Covariance

Let $X$ and $Y$ be random variables, possibly dependent.

Recall that the \textit{covariance} of $X$ and $Y$ is defined as

$$\text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y))$$

and that an alternate formula is

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Previously we used

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y)$$

and

$$\text{Var}(X_1 + X_2 + \cdots + X_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n)$$
**Covariance properties**

- \( \text{Cov}(X, X) = \text{Var}(X) \)
- \( \text{Cov}(X, Y) = \text{Cov}(Y, X) \)
- \( \text{Cov}(aX + b, cY + d) = ac \text{Cov}(X, Y) \)

**Sign of covariance**

\[ \text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y)) \]

- **When Cov(X, Y) is positive:**
  there is a tendency to have \( X > \mu_X \) when \( Y > \mu_Y \) and vice-versa, and \( X < \mu_X \) when \( Y < \mu_Y \) and vice-versa.

- **When Cov(X, Y) is negative:**
  there is a tendency to have \( X > \mu_X \) when \( Y < \mu_Y \) and vice-versa, and \( X < \mu_X \) when \( Y > \mu_Y \) and vice-versa.

- **When Cov(X, Y) = 0:**
  a) \( X \) and \( Y \) **might** be independent, but it’s not guaranteed.
  b) \( \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \)
Sample variance

Variance of a random variable:

\[ \sigma^2 = \text{Var}(X) = E((X - \mu_X)^2) = E(X^2) - (E(X))^2 \]

Sample variance from data \(x_1, \ldots, x_n\):

\[ s^2 = \text{var}(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n-1} \left( \sum_{i=1}^{n} x_i^2 \right) - \frac{n}{n-1} \bar{x}^2 \]

Vector formula:

*Centered data:* \(M = \begin{bmatrix} x_1 - \bar{x} & x_2 - \bar{x} & \cdots & x_n - \bar{x} \end{bmatrix}\)

\[ s^2 = \frac{M \cdot M}{n-1} = \frac{MM'}{n-1} \]
Sample covariance

Covariance between random variables $X, Y$:

$$\sigma_{XY} = \text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y)) = E(XY) - E(X)E(Y)$$

Sample covariance from data $(x_1, y_1), \ldots, (x_n, y_n)$:

$$s_{XY} = \text{cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n-1} \left( \sum_{i=1}^{n} x_i y_i \right) - \frac{n}{n-1} \bar{x}\bar{y}$$

Vector formula:

$$M_X = \begin{bmatrix} x_1 - \bar{x} & x_2 - \bar{x} & \cdots & x_n - \bar{x} \end{bmatrix}$$

$$M_Y = \begin{bmatrix} y_1 - \bar{y} & y_2 - \bar{y} & \cdots & y_n - \bar{y} \end{bmatrix}$$

$$s_{XY} = \frac{M_X \cdot M_Y}{n-1} = \frac{M_X M_Y'}{n-1}$$
Covariance matrix

For problems with many simultaneous random variables, put them into vectors:

\[
\vec{X} = \begin{bmatrix} R \\ S \end{bmatrix} \quad \vec{Y} = \begin{bmatrix} T \\ U \\ V \end{bmatrix}
\]

and then form a covariance matrix:

\[
\text{Cov}(\vec{X}, \vec{Y}) = \begin{bmatrix}
\text{Cov}(R, T) & \text{Cov}(R, U) & \text{Cov}(R, V) \\
\text{Cov}(S, T) & \text{Cov}(S, U) & \text{Cov}(S, V)
\end{bmatrix}
\]

In matrix/vector notation,

\[
\text{Cov}(\vec{X}, \vec{Y}) = E \left[ (\vec{X} - E(\vec{X})) (\vec{Y} - E(\vec{Y}))' \right]
\]

\[2 \times 3 \quad 2 \times 1 \quad (3 \times 1)' = 1 \times 3\]
Often there’s one vector with all the variables:

\[ \vec{X} = \begin{bmatrix} R \\ S \\ T \end{bmatrix} \]

\[
\text{Cov}(\vec{X}) = \text{Cov}(\vec{X}, \vec{X}) = E \left[ (\vec{X} - E(\vec{X})) (\vec{X} - E(\vec{X}))' \right] =
\begin{bmatrix}
\text{Cov}(R, R) & \text{Cov}(R, S) & \text{Cov}(R, T) \\
\text{Cov}(S, R) & \text{Cov}(S, S) & \text{Cov}(S, T) \\
\text{Cov}(T, R) & \text{Cov}(T, S) & \text{Cov}(T, T)
\end{bmatrix} =
\begin{bmatrix}
\text{Var}(R) & \text{Cov}(R, S) & \text{Cov}(R, T) \\
\text{Cov}(R, S) & \text{Var}(S) & \text{Cov}(S, T) \\
\text{Cov}(R, T) & \text{Cov}(S, T) & \text{Var}(T)
\end{bmatrix}
\]

This matrix is symmetric (it equals its own transpose). The diagonal entries are ordinary variances.
Covariance matrix properties

\[
\begin{align*}
\text{Cov}(\vec{X}, \vec{Y}) &= \text{Cov}(\vec{Y}, \vec{X})' \\
\text{Cov}(A\vec{X} + \vec{B}, \vec{Y}) &= A \text{Cov}(\vec{X}, \vec{Y}) \\
\text{Cov}(\vec{X}, C\vec{Y} + \vec{D}) &= \text{Cov}(\vec{X}, \vec{Y})C' \\
\text{Cov}(A\vec{X} + \vec{B}) &= A \text{Cov}(\vec{X})A' \\
\text{Cov}(\vec{X}_1 + \vec{X}_2, \vec{Y}) &= \text{Cov}(\vec{X}_1, \vec{Y}) + \text{Cov}(\vec{X}_2, \vec{Y}) \\
\text{Cov}(\vec{X}, \vec{Y}_1 + \vec{Y}_2) &= \text{Cov}(\vec{X}, \vec{Y}_1) + \text{Cov}(\vec{X}, \vec{Y}_2)
\end{align*}
\]

A, C are constant matrices, \(\vec{B}, \vec{D}\) are constant vectors, and all dimensions must be correct for matrix arithmetic.
Example (2D, but works for higher dimensions too)

Data \((x_1, y_1), \ldots, (x_{100}, y_{100})\):

\[
M_0 = \begin{bmatrix}
  x_1 & \cdots & x_{100} \\
  y_1 & \cdots & y_{100}
\end{bmatrix} = \begin{bmatrix}
  3.0858 & 0.8806 & 9.8850 & \cdots & 4.4106 \\
  12.8562 & 10.7804 & 8.7504 & \cdots & 13.5627
\end{bmatrix}
\]
Centered data
Computing sample covariance matrix

- **Original data:** 100 \((x, y)\) points in a \(2 \times 100\) matrix \(M_0:\)
  \[
  M_0 = \begin{bmatrix}
    x_1 & \cdots & x_{100} \\
    y_1 & \cdots & y_{100}
  \end{bmatrix} = \begin{bmatrix}
    3.0858 & 0.8806 & 9.8850 & \cdots & 4.4106 \\
    12.8562 & 10.7804 & 8.7504 & \cdots & 13.5627
  \end{bmatrix}
  \]

- **Centered data:** subtract \(\bar{x}\) from \(x\)'s and \(\bar{y}\) from \(y\)'s to get \(M\); here \(\bar{x} = 5, \bar{y} = 10:\)
  \[
  M = \begin{bmatrix}
    -1.9142 & -4.1194 & 4.8850 & \cdots & -0.5894 \\
    2.8562 & 0.7804 & -1.2496 & \cdots & 3.5627
  \end{bmatrix}
  \]

- **Sample covariance:**
  \[
  C = \frac{M M'}{100 - 1} = \begin{bmatrix}
    31.9702 & -16.5683 \\
    -16.5683 & 13.0018
  \end{bmatrix}
  \]
  \[
  = \begin{bmatrix}
    s_{XX} & s_{XY} \\
    s_{YX} & s_{YY}
  \end{bmatrix} = \begin{bmatrix}
    s_X^2 & s_{XY} \\
    s_{XY} & s_Y^2
  \end{bmatrix}
  \]
Recall that for vectors $\vec{v}, \vec{w}$, we have $\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos(\theta)$, where $\theta$ is the angle between the vectors.

*Orthogonal* means perpendicular. $\vec{v}$ and $\vec{w}$ are orthogonal when the angle between them is $\theta = 90^\circ = \frac{\pi}{2}$ radians. So $\cos(\theta) = 0$ and $\vec{v} \cdot \vec{w} = 0$.

Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are *orthonormal* when

- $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$ (different vectors are orthogonal)
- $\vec{v}_i \cdot \vec{v}_i = 1$ for all $i$ (each vector has length 1; they are all unit vectors)

In short: $\vec{v}_i \cdot \vec{v}_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$
Orthonormal matrix

- Form an $n \times n$ matrix of orthonormal vectors
  \[ V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \]
  by loading $n$-dimensional column vectors into the columns of $V$.
- Transpose it to convert the vectors to row vectors:
  \[ V' = \begin{bmatrix} \vec{v}_1' \\ \vec{v}_2' \\ \vdots \\ \vec{v}_n' \end{bmatrix} \]
  \((V'V)_{ij}\) is the $i^{th}$ row of $V'$ dotted with the $j^{th}$ column of $V$:
  \[(V'V)_{ij} = \vec{v}_i \cdot \vec{v}_j = \delta_{ij} \quad V'V = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \]

Thus, $V'V = I$ ($n \times n$ identity matrix), so $V' = V^{-1}$.
- An $n \times n$ matrix $V$ is orthonormal when $V'V = I$ (or equivalently, $VV' = I$), where $I$ is the $n \times n$ identity matrix.
Diagonalizing the sample covariance matrix $C$

$$C = V D V'$$

\[
\begin{bmatrix}
31.9702 & -16.5683 \\
-16.5683 & 13.0018
\end{bmatrix} =
\begin{bmatrix}
-0.8651 & -0.5016 \\
0.5016 & -0.8651
\end{bmatrix} \begin{bmatrix}
41.5768 & 0 \\
0 & 3.3952
\end{bmatrix} \begin{bmatrix}
-0.8651 & 0.5016 \\
-0.5016 & -0.8651
\end{bmatrix}
\]

$C$ is a real-valued symmetric matrix. It can be shown that:

- $C$ can be diagonalized (recall not all matrices are diagonalizable);
- in the special form $C = V D V'$ with $V$ orthonormal, so $V^{-1} = V'$;
- all eigenvalues are real numbers $\geq 0$. So we can put them on the diagonal of $D$ in decreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. 

Diagonalizing the sample covariance matrix $C$

- Since $C$ is symmetric, if $\mathbf{v}$ is a right eigenvector with eigenvalue $\lambda$, then $\mathbf{v}'$ is a left eigenvector with eigenvalue $\lambda$, and vice-versa:
  \[ C\mathbf{v} = \lambda \mathbf{v} \quad \text{so} \quad \lambda \mathbf{v}' = (C\mathbf{v})' = \mathbf{v}' C' = \mathbf{v}' C \]

- Diagonalization $C = VDV^{-1}$ loads right and left eigenvectors into $V$ and $V^{-1}$. Here those eigenvectors are transposes of each other, leading to the special form $C = VDV'$.

- Also, all eigenvalues are $\geq 0$ ("$C$ is positive semidefinite"): For all vectors $\mathbf{w}$,
  \[ \mathbf{w}' C \mathbf{w} = \frac{\mathbf{w}' M M' \mathbf{w}}{n-1} = \frac{|M' \mathbf{w}|^2}{n-1} \geq 0 \]

  Eigenvector equation $C \mathbf{w} = \lambda \mathbf{w}$ gives $\mathbf{w}' C \mathbf{w} = \lambda \mathbf{w}' \mathbf{w} = \lambda |\mathbf{w}|^2$.
  So $\lambda |\mathbf{w}|^2 = \mathbf{w}' C \mathbf{w} \geq 0$, giving $\lambda \geq 0$. 

Prof. Tesler
Principal Components Analysis
Math 283 / Fall 2018 15 / 40
Principal axes

- The columns of $V$ are the right eigenvectors of $C$.
- Multiply each eigenvector by the square root of its eigenvalue to get the principal components.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Eigenvector</th>
<th>PC</th>
</tr>
</thead>
<tbody>
<tr>
<td>41.5768</td>
<td>$\begin{bmatrix} -0.8651 \ 0.5016 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -5.5782 \ 3.2343 \end{bmatrix}$</td>
</tr>
<tr>
<td>3.3952</td>
<td>$\begin{bmatrix} -0.5016 \ -0.8651 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.9242 \ -1.5940 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

- Put them into the columns of a matrix:

$$ P = V \sqrt{D} = \begin{bmatrix} -5.5782 & -0.9242 \\ 3.2343 & -1.5940 \end{bmatrix} $$

- $C = VDV' = V \sqrt{D} \sqrt{D'}V' = (V \sqrt{D})(V \sqrt{D})' = PP'$
Plot the centered data with lines along the principal axes:

Sum of squared perpendicular distances of data points to first PC line (red) is minimum among all lines through origin.

$i$th PC is perpendicular to the previous ones, and the sum of squared perpendicular distances to the span (line, plane, ...) of the first $i$ PCs is minimum among all $i$-dim. spaces through origin.
Transform $M$ to $M_2 = V'M$ and plot points given by the columns of $M_2$:

From linear algebra, a linear transformation $M \mapsto AM$ does a combination of rotations, reflections, scaling, shearing, and orthogonal projections.

$V$ is orthonormal, so $M_2 = V'M$ rotates/reflects all the data.

$M = VM_2$ recovers centered data $M$ from rotated data $M_2$.

The sample covariance matrix of $M_2$ is

$$
\frac{M_2 M'_2}{n-1} = \frac{(V'M)(V'M)'}{n-1} = \frac{V'MM'V}{n-1} = V' \left( \frac{MM'}{n-1} \right) V = V'CV = D
$$

Note $C = VDV'$ and $V' = V^{-1}$, so $D = V^{-1}C(V')^{-1} = V'CV$. 
The rotated data has new coordinates \((t_1, u_1), \ldots, (t_{100}, u_{100})\) and covariance matrix \(D\):

\[
V'CV = D = \begin{bmatrix}
\text{var}(T) & \text{cov}(T, U) \\
\text{cov}(T, U) & \text{var}(U)
\end{bmatrix} = \begin{bmatrix}
41.5768 & 0 \\
0 & 3.3952
\end{bmatrix}
\]

In \(D\), the **total variance** is \(\text{var}(T) + \text{var}(U) = 44.9720\).
Note that this is the sum of the eigenvalues, \(\lambda_1 + \lambda_2 + \cdots\).

The **trace of a matrix** is the sum of its diagonal entries.
So the total variance is \(\text{Tr}(D) = \lambda_1 + \lambda_2 + \cdots\).

**General linear algebra fact:** \(\text{Tr}(X) = \text{Tr}(AXA^{-1})\).
So \(\text{Tr}(C) = \text{Tr}(VDV') = \text{Tr}(VDV^{-1}) = \text{Tr}(D)\).
Below, \(\text{Tr}(C) = \text{Tr}(D) = 44.9720\).

\[
C = \begin{bmatrix}
31.9702 & -16.5683 \\
-16.5683 & 13.0018
\end{bmatrix} \quad D = \begin{bmatrix}
41.5768 & 0 \\
0 & 3.3952
\end{bmatrix}
\]
Part of variance explained by each axis

The part of the variance explained by each axis is $\lambda_i/\text{total variance}$:

<table>
<thead>
<tr>
<th>Eigenvector</th>
<th>Eigenvalue</th>
<th>Explained</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8651]$</td>
<td>$0.5016$</td>
<td>$41.5768$</td>
</tr>
<tr>
<td>$[0.5016]  $</td>
<td>$41.5768$</td>
<td>$41.5768/44.9720 = 92.45%$</td>
</tr>
<tr>
<td>$[-0.5016]$</td>
<td>$[-0.8651]$</td>
<td>$3.3952$</td>
</tr>
<tr>
<td>$[3.3952]$</td>
<td>$[3.3952]$</td>
<td>$3.3952/44.9720 = 7.55%$</td>
</tr>
<tr>
<td>Total</td>
<td>$44.9720$</td>
<td>$100%$</td>
</tr>
</tbody>
</table>

This is an application of $\text{Cov}(A\overline{X}) = A\text{Cov}(\overline{X})A'$:

$$\text{Cov} \left( V' \begin{bmatrix} X \\ Y \end{bmatrix} \right) = V' \text{Cov} \left( \begin{bmatrix} X \\ Y \end{bmatrix} \right) V$$
Dimension reduction

To clean up “noise,” set all $u_i = 0$ and rotate back:

$$V \begin{bmatrix} t_1 & t_2 & t_3 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \cdots \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \cdots \end{bmatrix}$$

Project original centered data to 1st PC
Say we want to keep enough information to explain 90% of the variance.

Take enough top PCs to explain $\geq 90\%$ of the variance.

Let $M_3$ be $M_2$ (rotated data) with the remaining coordinates zeroed out.

Rotate it back to the original axes with $VM_3$.

In other applications, a dominant signal can be suppressed by zeroing out the coordinates for the top PCs instead of the bottom PCs.
Variations for PCA (and SVD, upcoming)

- Some people reverse the roles of rows and columns of $M$.
- In some applications, $M$ is “centered” (subtract off row means) and in others, it’s not.
- If the ranges on the variables (rows) are very different, the data might be rescaled in each row to make similar ranges. For example, replace each row by $Z$-scores for the row.
Sensitivity to scaling

- PCA was originally designed for measurements in ordinary space, so all axes had the same units (e.g., cm or inches) and equivalent results would be obtained no matter what units were used.

- It’s problematic to mix physical quantities with different units:
  - Length of \((a, b)\) in (seconds, mm): \(\sqrt{a^2 + b^2}\)
    (adding \(\text{sec}^2\) plus \(\text{mm}^2\) is not legitimate!)
  - Convert to (hours, miles): \(|\left(\frac{a}{3600}, \frac{b}{1609344}\right)| = \sqrt{\frac{a^2}{3600^2} + \frac{b^2}{1609344^2}}\)
  - Angles are also distorted by this unit conversion:
    \(\arctan\left(\frac{b}{a}\right) \neq \arctan\left(\frac{\frac{a}{3600}}{\frac{b}{1609344}}\right)\).
  - \(|(0 \degree C, 0 \degree C)| = 0\) vs. \(|(32 \degree F, 32 \degree F)| = 32 \sqrt{2}\).
    Both \(\degree C\) and \(\degree F\) use an arbitrary “zero” instead of absolute zero.

- PCA is sensitive to differences in the scale, offset, and ranges of the variables. Rescaling one row w/o the others changes angles and lengths nonuniformly, and changes eigenvalues and eigenvectors in an inequivalent way.

- Typically addressed by replacing each row with Z-scores.
Before we considered single genes where “red” or “green” (positive or negative expression level) distinguished the classes.

If $x_i$ is the expression level of gene $i$ then

$$L = a_1 x_1 + a_2 x_2 + \cdots$$

is a linear combination of genes.

Next up: a method that finds linear combinations of genes where $L > C$ and $L < C$ distinguish two classes, for some constant $C$. So $L = C$ is a line / plane / etc. that splits the multidimensional space of expression levels.

Different classes are not always separated in this fashion. In some situations, nonlinear relations may be required.
Consider an experiment with 80 microarrays, each with 10000 spots.

- $M$ is $10000 \times 80$.

- $C = \frac{MM'}{80-1}$ is $10000 \times 10000!$

- $M'M$ is $80 \times 80$.

We will see that $MM'$ has the same 80 eigenvalues as $M'M$, plus an additional $10000 - 80 = 9920$ eigenvalues equal to 0.

- Some of the 80 eigenvalues of $M'M$ may also be 0. For centered data, all row sums of $M$ are 0 so $[1, \ldots, 1]'$ is an eigenvector of $M'M$ with eigenvalue 0.

- We will see we can work with the smaller of $MM'$ or $M'M$. 


Singular Value Decomposition (SVD)

Let $M$ be a $p \times q$ matrix (not necessarily “centered”). The *Singular Value Decomposition* of $M$ is $M = U S V'$, where

- $U$ is orthonormal, $p \times p$.
- $V$ is orthonormal, $q \times q$.
- $S$ is a diagonal $p \times q$ matrix, $s_1 \geq s_2 \geq \cdots \geq 0$.
- If $M$ is $5 \times 3$, this would look like

\[
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix}
= 
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix}
\begin{pmatrix}
 s_1 & 0 & 0 \\
 0 & s_2 & 0 \\
 0 & 0 & s_3 \\
 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix}
\]

Prof. Tesler
Principal Components Analysis
Math 283 / Fall 2018 27 / 40
“Compact” SVD

For $p > q$: The bottom $p - q$ rows of $S$ are all 0. Remove them. Keep only the first $q$ rows of $S$ and first $q$ columns of $U$.

- $U$ is orthonormal, $p \times q$.
- $V$ is orthonormal, $q \times q$.
- $S$ is a diagonal $p \times q$ matrix, $s_1 \geq s_2 \geq \cdots \geq 0$.
- If $M$ is $5 \times 3$, this would look like

$$
\begin{bmatrix}
\ldots \\
\vdots \\
\ldots \\
\end{bmatrix}
\begin{bmatrix}
\ldots \\
\vdots \\
\ldots \\
\end{bmatrix}
= 
\begin{bmatrix}
\ldots \\
\vdots \\
\ldots \\
\end{bmatrix}
\begin{bmatrix}
s_1 & 0 & 0 \\
0 & s_2 & 0 \\
0 & 0 & s_3 \\
\end{bmatrix}
\begin{bmatrix}
\ldots \\
\vdots \\
\ldots \\
\end{bmatrix}
$$

- For $q > p$: keep only the first $p$ columns of $S$ and first $p$ rows of $V$.
- Matlab and R have options for full or compact form in $\text{svd}(M)$.
Computing the SVD

\[ M'M = (VS'U')(USV') = V(S'S)V' = V \begin{bmatrix} s_1^2 & 0 & 0 \\ 0 & s_2^2 & 0 \\ 0 & 0 & s_3^2 \end{bmatrix} V' \]

\[ MM' = (USV')(VS'U') = U(SS')U' = U \begin{bmatrix} s_1^2 & 0 & 0 & 0 & 0 \\ 0 & s_2^2 & 0 & 0 & 0 \\ 0 & 0 & s_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} U' \]

This diagonalization of \( M'M \) and \( MM' \) shows they have the same eigenvalues up to the dimension of the smaller matrix. The larger matrix has all additional eigenvalues equal to 0.

- **Compute the SVD using whichever gives smaller dimensions!**
Computing the SVD

\[ M'M = (VS'U')(USV') = V(S'S)V' = V \begin{bmatrix} s_1^2 & 0 & 0 \\ 0 & s_2^2 & 0 \\ 0 & 0 & s_3^2 \end{bmatrix} V' \]

\[ MM' = (USV')(VS'U') = U(SS')U' = U \begin{bmatrix} s_1^2 & 0 & 0 & 0 & 0 \\ 0 & s_2^2 & 0 & 0 & 0 \\ 0 & 0 & s_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} U' \]

First method (recommended when \( p \geq q \)):

- Diagonalize \( M'M = VDV' \).
- Compute \( p \times q \) matrix \( S \) with \( S_{ii} = \sqrt{D_{ii}} \) and 0’s elsewhere.
- The *pseudoinverse* of \( S \) is \( S^{-1} \): replace each nonzero diagonal entry of \( S \) by its reciprocal, and transpose to get a \( q \times p \) matrix.
- Compute \( U \): \( M = USV' \implies U = M(V')^{-1}S^{-1} = MVS^{-1} \).

\( q \geq p \) is analogous: diagonalize \( MM' = UDU' \); compute \( S \) from \( D \); then compute \( V = (S^{-1}U'M)' \).

**svd(M)** in both Matlab and R.
Singular values and singular vectors

Let $M$ be a $p \times q$ matrix (not necessarily centered). Suppose
- $s$ is a scalar.
- $\vec{v}$ is a $q \times 1$ unit vector (column vector).
- $\vec{u}$ is a $p \times 1$ unit vector (column vector).

$s$ is a **singular value** of $M$ with **right singular vector** $\vec{v}$ and **left singular vector** $\vec{u}$ if

$$M\vec{v} = s\vec{u} \quad \text{and} \quad \vec{u}'M = s\vec{v}' \quad \text{(same as} \ M'\vec{u} = s\vec{v})\text{).}$$

**Break** $U$ and $V$ into columns

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p \end{bmatrix}$$

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_q \end{bmatrix}$$

Then $M\vec{v}_i = s_i\vec{u}_i$ and $M'\vec{u}_i = s_i\vec{v}_i$ for $i \leq \min(p, q)$.

If $p > q$: $M'\vec{u}_i = \vec{0}$ for $i > q$.
If $q > p$: $M\vec{v}_i = \vec{0}$ for $i > p$.

**To get full-sized $M = USV'$ from compact ($p \geq q$ case):** choose the remaining columns of $U$ from the nullspace of $M'$ in such a way that the columns of $U$ are an orthonormal basis of $\mathbb{R}^p$. 
Relation between PCA and SVD

Previous computation for PCA
- Start with centered data matrix $M$ ($n$ columns).
- Compute covariance matrix, diagonalize it, compute $P$:
  \[ C = \frac{MM'}{n-1} = VDV' = PP' \quad \text{where} \quad P = V \sqrt{D} \]

Computing PCA using SVD
- In terms of the SVD factorization $M = USV'$, covariance is
  \[ C = \frac{MM'}{n-1} = \frac{(USV')(VS'U')}{n-1} = \frac{U(SS')U'}{n-1} = UDU' \quad \text{where} \quad D = \frac{SS'}{n-1} \]
  \[ = PP' \quad \text{where} \quad P = \frac{US}{\sqrt{n-1}} \]
- Variance for $i$th component is $\frac{s_i^2}{n-1}$
- Note: there were minor notation adjustments to deal with $n - 1$. 
Nielsen et al.\textsuperscript{1} studied tumors in six types of tissue.
41 tissue samples and 46 microarray slides
They switched microarray platforms in the middle of the experiment:
- The first 26 slides have 22,654 spots (22K).
- The next 20 slides have 42,611 spots (42K) (mostly a superset).
- Five of the samples were done on both 22K and 42K platforms.
7425 spots were in common to both platforms, had good signal across all slides, and had sample variance above a certain threshold. So $M$ is $7425 \times 46$.

The compact form $M = USV'$ is shown above.

They call the columns of $U$ "eigenarrays" and the columns of $V$ (rows of $V'$) "eigengenes."

- An eigenarray is a linear combination of arrays.
- An eigengene is a linear combination of genes.
SVD in microarrays

Sample covariance matrix: \( C = MM' / 45 = U S S' U'/45 \)

Sample variance of \( i \)th component: \( s_i^2 / 45 \).

Total sample variance: \( (s_1^2 + \cdots + s_{46}^2) / 45 \).

Here is \( V' \) and the explained fractions \( s_i^2 / (s_1^2 + \cdots + s_{46}^2) \).

Nielsen et al., supplementary material.
The expression level of gene $i$ on array $j$ is $M_{ij}$.

**Interpretation of change of basis** $S = U'MV$:
the $i$th eigenarray only detects the $i$th eigengene, and has 0 response to other eigengenes.

**Interpretation of** $V' = U'MS^{-1}$:
The “expression level” of eigengene $i$ on array $j$ is $(V')_{ij} = V_{ji}$.

Let $\tilde{m}$ represent a new array (e.g., a column vector of expression levels in each gene).
The expression level of eigengene $i$ is $(U'\tilde{m})_i/s_i$. 
Platform bias

- They re-ordered the arrays according to the expression level $V_{j1}$ in the first eigengene (largest eigenvalue).
- They found that $V_{j1}$ tends to be larger in the 42K arrays and smaller in the 22K arrays. This is an experimental artifact, not a property of the specimens under study.

Nielsen et al., supplementary material.
Let \( \tilde{S} \) be \( S \) with the (1,1) entry replaced by 0.

Let \( \tilde{M} = U \tilde{S} V' \).

This reduces the signal and variance in many spots. After removing weak spots, they cut down to 5520 spots, giving a 5520 \( \times \) 46 data matrix.
Classification — Eigengenes — 1D

For the $5520 \times 46$ matrix, the expression levels of the top three eigengenes can be used to classify some tumor types.

Prof. Tesler
Principal Components Analysis
Math 283 / Fall 2018 39 / 40
Classification — Eigengenes — 2D

$\lambda_1, \lambda_2, \lambda_3$ help distinguish between tumor types