10 = 0 \]
  \[ A_y = x - \frac{1000}{y^2} = 10 - \frac{1000}{10^2} = 10 - 10 = 0 \]

- Yes, it’s a critical point.
- Solution of original problem:

  Dimensions \( x = y = 10 \text{ cm}, \ z = 5 \text{ cm} \)
  Volume \( V = xyz = (10)(10)(5) = 500 \text{ cm}^3 \)
  Area \( A = xy + 2xz + 2yz \)
  \[ = (10)(10) + 2(10)(5) + 2(10)(5) = 300 \text{ cm}^2 \]
Example: Rectangular box

Method 1: Critical points

\[
A = xy + \frac{1000}{y} + \frac{1000}{x}
\]

Second derivatives test at \((x, y) = (10, 10)\):

\[
A_{xx} = \frac{2000}{x^3} = \frac{2000}{10^3} = 2
\]

\[
A_{yy} = \frac{2000}{y^3} = \frac{2000}{10^3} = 2
\]

\[
A_{xy} = 1
\]

\[
D = (2)(2) - 1^2 = 3 > 0 \text{ and } A_{xx} > 0 \text{ so local minimum}
\]
Example: Rectangular box

Method 1: Critical points

Using gradients instead of 2\textsuperscript{nd} derivatives test

\[ A = xy + \frac{1000}{y} + \frac{1000}{x} \]

\[ A_x = y - \frac{1000}{x^2} \quad A_y = x - \frac{1000}{y^2} \]

- The signs of \(A_x, A_y\) split the first quadrant into four regions.
- \(\nabla A(x, y)\) points away from \((10, 10)\) in each region.
- \(A(x, y)\) increases as we move away from \((10, 10)\) in each region.
- So \((10, 10)\) is the location of the global minimum.
Example: Rectangular box
Method 2: Lagrange Multipliers

An open rectangular box (5 sides but no top) has volume 500 cm$^3$. What dimensions give the minimum surface area, and what is that area?

Dimensions $x, y, z > 0$

Volume $V = xyz = 500$

Area $A = xy + 2xz + 2yz$

- Solve $\nabla A = \lambda \nabla V$ and $V = xyz = 500$ for $x, y, z, \lambda$.
- Solve $\langle y + 2z, x + 2z, 2x + 2y \rangle = \lambda \langle yz, xz, xy \rangle$ and $V = xyz = 500$
- Solve for $\lambda$:

$$\lambda = \frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2x + 2y}{xy}$$

$$\lambda = \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

There is no division by 0 since $xyz = 500$ implies $x, y, z \neq 0$. 
Example: Rectangular box
Method 2: Lagrange Multipliers

\[ \lambda = \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x} \]

- Taking any two of those at a time gives
  \[ \frac{1}{z} = \frac{2}{y} = \frac{2}{x} \]
  so \( x = y = 2z \).

- Combine with \( xyz = 500 \):
  \[ (2z)(2z)(z) = 4z^3 = 500 \]
  \[ z^3 = 500/4 = 125 \] and \( z = 5 \)
  \[ x = y = 2z = 10 \]
  \( (x, y, z) = (10, 10, 5) \) cm

- Area: \( (10)(10) + 2(10)(5) + 2(10)(5) = 300 \) cm\(^2\).

- This method doesn’t tell you if it’s a minimum or a maximum! Use your intuition (in this case, there is a minimum area that can encompass the volume, but not a maximum) or test nearby values.
Example: Rectangular box
Method 2: Lagrange Multipliers

This method doesn’t tell you if it’s a minimum or a maximum!

- Use your intuition (in this case, there is a minimum area that can encompass the volume, but not a maximum) or test nearby values.

- Surface $xyz = 500$ (with $x, y, z > 0$) is not bounded, so Extreme Value Theorem doesn’t apply. No guarantee there’s a global min/max in the region.

- Only one candidate point, so we can’t compare candidates.

- Pages 197–201 extend the $2^{nd}$ derivatives test to constraint equations, but it uses Linear Algebra (Math 18).
Example: Function of 10 variables

Find 10 positive #'s whose sum is 1000 and whose product is maximized:

Maximize \( f(x_1, \ldots, x_{10}) = x_1 x_2 \cdots x_{10} \)

Subject to \( g(x_1, \ldots, x_{10}) = x_1 + \cdots + x_{10} = 1000 \)

\[ \nabla f = \left\langle \frac{f}{x_1}, \ldots, \frac{f}{x_{10}} \right\rangle \]

\[ \nabla g = \langle 1, \ldots, 1 \rangle \]

- Solve \( \nabla f = \lambda \nabla g \):
  \[ \frac{f}{x_1} = \cdots = \frac{f}{x_{10}} = \lambda \cdot 1 \]
  \[ x_1 = \cdots = x_{10} \]

- Combine with constraint \( g = x_1 + \cdots + x_{10} = 1000 \):
  \[ 10x_1 = 1000 \quad \text{so} \quad x_1 = \cdots = x_{10} = 100 \]

- The product is \( 100^{10} = 10^{20} \). This turns out to be the maximum.

- Minimum: as any of the variables approach 0, the product approaches 0, without reaching it. So, in the domain \( x_1, \ldots, x_{10} > 0 \), the minimum does not exist.
What point on the plane \( x + 2y + z = 4 \) is closest to the origin?

- Physical intuition tells us there is a minimum but not a maximum.
- No max: plane has infinite extent, with points arbitrarily far away.
- Approaches: vector projections (Chapter 1.2), critical points (3.3), and Lagrange Multipliers (3.4).

**Generalization:** Given a point \( A \), find the closest point to \( A \) on surface \( z = f(x, y) \).
What point on the plane $x + 2y + z = 4$ is closest to the origin?

- Pick any point $Q$ on the plane; let’s use $Q = (1, 1, 1)$.
- Form the projection of $\vec{a} = \overrightarrow{OQ} = \langle 1, 1, 1 \rangle$ along the normal vector $\vec{n} = \langle 1, 2, 1 \rangle$ to get $\overrightarrow{OP}$, where $P$ is the closest point:

$$\overrightarrow{OP} = \frac{(\vec{a} \cdot \vec{n})\vec{n}}{\|\vec{n}\|^2} = \frac{(1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1)\vec{n}}{1^2 + 2^2 + 1^2} = \frac{4\vec{n}}{6} = \left\langle \frac{2}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle$$

- Closest point is $P = O + \overrightarrow{OP} = \left( \frac{2}{3}, \frac{4}{3}, \frac{2}{3} \right)$. 
Closest point on a plane to the origin

Method 2: Critical points

What point on the plane \( x + 2y + z = 4 \) is closest to the origin?

- For \((x, y, z)\) on the plane, the distance to the origin is
  \[
  f(x, y, z) = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}
  \]
  This is minimized at the same place as its square:
  \[
  g(x, y, z) = x^2 + y^2 + z^2
  \]

- On the plane, \( z = 4 - x - 2y \). So find \((x, y)\) that minimize
  \[
  h(x, y) = x^2 + y^2 + (4 - x - 2y)^2
  \]

  Then plug the solution(s) of \((x, y)\) into \( z = 4 - x - 2y \).
Closest point on a plane to the origin

Method 2: Critical points

What point on the plane \( x + 2y + z = 4 \) is closest to the origin?

- Minimize \( h(x, y) = x^2 + y^2 + (4 - x - 2y)^2 \).

First derivatives:

\[
\begin{align*}
    h_x &= 2x - 2(4 - x - 2y) = 4x + 4y - 8 \\
    h_y &= 2y + 2(-2)(4 - x - 2y) = 4x + 10y - 16
\end{align*}
\]

Critical points: solve \( h_x = h_y = 0 \):

\[
\begin{align*}
    h_x &= 0 \quad \text{gives} \quad y = 2 - x \\
    h_y &= 0 \quad \text{becomes} \quad 4x + 10(2 - x) - 16 \\
    & \quad \quad \quad \quad = 4x + 20 - 10x - 16 = -6x + 4 = 0 \\
    & \quad \quad \quad \quad \text{so} \quad x = 2/3 \quad \text{and} \quad y = 2 - 2/3 = 4/3
\end{align*}
\]

This gives \( z = 4 - x - 2y = 4 - (2/3) - 2(4/3) = 2/3 \).

The point is \( \left( \frac{2}{3}, \frac{4}{3}, \frac{2}{3} \right) \).

Its distance to the origin is \( \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{24}}{3} = \frac{2\sqrt{6}}{3} \).
Closest point on a plane to the origin

Method 2: Critical points

2nd derivative test

\[ h(x, y) = x^2 + y^2 + (4-x-2y)^2 \]

\[ h_x = 4x + 4y - 8 \]
\[ h_y = 4x + 10y - 16 \]

\[ h_{xx} = 4 \quad h_{yy} = 10 \quad h_{xy} = 4 \]

\[ D = (4)(10) - 4^2 = 24 \]

Since \( D > 0 \) and \( h_{xx} > 0 \), it’s a local minimum.

Gradient diagram

The plane is split into four regions, according to the signs of \( h_x \) and \( h_y \).

\( h \) increases as we move away from \( \left( \frac{2}{3}, \frac{4}{3} \right) \), so it’s an absolute minimum.
What point on the plane $z = 4 - x - 2y$ is closest to the origin?

Rewrite this as a constraint function = constant: $x + 2y + z = 4$

Minimize $f(x, y, z) = x^2 + y^2 + z^2$ (square of distance to origin)
Subject to $g(x, y, z) = x + 2y + z = 4$ (constraint: on plane)

Solve $\nabla f = \lambda \nabla g$ and $x + 2y + z = 4$:

$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 2, 1 \rangle$

$x + 2y + z = 4$

$2x = \lambda \cdot 1 \quad 2y = \lambda \cdot 2 \quad 2z = \lambda \cdot 1$

$x = \frac{\lambda}{2} \quad y = \lambda \quad z = \frac{\lambda}{2} \quad \frac{\lambda}{2} + 2\lambda + \frac{\lambda}{2} = 3\lambda = 4 \quad \text{so} \quad \lambda = \frac{4}{3}$

$x = \frac{2}{3} \quad y = \frac{4}{3} \quad z = \frac{2}{3}$

The closest point is $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$.

Its distance to the origin is $\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{24}{9}} = \frac{2\sqrt{6}}{3}$. 
What point $Q$ on the paraboloid $z = x^2 + y^2$ is closest to $P = (1, 2, 0)$?
Closest point on a surface to a given point

What point $Q$ on the paraboloid $z = x^2 + y^2$ is closest to $P = (1, 2, 0)$?

- Minimize the square of the distance of $P$ to $Q = (x, y, z)$

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 0)^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 - z = 0$$

- $\nabla f = \langle 2(x - 1), 2(y - 2), 2z \rangle$ \quad $\nabla g = \langle 2x, 2y, -1 \rangle$

- Solve $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$ for $x, y, z, \lambda$:

$$2(x - 1) = \lambda(2x) \quad 2(y - 2) = \lambda(2y) \quad 2z = -\lambda$$

$$x^2 + y^2 - z = 0$$

- Note $x \neq 0$ since the $1^{\text{st}}$ equation would be $-2 = 0$. Similarly, $y \neq 0$. So we may divide by $x$ and $y$.

- The first three give $\lambda = 1 - \frac{1}{x} = 1 - \frac{2}{y} = -2z$ \quad so $y = 2x$

- Constraint gives $z = x^2 + y^2 = x^2 + (2x)^2 = 5x^2$
Closest point on a surface to a given point

What point $Q$ on the paraboloid $z = x^2 + y^2$ is closest to $P = (1, 2, 0)$?

- So far, $y = 2x$, $z = 5x^2$, and $\lambda = 1 - \frac{1}{x} = 1 - \frac{2}{y} = -2z$.
- Then $1 - \frac{1}{x} = -2z = -2(5x^2)$ gives $1 - \frac{1}{x} = -10x^2$, so

$$10x^3 + x - 1 = 0$$

- Solve exactly with the cubic equation or approximately with a numerical root finder.

https://en.wikipedia.org/wiki/Cubic_function#Roots_of_a_cubic_function

It has one real root (and two complex roots, which we discard):

$$x = \frac{\alpha}{30} - \frac{1}{\alpha} \approx 0.3930027 \quad \text{where} \quad \alpha = \frac{3}{2} \sqrt[3]{1350 + 30 \sqrt[4]{2055}}$$

$$y = 2x \approx 0.7860055 \quad z = 5x^2 \approx 0.7722557$$

$$Q = (x, 2x, 5x^2) \approx (0.3930027, 0.7860055, 0.7722557)$$