# $H^\infty$ Control for Nonlinear Systems with Output Feedback

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***Abstract*—The basic question of nonlinear $H^\infty$ control theory is to decide, for a given two port system, when does feedback exist which makes the full system dissipative and internally stable. This problem can also be viewed as an interesting question about circuits. Also, after translation, the problem has a game theoretic statement. This paper presents several necessary conditions for solutions to exist and gives sufficient conditions for a certain construction to lead to a solution.**

## I. Introduction

The basic question is, given a two port system, when does feedback exist which makes the full system dissipative and internally stable? This while an interesting question about circuits is also the central question in $H^\infty$ control.

### A. The System We Treat

Here, $W$ includes all command and disturbance signals, $U$ is the control signal, $Z$ is the error signal, $Y$ is the measurement signal, and $x \in \mathscr{X} = R^n$ is the state of the system (see Fig. 1). The given system, described by state space equations

$$dx/dt = F(x, W, U), \qquad Z = G_1(x, W, U), \qquad Y = G_2(x, W, U), \tag{1}$$

we take to be nonlinear but time invariant. We wish to find a nonlinear time-invariant feedback system

$$dz/dt = f(z, Y), \qquad U = g(z, Y) \tag{2}$$

which improves performance. We assume these systems are homogeneous throughout the entire paper, that is, that

$$F(0,0,0) = 0, \quad G_1(0,0,0) = 0, \quad \text{and} \quad G_2(0,0,0) = 0 \tag{3}$$

so $(0,0)$ is an equilibrium point and that $G_2(x, W, U)$ does not depend on $U$. The standard problem of $H^\infty$ control in the nonlinear setting is to find a stabilizing feedback law

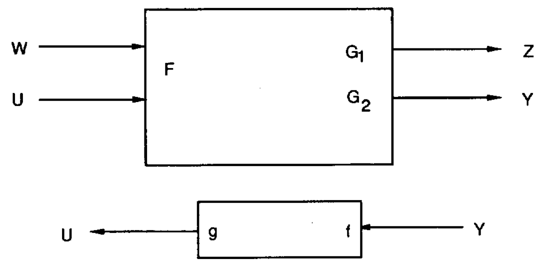


Fig. 1

$(f, g)$ so that the resulting closed-loop system satisfies

$$\|Z\|_2^2 \le K \|W\|_2^2$$

for a preassigned tolerance level $K$, i.e., in the terminology of [34] and [25], the closed-loop system is *dissipative* with respect to the particular energy supply rate $\varphi(W, Z) = K\|W\|_2^2 - \|Z\|_2^2$.

A special case we emphasize is that of an *Input Affine* (IA) system where

$$\begin{aligned} F(x, W, U) &= A(x) + B_1(x)W + B_2(x)U \\ G_1(x, W, U) &= C_1(x) + D_{12}(x)U, \\ G_2(x, W, U) &= C_2(x) + D_{21}(x)W \end{aligned} \tag{4}$$

is an IA plant system, and

$$f(z, Y) = a(z) + b(z)Y, \qquad g(z, Y) = c(z) + d(z)Y \tag{5}$$

is an IA compensator system. Homeogeneity for IA systems is equivalent to

$$\begin{aligned} &A(0) = 0, \quad C_1(0) = 0, \quad C_2(0) = 0, \\ &a(0) = 0, \quad c(0) = 0. \end{aligned} \tag{6}$$

Also of significant physical importance is a (plant) system which is affine linear only in the disturbance $W$. We shall call such systems *W-Input Affine* (WIA) systems.

$$\begin{aligned} F(x, W, U) &= AB(x, U) + B_1(x)W \\ G_1(x, W, U) &= C_1(x, U), \\ G_2(x, W, U) &= C_2(x) + D_{21}(x)W \end{aligned} \tag{7}$$

is a WIA plant system. WIA systems include fully nonlinear classical control problems. Also, they have the extremely appealing property that certain basic computions are possible for them. In this paper, we shall always seek IA compensators even for this very general class of plants.

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### B. Perspective

A recent breakthrough in the linear $H^\infty$ theory was the derivation of elegant state space formulas for the solution of the standard linear $H^\infty$-control problem in terms of the solutions of two Riccati equations (see [15]). This work, unlike earlier work in the $H^\infty$ theory which emphasized factorization of transfer functions and Nevanlinna–Pick interpolation in the frequency domain, operated exclusively in the time domain and drew strong parallels between the $H^\infty$-theory and the more established LQG control theory; in particular a separation principle, whereby the output feedback problem can be split into uncoupled state feedback and observation based state estimation problems as in the LQG case was presented. Now there have appeared a number of alternative derivations of the formulas from [15], most also in the time domain. We mention in particular [31] which emphasizes the bounded real lemma and which was particularly influential for the present paper; indeed, one level at which to read this paper is to specialize to the linear case and obtain an alternative motivation for the steps in [31]. There now have also appeared improved versions of the approach through factorization of transfer functions [9], [17], [19]); we expect that some of these may also have extensions to nonlinear settings.

The $H^\infty$ theory for the nonlinear setting is much less developed. The operator factorization approach of [3]–[6] constructs a nonlinear fractional map to parameterize a large set of solutions of certain special cases of the nonlinear measurement feedback $H^\infty$-control problem in the discrete time setting. Construction of the nonlinear system giving rise to the desired nonlinear fractional map was based on the assumption that it be a lossless dynamical system (with a nonnegative energy function on the state space balancing the integrated power consumed or put out by the input–output behavior). The authors later found (from C. Byrnes and [32], [33]) that a general theory for such dynamical systems (both lossless and dissipative) has been laid out by Willems [34] and Hill and Moylan [25].

The first systematic use of the work of Hill–Moylan [25] on dissipative systems in $H^\infty$ control was by van der Schaft who in extremely valuable papers gave a coherent general theory as well as derivations of the Hamilton–Jacobi–Isaacs equations for IA systems with state feedback. Similar work was done in [12]; there the performance measure was taken to be the supply rate associated with passivity rather than with finite gain, and hence the $H^\infty$-control interpretation was missing. Closely related results appear in [7] (see also [4]) for the problem in the discrete time setting; there, the authors ignorant of the work of Hill and Moylan, derived results very close to parts of [25]–[27] in the more involved context of making a system dissipative after feedback. This paper also treated a special case of the output feedback problem. The formula for the desired feedback involves the solution of a Hamilton–Jacobi equation and also can be derived directly from game theory ideas. The general theory was used to work out explicit formulas for the case of linear systems composed with mild memoryless nonlinearities rather than for IA systems as in [12] and [32], [33].

A comprehensive treatment of $H^\infty$-control theory from the point of view of game theory can now be found in [1]. In [32], [33] the general interpretation of the nonlinear $H^\infty$-control problem as that of finding a feedback which makes the system dissipative in the sense of [34] was formulated, and the Hamilton–Jacobi equation for the state feedback problem was derived from this point of view.

Most recently [29] presents sufficient conditions for a particular construction to yield a local solution of the output feedback nonlinear $H^\infty$-control problem. There also the interpretation of the $H^\infty$-problem as construction of a feedback which makes the system dissipative is prominent.

The report [8] summarizes the work in nonlinear $H^\infty$-control theory up to 1989, in particular [3]–[5] and the nonlinear commutant lifting method of [2] and [16], while [23] includes a summary of [7].

The formulation of the nonlinear $H^\infty$-control problem as presented here demands a choice of control law (state feedback or more generally output feedback) which guarantees 1) asymptotic stability of the internal state of the closed-loop system when subjected to an arbitrary initial condition and zero external input, and 2) that the size of an error signal be bounded uniformly with respect to the worst case size of a disturbance command signal. The approach here (as well as in [32], [33], [29]) is to guarantee the latter dissipative inequality by the construction of an energy or storage function for the putative closed-loop system. Once this storage function is found, it can also be used (under sufficient observability assumptions) as a Lyapunov function to guarantee the internal stability requirement. This dual use of the storage function was exploited systematically probably for the first time in the work of Hill and Moylan [26], [27] and later also in [12]. Starting in the 1970's there appeared the work of Gutman, Leitman, and Corless (see [20], [22], [13], [14]) which in some sense anticipated the $H^\infty$-control theory in the nonlinear context. There it is assumed that a known Lyapunov function guarantees stability for a nominal plant which is subject to disturbances and parameter variations of some assumed size and depending on the state. The goal is to construct a state feedback which guarantees asymptotic stability for all admissible choices of the disturbances and parameter variations. The uncertainties are assumed to be of a deterministic rather than statistical form (just as in the $H^\infty$-theory) and the goal is to guarantee stability (rather than a quantitative performance measure as in the $H^\infty$-theory) over all admissible uncertainties (i.e., in the worst case). The strategy is to find a feedback (unfortunately possibly discontinuous) for which the assumed Lyapunov function for the nominal system also serves as a Lyapunov function for the closed-loop system for all admissible uncertainties; this leads to a min-max criterion on the

Lyapunov inequality as opposed to a min-max criterion on the dissipation inequality in the $H^\infty$-theory.

In this paper, we formulate the nonlinear $H^\infty$-problem as that of finding a stabilizing compensator so that the closed-loop system satisfies the hypotheses of the nonlinear bounded real lemma (see [25]). This leads us to a systematic analysis of possible interchanges of max and min and the derivation of several necessary conditions analogous to the two Riccati equations in the linear case for solutions to exist. We also present a recipe for a candidate solution and sufficient conditions for the recipe to give a solution.

### C. A Symbolic Algebra Package for Systems Theory

The paper is also available in computer executable form for those who have Mathematica. It is a package which does noncommutative algebra, noncommutative directional differentiation, etc., symbolically. Indeed all formulas in this paper were first derived using this and the original version of this paper was [10] which contained statements of the theorems here with formulas which could be all manipulated inside our noncommuting algebra package. Obtain [10] from ncalg@osiris.ucsd.edu.

### D. Conventions

Assume $z \in R^n$. The gradient of a scalar-valued function $g(z)$ will be a (column) vector $\nabla_z g(z)$ with action on (column) vectors $h \in R^n$ denoted by $\nabla_z(g(z)) \cdot h \triangleq (\nabla_z g(z))^T h$. $D_z$ denotes differential in a variable $z$. For a scalar-valued function $g(z)$, $D_z g(z) = \nabla_z^T g(z)$. For a (column) vector-valued function

$$\psi(z) \triangleq \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \\ \vdots \\ \psi_m(z) \end{pmatrix},$$

$$\text{define } D_z(\psi(z)) = \begin{pmatrix} \frac{\partial\psi_1}{\partial z_1} & \frac{\partial\psi_1}{\partial z_2} & \cdots & \frac{\partial\psi_1}{\partial z_n} \\ \frac{\partial\psi_2}{\partial z_1} & \frac{\partial\psi_2}{\partial z_2} & \cdots & \frac{\partial\psi_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\psi_m}{\partial z_1} & \frac{\partial\psi_m}{\partial z_2} & \cdots & \frac{\partial\psi_m}{\partial z_n} \end{pmatrix}.$$

We will not define $D_z$ for row vectors.

The paper is organized as follows. Section II recalls the theory of dissipative systems, Section III analyzes the nonlinear $H^\infty$-control problem from the point of view of dissipative systems. Section IV develops some interchanges of max. and min. to obtain some necessary conditions for solutions to exist. Section V develops the consequences of assuming the energy function has some special forms and presents our recipe with sufficient conditions for it to yield a solution. Section VI presents theorems for WIA systems analogous to those derived in Section IV for IA systems. Finally, Section VII presents a generalization of the separation principle to a general nonlinear setting.

Most of the paper could have been presented at the more general level of WIA systems. For tutorial purposes, however, we chose to first present the results using IA systems in order to provide expressions with close connections to the linear results in [15] and [31]. This has resulted in some redundancy, which we have attempted to minimize, in the results presented in Sections IV and VI.


Fig. 2

## II. Dissipative Systems

First we recall the bounded real lemma but we do so at a high level of generality (see Fig. 2). The system definition is:

$$dx/dt = F(x, W), \qquad Z = G_1(x, W). \tag{8}$$

For linear systems, this is

$$F(x, W) = Ax + B_1 W, \qquad G_1(x, W) = C_1 x + D_{11} W. \tag{9}$$

Define a finite-gain dissipative system with gain $K$ to be a system for which

$$\int_{t_0}^{t_1} \|Z\|^2 \, dt \le K \int_{t_0}^{t_1} \|W\|^2 \, dt \tag{10}$$

where $K$ is a constant, and $x(t_0) = 0$. If $K = 1$ the system will be called dissipative. In circuit theory these would be called passive. This agrees with the notion of dissipative in [34], [25] with respect to the specific supply rate $\|W\|^2 - \|Z\|^2$.

Define a storage or energy function on the state space to be a nonnegative function $\epsilon$ satisfying

$$\int_{t_0}^{t_1} \{\|Z(t)\|^2 - \|W(t)\|^2\} \, dt \le \epsilon(x(t_0)) - \epsilon(x(t_1)) \tag{11}$$

and $\epsilon(0) = 0$. Hill and Moylan ([25]) showed that a system is dissipative if and only if an energy (storage) function (possibly extended real valued) exists. Under controllability assumptions, there exists an energy function with finite values.

Given a differentiable real-valued function $\epsilon$ on the state-space $\mathscr{X}$, we say that a system of the form (8) is $\epsilon$-dissipative provided that the energy Hamiltonian $H$ defined by

$$H = \|G_1(x, W)\|^2 - \|W\|^2 + \nabla\epsilon(x) \cdot F(x, W) \tag{12}$$

is nonpositive. That is, $0 \ge H$ for all $W$ and all $x$ in the set of states reachable from 0 by the system.

*Theorem 2.1:* (see [34], [25]) Let $\epsilon$ be a given nonnegative differentiable function with $\epsilon(0) = 0$. Then a system is $\epsilon$-dissipative if and only if $\epsilon$ is a storage function for the system. In this case, the system is dissipative.

Therefore the key issue in determining dissipativity in many cases is to find a nonnegative energy function $\epsilon$ which makes a system $\epsilon$-dissipative. Note in the linear case the energy function is quadratic: $\epsilon(x) = x^T X x$, $X^T = X$. (To see this, use the Hill–Moylan minimal energy function; the infimum of quadratics is quadratic.) Thus, $\nabla\epsilon(x) = 2Xx$. This gives the classical linear bounded real lemma ([31]).

## III. Output Feedback to Make Systems Dissipative

We wish to analyze the dissipativity condition on two port systems with a one port system in feedback. The basic question is when does feedback exist which makes the full system dissipative and internally stable? This is the central question in $H^\infty$ control.

### A. Energy Balance Equations

We begin with notation for analyzing the dissipativity of the systems obtained by connecting $f, g$ to $F, G$. The energy function on the statespace is denoted by $\epsilon$. $H$ below is the Hamiltonian of the two systems where inputs are $W$, $U$, and $Y$

$$H = \|Z\|^2 - \|W\|^2 + \nabla_z \epsilon(x,z) \cdot f(z,Y) + \nabla_x \epsilon(x,z) \cdot F(x,W,U). \quad (13)$$

By definition (see Section II) the closed-loop system being $\epsilon$-dissipative corresponds to the Hamiltonian function $H$ above being nonpositive.

To construct the closed-loop system, we connect the two systems in feedback, that is tie off $U$ and $Y$ with the substitutions $Y \to G_2(x,W,U)$ and $U \to g(z,Y)$. In the following, when we impose the IA assumptions [see (5) and (6)], we will specialize to a plant which satisfies

$$D_{12}(x)^T D_{12}(x) = e_1(x) > 0,$$
$$D_{21}(x) D_{21}(x)^T = e_2(x) > 0 \quad (14)$$

and often to a compensator with

$$d(z) = 0. \quad (15)$$

We will use the notations $\mathscr{H}_{f,g}(W,x,z)$, $H_{a,b,c,d}(W,x,z)$ to represent the Hamiltonian $H$ (13) for a closed-loop system consisting of a general plant (1) with a general compensator (2), respectively with an IA compensator (5). In this notation $f, g$ represent functions of $z$ and $Y$ and $a, b, c, d$ are variables, not functions, which may be replaced by the *values* of functions defining the compensator at particular values of $z$. The two notations are related in the case of IA compensators (5) by $\mathscr{H}_{f,g}(W,x,z) = H_{a(z),b(z),c(z),d(z)}(W,x,z)$. The Hamiltonian for the closed-loop system consisting of the plant (4) with compensator (5) under the assumptions (14) and (15) is given by

$$\begin{aligned} H_{a(z),b(z),c(z),0}&(W,x,z) \\ &= \nabla_x \epsilon(x,z)^T (A(x) + B_1(x)W + B_2(x)c(z)) \\ &\quad - W^T W + \|C_1(x) + D_{12}(x)c(z)\|^2 \\ &\quad + \nabla_z \epsilon(x,z)^T (b(z)(C_2(x) + D_{21}(x)W) + a(z)). \end{aligned} \quad (16)$$

### B. $H^\infty$ Problem

Find $f = f(z,Y)$, $g = g(z,Y)$ which make the closed-loop system dissipative and internally stable.

This discussion and results about the linear problem, notably Peterson–Anderson–Jonckheere ([31]), lead us to formulate our $H^\infty$ control problem or dissipative feedback problem as follows:

> Find a nonnegative differentiable function $\epsilon$ on $\mathscr{X} \times \mathscr{Z}$ with $\epsilon(0) = 0$ so that there exist functions $f = f(z,Y)$, $g = g(z,Y)$ which satisfy the well-known dissipation inequality
>
> $$(\epsilon - DISFBK) \qquad 0 \geq \max_{x,z,W} \mathscr{H}_{f,g}(W,x,z)$$
>
> where $Y$ is given by $G_2(x,W,U)$.

Also we wish to find formulas for or properties of the functions $f, g$.

We refer to the above statement as the $\epsilon - DISFBK$ problem. We shall say that $\epsilon$ is strictly positive if $\epsilon(x) > 0$ whenever $x \neq 0$. To meet the internal stability constraint, it is often useful to have $\epsilon$ proper or strictly positive. In our formulation of $\epsilon - DISFBK$, we make no such stipulation. In this paper, we shall separate the requirement that $\epsilon$ be strictly positive and proper from other restrictions on $\epsilon$. As we shall see, this is natural and informative.

In practice, it may be difficult or nonessential that we find functions $f, g$ so that $\max_W \mathscr{H}_{f,g}(W,x,z) \leq 0$ for all $x, z$; we will be satisfied if $\max_W \mathscr{H}_{f,g}(W,x,z) \leq 0$ for all $x, z$ in some large region $\Omega \subseteq \mathscr{X} \times \mathscr{Z}$ containing the equilibrium point $(0,0)$. Then the closed-loop system still satisfies the input–output dissipation inequality as long as the state trajectory stays inside $\Omega$. We refer to this modification of $(\epsilon - DISFBK)$ as the *regional* $(\epsilon - DISFBK)$ problem.

Positivity and properness are essential to the $H^\infty$ control problem because they guarantee stability (but not necessarily asymptotic stability) of the closed-loop system for arbitrary $L^2$ inputs. This is the technique which has been used in [33] and [12].

*Theorem 3.1 [25]:* Suppose $\epsilon$ is a proper nonnegative function on $\mathscr{X} \times \mathscr{Z}$ and functions $f, g$ are such that

$$0 \geq \mathscr{H}_{f,g}(W,x,z)$$

for all $W, x, z$. Then the closed-loop system of Fig. 1 has the property that $(x(t), z(t))$ remains in a bounded subset of $\mathscr{X} \times \mathscr{Z}$ for each choice of input function $W \in L^2(0,\infty)$ when started in any state $(x_0, z_0)$.

### C. The Max of H in W

In the remainder of the paper (with the exception of Section VII, the Separation Principle), we will specialize to IA compensators having $d(z) = 0$. Often one finds that

$$H^{*W}_{a,b,c}(x,z) := \max_W H_{a,b,c,0}(W,x,z) \tag{17}$$

is well behaved and is the first max taken in many approaches to solving the problem. We sometimes relax the notation and write

$$H^{*W} = H^{*W}_{a,b,c}(x,z). \tag{18}$$

In light of the discussion in Section I, $H^{*W}$ has a physical interpretation. For a system with energy function $\epsilon$, one fixes a state $(x,z)$ and drives the system with the input $W$ making the energy balance $H^{*W}$ for the system the least dissipative (at that instant). Thus, it is appropriate to call $H^{*W}$ the *worst $\epsilon$-dissipation rate*, which we abbreviate to *worst $\epsilon$-dissipation.*

$H^{*W}$ can be computed concretely for IA and WIA systems by taking the gradient of $H$ (16) in $W$ and setting it to 0 to find the critical point $W^*$. Substitute this back into (16) to get $H^{*W}$. One obtains the following, under the assumptions $G_1 = G_1(x,U)$, $G_2 = G_2(x,W)$ and $d(z) = 0$ (i.e., (14) and (15) for IA systems) for both IA and WIA plants:

$$W^*(x,z,b) = \tfrac{1}{2}\big(B_1(x)^T\nabla_x\epsilon(x,z) + D_{21}(x)^T b^T \nabla_z\epsilon(x,z)\big) \tag{19}$$

and for WIA systems:

$$\begin{aligned} H^{*W} = {} & C_1(x,c)^T C_1(x,c) + \nabla_x\epsilon(x,z)^T AB(x,c) \\ & + \nabla_z\epsilon(x,z)^T a + \nabla_z\epsilon(x,z)^T bC_2(x) \\ & + \tfrac{1}{4}\nabla_x\epsilon(x,z)^T B_1(x)B_1(x)^T\nabla_x\epsilon(x,z) \\ & + \tfrac{1}{2}\nabla_x\epsilon(x,z)^T B_1(x)D_{21}(x)^T b^T\nabla_z\epsilon(x,z) \\ & + \tfrac{1}{4}\nabla_z\epsilon(x,z)^T be_2(x)b^T\nabla_z\epsilon(x,z). \end{aligned} \tag{20}$$

Note that $W^*$ does not depend on $a$ or $c$.

### D. The Doyle–Glover–Khargonekar–Francis Simplifying Assumptions

A special class of IA systems are those satisfying

$$D_{12}(x)^T C_1(x) = 0, \quad B_1(x)D_{21}(x)^T = 0, \quad e_1(x) = I, \quad e_2(x) = I \tag{21}$$

denoted in this paper as the Doyle–Glover–Khargonekar–Francis (DGKF) simplifying assumptions (see [15]). These simplify algebra substantially so are good for tutorial purposes even though they are not satisfied in actual control problems.

## IV. Necessary Conditions for Smooth Solutions of $\epsilon$-*DISFBK* for Input Affine Plants

In this section, we present conditions necessary for a smooth solution to the $\epsilon - DISFBK$ problem for an IA plant to exist. These necessary conditions parallel those known in the linear case and give similar algebraic expressions. We also provide candidate functions $a^*(z)$ and $c^*(z)$ for a feedback compensator and give plausible conditions under which the compensator *must* be given by these functions. This is a bit surprising. The function $b(z)$ is not uniquely defined by these conditions.

We begin by assuming we have a smooth solution to $\epsilon$-*DISFBK*. As we shall see, crucial to the problem are the two sets

$$Z_\epsilon := \{(x,z): \nabla_z\epsilon(x,z) = 0\} \quad \text{and} \tag{22}$$

$$Nz := \{(x,z): z = 0\}. \tag{23}$$

We now list the assumptions which will be used in this section. Later there is a paragraph (after Theorem 4.2) which motivates these and some stronger assumptions. In this section we deal exclusively with IA systems which satisfy (14) and (15). We also assume the following:

A1) Energy functions are differentiable.
A2) $Z_\epsilon$ is a graph over $\mathscr{X}$, i.e., $Z_\epsilon = \{(x, \varphi(x)): x \in \mathscr{X}\}$ for some smooth function $\varphi$.
A3) Vectors $z$ and $x$ are of the same dimension so the compensator state space $\mathscr{Z}$ can be identified with the plant state space $\mathscr{X}$.
A4) $D_z(\nabla_z\epsilon(x,z))|_{z=\varphi(x)}$ has full rank.

For linear systems the energy function can be assumed to be a quadratic form which satisfies the assumption that $Z_\epsilon$ is a graph over $x$ if, for example, the form is positive definite.

*Lemma 4.1:* Fix $x$ and $z$. Then for homogeneous IA functions $f(z,Y), g(z)$ (5),

$$\inf_{f,g}\max_W \mathscr{H}_{f,g}(W,x,z) = \begin{cases} \inf_{a,b,c} H^{*W}_{a,b,c}(x,z), & z \neq 0 \\ \inf_b H^{*W}_{0,b,0}(x,z), & z = 0 \end{cases}$$

which gives the more explicit formulas

$$\inf_{a,b,c} H^{*W}_{a,b,c}(x,z) = \left\{\begin{matrix} -\infty & z \neq \varphi(x) \\ \min_c H^{*W} = IAX(x), & z = \varphi(x) \end{matrix}\right\}, \quad z \neq 0 \tag{24}$$

$$\inf_b H^{*W}_{0,b,0}(x,0) = \min_b H^{*W}_{0,b,0}(x,0) = IAYI(x), \quad z = 0. \tag{25}$$

Here we define

$$\begin{aligned} IAX(x) = {} & \big[-C_1(x)^T D_{12}(x)e_1(x)^{-1}B_2(x)^T + A(x)^T\big]X(x) \\ & + X(x)^T\big[A(x) - B_2(x)e_1(x)^{-1}D_{12}(x)^T C_1(x)\big] \\ & + X(x)^T\big[B_1(x)B_1(x)^T - B_2(x)e_1(x)^{-1}B_2(x)^T\big]X(x) \\ & + C_1(x)^T\big(I - D_{12}(x)e_1(x)^{-1}D_{12}(x)^T\big)C_1(x) \end{aligned} \tag{26}$$

$$
\begin{aligned}
&IAYI(x)\\
&= \left[-C_2(x)^T e_2(x)^{-1} D_{21}(x) B_1(x)^T + A(x)^T\right] Y_I(x)\\
&\quad + Y_I(x)^T\left[A(x) - B_1(x) D_{21}(x)^T e_2(x)^{-1} C_2(x)\right]\\
&\quad + C_1(x)^T C_1(x) - C_2(x)^T e_2(x)^{-1} C_2(x)\\
&\quad + Y_I(x)^T\left[B_1(x) B_1(x)^T - B_1(x) D_{21}(x)^T\right.\\
&\quad \left.\cdot e_2(x)^{-1} D_{21}(x) B_1(x)^T\right] Y_I(x) \qquad (27)
\end{aligned}
$$

where the functions $X$ and $Y_I$ are defined by

$$X(x) = \tfrac{1}{2}\nabla_x \epsilon(x,\varphi(x)) \quad \text{and} \quad Y_I(x) = \tfrac{1}{2}\nabla_x \epsilon(x,0). \qquad (28)$$

The minimizing $c$ when $z = \varphi(x)$ can be computed explicitly to be

$$c^*(z) = -e_1(x)^{-1}\big(B_2(x)^T X(x) + D_{12}(x)^T C_1(x)\big). \qquad (29)$$

The minimizing $b$ when $z = 0$ will be computed explicitly in the proof.

*Proof:* Fix $x, z$.

A) For $z \neq 0$ and any $b, c$: $\inf_a H^{*W} = -\infty$ unless $z = \varphi(x)$. This is because the explicit form (20) for $H^{*W}$ contains $a$ linearly, unless the coefficient of $a$ is 0 (i.e., $\nabla_z \epsilon(x, \varphi(x)) = 0$). Here we have assumed that only $z = \varphi(x)$ satisfies $\nabla_z \epsilon(x,z) = 0$.

B) If $\varphi(x) = z \neq 0$, then $H^{*W}$ is independent of $a, b$, and $\min_c H^{*W} = IAX(x)$. This identity and the minimizer $c^*(z)$ are calculated by applying the change of notation (28) and calculating the critical $c$ in the resulting expression. The critical $c$ is substituted back into the expression to obtain $IAX(x)$. The main observation is that in the explicit form (20) for $H^{*W}$, we have that $\nabla_z \epsilon(x,z)|_{z=\varphi(x)}$ vanishes, thereby eliminating dependence on both $a$ and $b$.

C) If $z = 0$, then define $q(x,0) = \nabla_z \epsilon(x,o)^T b$ and minimize $H^{*W}$ over $q$ to obtain that the minimizer $b$ is given by

$$\nabla_z \epsilon(x,0)^T b = -2\big(C_2(x)^T + Y_I(x)^T B_1(x) D_{21}(x)^T\big) \cdot e_2(x)^{-1}. \qquad (30)$$

Substituting this into $H^{*W}$, we obtain $\min_b H^{*W}_{0,b,0}(x,z) = IAYI(x)$. ■

*Theorem 4.2:*

a) If there is a function $\epsilon(x,z)$ and an IA compensator system $a(z), b(z), c(z)$ making the closed-loop system $\epsilon$-dissipative, then the inequalities

$$IAX(x) \le 0 \quad \text{and} \quad IAYI(x) \le 0$$

for all $x$ have solutions $X$ and $Y_I$ given by $X(x) = (1/2)\nabla_x \epsilon(x, \varphi(x))$ and $Y_I(x) = (1/2)\nabla_x \epsilon(x,0)$.

b) If the function $\epsilon$ in part a) is nonnegative with $\epsilon(0,0) = 0$, i.e., $\epsilon$-*DISFBK* has a solution, then $X(x)$ and $Y_I(x)$ are gradients of nonnegative functions, and $Y_I(x) - X(x)$ is the gradient of a function which is nonnegative near 0. In particular, the linearized problem has a solution.

Conversely, suppose the ($\epsilon$-*DISFBK*) expression has a saddle value in $x$, that is

$$
\begin{aligned}
&\inf_{f,g} \max_{x,W} \mathscr{H}_{f,g}(W,x,z)\\
&= \begin{cases} \inf\limits_{a,b,c} \max\limits_{x} H^{*W}_{a,b,c}(x,z), & z \neq 0\\ \min\limits_{b} \max\limits_{x} H^{*W}_{0,b,0}(x,0), & z = 0 \end{cases} \qquad (31)\\
&= \begin{cases} \max\limits_{x} \inf\limits_{a,b,c} H^{*W}_{a,b,c}(x,z), & z \neq 0\\ \max\limits_{x} \min\limits_{b} H^{*W}_{0,b,0}(x,0), & z = 0. \end{cases} \qquad (32)
\end{aligned}
$$

If a particular nonnegative $\epsilon$ defines $X(x), Y_I(x)$ as in (28) which satisfy $IAX(x) \le 0$ and $IAYI(x) \le 0$ for all $x$ and minimizing $a, b, c$ for the expressions in (31) exist, then there is a solution to $\epsilon$-*DISFBK*.

It is worthwhile to note here that linear systems actually satisfy the saddle *point* condition that the optimum values of $x, a, b, c$ in the expressions in (31) are also optimal in the expressions (32). The converse statement of this theorem illustrates that it is sufficient to have the slightly weaker saddle *value* condition with compensator functions defined by the minimizing $a, b, c$ of the left hand expressions for each $z$. In the nonlinear case, a saddle point will not exist in general.

*Proof of Theorem 4.2: Part a)* The forward side of the theorem follows from (24), (25) and the fact that min max $\ge$ max min. To be more explicit, if the compensator functions $a(z), b(z), c(z)$ make the closed-loop system $\epsilon$-dissipative, then for each fixed $z \in range(\varphi)$, $z \neq 0$,

$$
\begin{aligned}
0 &\ge \max_{x,W} H_{a(z),b(z),c(z),0}(W,x,z) \ge \max_{x} \inf_{a,b,c} H^{*W}\\
&= \max_{x \in \varphi^{-1}(z)} IAX(x)
\end{aligned}
$$

and for $z = 0$,

$$
\begin{aligned}
0 &\ge \max_{x,W} H_{0,b(z),0,0}(W,x,0) \ge \max_{x} \min_{b} H^{*W}\\
&= \max_{x} IAYI(x).
\end{aligned}
$$

(When $z = 0, c(z)|_{z=0} = 0 = a(z)|_{z=0}$ eliminates the dependence of $H$ on $a$ and $c$.) Consequently, $IAX(x) \le 0$ and $IAYI(x) \le 0$ for all $x$ as required. *Part b)* We have $Y_I = \nabla\psi_1$, where $\psi_1(x) = (1/2)\epsilon(x,0)$, and $X = \nabla\psi_2$ where $\psi_2(x) = (1/2)\epsilon(x, \varphi(x))$ (using $\nabla_z \epsilon(x, \varphi(x)) = 0$). Denote the linearization of $\epsilon(x,z)$ about $(0,0)$ by

$$\epsilon_{\text{Lin}}(x,z) = [x^T \quad z^T]\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22}\end{bmatrix}\begin{bmatrix} x \\ z\end{bmatrix}.$$

Here $P_{11} = Y_{I1}$ where $Y_{I1}x$ is the linearization of $Y_I(x)$ at 0. Using the chain rule on $X = \nabla\psi_2$, we obtain that $X_1 = P_{11} + P_{12} D_x \varphi(0)$ where $X_1 x$ is the linearization of $X(x)$ at 0. Using $\nabla_z \epsilon(x, \varphi(x)) = 0$, we obtain similarly that $P_{12}^T + P_{22} D_x \varphi(0) = 0$. Then $Y_{I1} - X_1 = P_{11} - (P_{11}$

$+ P_{12}D_x\varphi(0)) = -P_{12}D_x\varphi(0)$ and thus $(Y_{I1} - X_1)^T = -D_x\varphi(0)^T P_{12}^T = D_x\varphi(0)^T P_{22} D_x\varphi(0) \geq 0$. The quantity $Y_{I1} - X_1$ is the Hessian of $\psi_1 - \psi_2$ and thus $Y_I - X$ is the gradient of a locally nonnegative function. The proof that the linearization has a solution will be deferred until the discussion at the end of this section.

For the converse, we need the following lemma.

*Lemma 4.3:* Let $F, G_1, G_2$ define an arbitrary (plant) system of the form (1), (2) and let $\epsilon$ be a fixed nonnegative function on $\mathscr{X} \times \mathscr{Z}$. If

$$\max_{z \neq 0} \min_{a,b,c,d} \max_x \max_W H_{a,b,c,d}(W, x, z) \leq 0,$$

and

$$\min_{b,c} \max_x \max_W H_{0,b,0,d}(W, x, 0) \leq 0$$

then there exists an homogeneous IA system (compensator) $f^*(z, Y) = a^*(z) + b^*(z)Y$, $g^*(z, Y) = c^*(z) + d^*(z)Y$ such that the closed-loop system solves $\epsilon$-*DISFBK*.

Note that the letters $a$, $b$, $c$ and $d$ represent free parameters so that minimization is with respect to real numbers rather than functions. Combine this lemma with [lemma 4.1, (24), (25)] to see that $\epsilon$ provides a solution to $\epsilon$-*DISFBK*. ■

In the remainder of this paper, we will make the simplifying assumption that function $\varphi$ defining the graph $Z_\epsilon$ is invertible. For linear systems, the standard Doyle–Glover–Khargonekar–Francis (DGKF) solution (maximum-entropy solution) with $X$ and $Y$ invertible has this property. As a consequence nonlinear solutions linearizing to it will also have the property near 0. When $\varphi$ is invertible, the energy function can be transformed through a change of $z$ coordinates so that $Z_\epsilon = \{(x, z) \colon x = z\}$ since under the change of variables $\varphi(z)$, the system

$$\dot{z} = (\nabla\varphi(z))^{-1}(a \circ \varphi)(z) + (\nabla\varphi(z))^{-1}(b \circ \varphi)(z)Y,$$
$$U = c \circ \varphi(z)$$

provides the same feedback as the system $a(z), b(z), c(z)$ in the original $z$ coordinates but satisfies in addition $Z_\epsilon = \{(x, z) \colon x = z\}$. Indeed henceforth we always use

$$\nabla_z \epsilon(x, z) = 0 \quad \text{if and only if} \quad z = x. \tag{33}$$

For linear systems, this implies that the gradient of $\epsilon$ can be expressed entirely in terms of the DGKF $X$ and $Y$:

$$\tfrac{1}{2}\nabla_x \epsilon(x, 0) = Y_I(x) = Y^{-1}x,$$
$$\tfrac{1}{2}\nabla_x \epsilon(x, x) = X(x) = Xx$$

and that

$$IAX(x) = 0, \qquad IAYI(x) = 0$$

are the DGKF Riccati equations for $X$ and for the inverse of $Y$, respectively, (for the case where $Y$ is invertible). Also, the minimizing $b$ and $c$ in Lemma 4.1 (linear case) are independent of $x$ and thus provide functions with which to construct the central compensator (see e.g., [31]).

In the nonlinear case (IA system), assumption A1) implies that the minimizing $c$ in Lemma 4.1 occurs when $z = x$ and thus defines a function $c^*$ of $z$ only [see (29)] which may be used as a candidate function for constructing the compensator. This formula agrees with that found for the state feedback problem by [1], [32], [33], [12] for IA systems and [7]. Unfortunately, the minimizing $b$ occurring when $z = 0$ is a function of $x$ so that a candidate function $b(z)$ for the compensator is not determined by the minimization process.

The next result gives certain conditions which force the form of $a(z)$ and $c(z)$ when an IA compensator $(a(z), b(z), c(z))$ is a solution of $(\epsilon - DISFBK)$.

*Theorem 4.4: Separation Principle.* Suppose that an energy function $\epsilon(x, z)$ satisfies assumptions A1)–A4) and $IAX(x) = 0$ for all $x$. If the functions $a(z), b(z), c(z)$ solve ($\epsilon$-*DISFBK*), then $c(z)$ is given by $c^*(z)$ in (29) (with $\varphi(x) = x$) and $a(z)$ is given by

$$\begin{aligned} a^*(z) &= F(z, W^*(z, z, b(z)), c^*(z)) \\ &\quad + b(z)[-G_2(z, W^*(z, z, b(z)))] \\ &= A(z) + B_2(z)c^*(z) - b(z)C_2(z) \\ &\quad + [B_1(z) - b(z)D_{21}(z)]B_1(z)^T X(z). \end{aligned} \tag{34}$$

The theorem has a physical interpretation. The key hypothesis $IAX = 0$ says that if one chooses memoryless state feedback which produces the most negative $\epsilon$-dissipation rate possible and obtains an $\epsilon$-dissipation rate equal to 0 (recall, this means $H^{*W}_{a(z), b(z), c^*(z)}(z, z) = IAX(z) = 0$, for all states), then any solution to ($\epsilon$-*DISFBK*) has $a(z)$ and $c(z)$ prescribed as above. In the linear case, the result has the interpretation that the output feedback problem can be split into two separate pieces: the state feedback problem and the problem of state estimation via output injection (see [31], [15]).

*Proof:* Let $a(z), b(z), c(z)$ denote a solution to ($\epsilon$-*DISFBK*) whose existence is guaranteed by assumption. Then $c(z)$ must satisfy

$$c(z) = \arg\min_c H^{*W}_{a,b,c}(z, z)$$

because another $c(z)$ will, for some $z$, make

$$H^{*W}_{a,b,c(z)}(z, z) > \min_c H^{*W}_{a,b,c}(z, z) = IAX(z) = 0.$$

Thus, $c(z) = c^*(z)$.

As a solution to ($\epsilon$-*DISFBK*), $a(z), b(z), c^*(z)$ will also satisfy

$$H^{*W}_{a(z), b(z), c^*(z)}(x, z) \leq 0 \tag{35}$$

for all $x, z$. Lemma 4.1 implies that $H^{*W}_{a(z), b(z), c^*(z)}(x, z) = IAX(x) = 0$ on the diagonal $z = x$, so for each fixed $x$ it achieves its maximum as a function of $z$ at $z = x$. Hence,

$$D_z \left\{ H^{*W}_{a(z), b(z), c^*(z)}(x, z) \right\} \Big|_{z=x} = 0.$$

So

$$D_U\left\{H^{*W}_{a(z),b(z),U}(x,z)\right\}\Big|_{\substack{z=x\\ U=c^*(z)}} D_z c^*(z) + D_z\left\{H^{*W}_{a(z),b(z),U}(x,z)\right\}\Big|_{\substack{z=x\\ U=c^*(z)}} = 0.$$

But

$$D_U\left\{H^{*W}_{a(z),b(z),U}(x,z)\right\}\Big|_{\substack{z=x\\ U=c^*(z)}} = 0$$

by definition of $c^*(z)$. Hence

$$D_z\left\{H^{*W}_{a(z),b(z),U}(x,z)\right\}\Big|_{\substack{z=x\\ U=c^*(z)}} = 0. \tag{36}$$

Now from (35) and (36) combined with assumptions A1)–A4), we have the hypotheses for the Separation Principle presented in Section VII (specialized to the IA case). Equation (34) now follows from (48). ■

*Theorem 4.5:* A necessary condition for the system $a^*(z), b(z), c^*(z)$ with functions $a^*, c^*$ given by (34) and (29) to solve $\epsilon$-*DISFBK* is $k(x,z) \le 0$ for all $x, z$ where $k$ is

$$\begin{aligned} k(x,z) &:= A(x)^T \nabla_x \epsilon(x,z) + A(z)^T \nabla_z \epsilon(x,z) \\ &\quad + C_1(x)^T C_1(x) \\ &\quad + 2\big[C_1(x)^T D_{12}(x) + \tfrac{1}{2}\nabla_x \epsilon(x,z)^T B_2(x) \\ &\quad + \tfrac{1}{2}\nabla_z \epsilon(x,z)^T B_2(z)\big] \\ &e_1(z)^{-1}\big[-B_2(z)^T X(z) - D_{12}(z)^T C_1(z)\big] \\ &\quad -\big[C_2(x) - C_2(z) + \tfrac{1}{2} D_{21}(x) B_1(x)^T \nabla_x \epsilon(x,z) \\ &\quad - D_{21}(z) B_1(z)^T X(z)\big]^T e_2(x)^{-1} \\ &\quad \cdot\big[C_2(x) - C_2(z) + \tfrac{1}{2} D_{21}(x) B_1(x)^T \\ &\quad \cdot \nabla_x \epsilon(x,z) - D_{21}(z) B_1(z)^T X(z)\big] \\ &\quad + \tfrac{1}{4}\nabla_x \epsilon(x,z)^T B_1(x) B_1(x)^T \nabla_x \epsilon(x,z) \\ &\quad + X(z)^T B_1(z) B_1(z)^T \nabla_z \epsilon(x,z) \\ &\quad + \big[C_1(z)^T D_{12}(z) + X(z)^T B_2(z)\big] e_1(z)^{-1} \\ &\quad \cdot\big[B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\big]. \end{aligned} \tag{37}$$

Note that $k$ is independent of $b(z)$ and $k(x,x) = IAX(x)$ where $X(x) = (1/2)\nabla_x \epsilon(x,x)$ and (if $X(0)=0$) $k(x,0) = IAYI(x)$ where $Y_I(x) = (1/2)\nabla_z \epsilon(x,0)$.

*Proof:* When $a^*(z), c^*(z)$ are given by (34) and (29), then

$$H^{*W}_{a^*(z),b(z),c^*(z)}(x,z) = \hat{b} e_2(x) \hat{b}^T + k(x,z) \tag{38}$$

where

$$\begin{aligned}\hat{b} &= \tfrac{1}{2}\nabla_z \epsilon(x,z)^T b(z) \\ &\quad + \big(\tfrac{1}{2}\nabla_x \epsilon(x,z)^T B_1(x) D_{21}(x)^T - X(z)^T B_1(z) D_{21}(z)^T \\ &\quad + C_2(x)^T - C_2(z)^T\big) e_2(x)^{-1}.\end{aligned} \tag{39}$$

Since $\hat{b} e_2(x) \hat{b}^T \ge 0$ for any $b(z), x, z$, and

$$H^{*W}_{a^*(z),b(z),c^*(z)}(x,z) \le 0$$

by assumption, the conclusion follows. ■

We conclude this section with some connections between our results on nonlinear closed-loop systems given by (1)–(5) [satisfying (14), (15)] and their linearizations

$$\begin{aligned} dx/dt &= A_1 x + B_1(0) W + B_2(0) U \\ Z &= C_{11} x + D_{12}(0) U, \qquad Y = C_{21} x + D_{21}(0) W \\ dz/dt &= a_1 z + b_0 Y, \qquad U = c_1 z \end{aligned} \tag{40}$$

where

$$A_1 = D_x A(0), \quad C_{11} = D_x C_1(0), \quad C_{21} = D_x C_2(0) \tag{41}$$

$$a_1 = D_z a(0), \quad b_0 = b(0), \quad c_1 = D_z c(0). \tag{42}$$

For the linearized system (40) we denote the equations (26), (27) by $IAX(x) = x^T L_X x$ and $IAYI(x) = x^T L_Y x$. (See e.g., (4.3), (4.4) of [31].)

*Corollary 4.6:* Each of the statements below implies the statement which follows it.

a) There exists a neighborhood of the origin $\Omega \subseteq \mathscr{X} \times \mathscr{Z}$, a function $\epsilon$ with strictly positive Hessian satisfying assumptions A1)–A4), and functions $a(z), b(z), c(z)$ solving the regional $\epsilon$-*DISFBK* problem for the closed-loop system (1)–(5) in $\Omega$.

b) There exists a neighborhood of the origin in which $IAX(x) \le 0$, $IAYI(x) \le 0$ have solutions $X(x), Y_I(x)$ defined by (28) such that $X, Y_I$, and $Y_I - X$ are locally gradients of positive functions with strictly positive Hessians.

c) The DGKF Riccati's $L_X \le 0$ and $L_Y \le 0$ have solutions $X_1 > 0, Y_{I1} > 0$ with $Y_{I1} - X_1 > 0$.

d) There exists a positive definite quadratic function $\epsilon_{\text{Lin}}$ satisfying assumptions A1)–A4) and matrices $a_1, b_0, c_1$ solving $\epsilon$-*DISFBK* for the linearized closed-loop system (40).

In addition c) ⇔ d). Moreover, if a) is satisfied, then a compensator satisfying d) is given by the linearizations (42) of the compensator satisfying a).

The corollary was stated in terms of inequalities, e.g., $L_X \le 0$, while equalities $L_X \equiv 0$ are more common in the literature. For the linear case if a positive definite solution $X$ to a Riccati such as $L_X \le 0$ exists, then there is a positive semidefinite solution $\tilde{X}$ to $L_X \equiv 0$. This is a phenomenon (which implicitly includes differentiability) not yet demonstrated in the nonlinear case. Using this the interested reader could produce an equality version of the linear statements.

An identical argument to the separation principle (Theorem 4.4) also shows that if $L_X \equiv 0$, then $a_1$ and $c_1$ are given by the linearized versions of (29) and (34). (See also [15].)

Similarly, minimizing the linearized $H^{*W}$ constructed from (40) over $q = (Y_{I1} - X_1)^T b_0$ gives the same answer as linearizing the expression $k(x,z)$ in (37). The minimizing $b_0$ above can be solved for in the case that $Y_{I1} - X_1 > 0$ to obtain

$$b_0 = (Y_{I1} - X_1)^{-1}\left(C_{21}^T + Y_{I1}B_1(0)D_{21}(0)^T\right)e_2(0)^{-1} \tag{43}$$

a constant. With this choice of $b_0, \hat{b} = 0$ in (38) so that the minimum $k(x,z)$ is attained. For the linearization (and linear systems in general), $k(x,z)$ has the particularly useful form

$$k(x,z) = [x^T \quad z^T]\begin{bmatrix} L_Y & L_X - L_Y \\ L_X - L_Y & L_Y - L_X \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}$$

from which the conditions $L_Y \leq 0, L_Y - L_X \leq 0$, and (through a congruence transformation) $L_X \leq 0$ may be read off. Note that this reflects the value of the Hamiltonian upon replacement of $a$, $b$, and $c$ according to the linear versions of (34), (43), and (29), respectively.

## V. Energy Ansatzes

To make further progress we make assumptions on the storage functions which we permit. It is a valid question to ask whether we can we find feedback $f, g$ to make the system $\epsilon$-dissipative where $\epsilon$ is restricted in some way. Of course we hope the restriction not only makes computation possible but that optimal or near optimal storage functions have this form. In order of liberality, natural conditions are as follows:

C1) $\epsilon(x,z) = \eta_1(x) + \eta_2(x - z)$.

C2) $\epsilon(x,z) = \eta_1(x) + \eta_2(x - z) + \sum_{j>2}\gamma_j(z)\eta_j(x - z)$.

C3) $\epsilon(x,z) = \eta_1(x) + \eta_2(x - z) + r(x,z)$ where $r$ is a smooth nonnegative function such that $r$ vanishes on $z = 0$ and on the diagonal $x = z$ and $\nabla_z r(x,x) = 0$.

C4) $\epsilon(x,z) = \eta_1(x) + \eta_2(x - z) + \sum_{j>2}\gamma_j(x,z)\eta_j(x - z)$.

C5) $\epsilon(x,z) = \eta_1(x) + \eta_2(x - z) + r(x,z)$ where $r$ is a smooth nonnegative function such that $r$ and $\nabla r$ vanish on the diagonal $x = z$.

In all cases, we shall assume that each function $\eta_k, \gamma_k$ is nonnegative, smooth, and vanishing at 0. The motivation for these conditions is primarily mathematical, in that linear systems produce $\epsilon$ of the form C1), and without the weakest assumption C5) the mathematical problem is vastly more complicated than with it. The following proposition describes the relationships between these "energy ansatzes".

*Proposition 5.1:* An energy function satisfying condition (5.$m$) also satisfies condition (5.$n$) if $m < n < 3$ or $m = 4, n = 5$.

Note that the hypotheses 4) and 5) of the Separation Principle (Section VII) are implied by C3).

Ansatzes on $\epsilon$ convert directly to conditions on gradients which come from differentiating $\epsilon$. For example, in C1) we recover $\eta_1$ and $\eta_2$ from $\eta_1(x) = \epsilon(x,x)$, and $\eta_2(z) = \epsilon(0, -z)$. Thus,

$$\nabla_x\epsilon(x,z) = (\nabla_x\epsilon(x,x) + \nabla_x\epsilon(x - z, 0)) - \nabla_x\epsilon(x - z, x - z)$$
$$\nabla_z\epsilon(x,z) = -\nabla_x\epsilon(x - z, 0) + \nabla_x\epsilon(x - z, x - z). \tag{44}$$

By assuming an ansatz for $\epsilon$, we get *necessary* conditions for the solution of the $H^\infty$ problem which do not involve the unknown function $\epsilon$, but rather only solutions of certain Hamilton–Jacobi inequalities.

*Theorem 5.2:* If ($\epsilon$-*DISFBK*) has a strictly positive solution $\epsilon$ for the closed-loop IA system, where $\epsilon$ is

a) of the form C3), then there exist solutions $X(x)$ and $Y_I(x)$ of $IAX(x) \leq 0$ and $IAYI(x) \leq 0$ [see (26) and (27)] such that $X(x)$, $Y_I(x)$, and $Y_I(x) - X(x)$ are gradients of positive functions.

b) of the form C1) then for all $x, z$

$$\begin{aligned}
0 \geq\ & 2\Big\{X(x - z)^T\big[B_1(z)B_1(z)^T + B_2(x)B_2(z)^T \\
& -B_2(z)B_2(z)^T\big] + Y_I(x - z)^T\big[-B_1(z)B_1(z)^T \\
& -B_2(x)B_2(z)^T + B_2(z)B_2(z)^T\big]\Big\}X(z) \\
& + 2\big[-X(x-z)^TB_1(x)B_1(x)^T \\
& -X(z)^TB_2(z)B_2(x)^T \\
& +Y_I(x - z)^TB_1(x)B_1(x)^T + A(x)^T\big]X(x) \\
& + 2\big[A(x)^T - A(z)^T\big]Y_I(x - z) \\
& +2\big[-Y_I(x - z)^TB_1(x)B_1(x)^T - A(x)^T \\
& +A(z)^T\big]X(x - z) + C_1(x)^TC_1(x) \\
& - C_2(x)^TC_2(x) + C_2(x)^TC_2(z) \\
& + C_2(z)^TC_2(x) - C_2(z)^TC_2(z) \\
& + X(x)^TB_1(x)B_1(x)^TX(x) \\
& + X(x - z)^TB_1(x)B_1(x)^TX(x - z) \\
& + X(z)^TB_2(z)B_2(z)^TX(z) \\
& + Y_I(x - z)^TB_1(x)B_1(x)^TY_I(x - z)
\end{aligned}$$

provided the DGKF simplifying assumptions (21) hold.

*Proof:* i) Note that Theorem 4.2 guarantees all of this except $Y_I - X$ being the gradient of a positive function outside of some neighborhood of the origin. We have

$$X(x) = \tfrac{1}{2}\nabla_x\epsilon(x,x) = \tfrac{1}{2}\nabla\eta_1(x)$$
$$Y_I(x) = \tfrac{1}{2}\nabla_x\epsilon(x,0) = \tfrac{1}{2}\nabla\eta_1(x) + \tfrac{1}{2}\nabla\eta_2(x).$$

Here, $\nabla_x r(x,x) = 0$ follows from $\nabla_z r(x,x) = 0$ and $r(x,x) \equiv 0$. Similarly, $r(x,0) \equiv 0$ implies that $\nabla_x r(x,0) = 0$. Then $Y_I(x) - X(x) = (1/2)\nabla\eta_2(x)$ and the theorem follows, since $\eta_1$, $\eta_2$, and $\eta_1 + \eta_2$ are positive. ii) The expression in b) is just the condition $k \leq 0$ in Theorem 4.5 with expressions (44) substituted in for the respective gradients of $\epsilon$, where the DGKF simplifying assumptions were used to simplify some expressions. ∎

To this point our results have been in the direction of necessary conditions for a solution. We now suggest some constructions to the reader for producing solutions. They probably are not optimal at all, since there is a considerable gap between our necessary and our sufficient conditions. The main weakness in our understanding lies with $b(z)$, so the recipe below just picks it in one sensible way. Often there will be better ways. The solutions given by the recipe in the linear case is the maximum entropy solution.

*RECIPE*

1) Find a solution $X(x)$ to $IAX(x) \le 0$, where $IAX(x)$ is given by (26).

2) Choose $c^*(z)$ and $a^*(z)$ as in (29) and (34), that is

$$c^*(z) = -e_1(z)^{-1}\big(B_2(z)^T X(z) + D_{12}(z)^T C_1(z)\big)$$

$$a^*(z) = A(z) + B_2(z)c^*(z) - b(z)C_2(z) + (B_1(z) - b(z)D_{21}(z))B_1(z)^T X(z).$$

3) For a suitable energy function, define $b(z)$ by

$$b(z)^T = -2(D_x \nabla_x \epsilon(x,z)|_{x=z})^{-1} \cdot D_x\Big\{e_2(x)^{-1}\big[\tfrac{1}{2}D_{21}(x)B_1^T(x)\nabla_x \epsilon(x,z) - D_{21}(z)B_1^T(z)X(z) + C_2(x) - C_2(z)\big]\Big\}|_{x=z}.$$

Here we assume that $D_x \nabla_z \epsilon(x,z)|_{x=z}$ is invertible so that we can solve for $b(z)$.

*Remark 1:* Motivation for the formulas in the RECIPE for $a^*$ and $c^*$ arises as follows. First of all, in the linear case, the maximum entropy or central solution arises via the recipe with $IAX(x) = 0$, $IAYI(x) = 0$ and with $\epsilon$ of the form C1) with

$$\eta_1(x) = x^T X(x), \qquad \eta_2(x) = x^T(Y_I(x) - X(x))$$

where $X(x)$ and $Y_I(x)$ are linear. For the general nonlinear case, if $\epsilon$ has the form C3) with $X(x) = (1/2)\nabla_x \epsilon(x,x)$ and $Y_I(x) = (1/2)\nabla_x \epsilon(x,0)$ and if $IAX(x) = 0$, then the form of $a^*$ and $c^*$ is forced on us by the Separation Principle, so it is natural to look for solutions of this form even if we only have $IAX(x) \le 0$.

*Remark 2:* We would like to choose $b(z)$ so that the dissipation inequality $H^{*W}_{a^*(z),b(z),c^*(z)} \le 0$, i.e., [from (38)] $\hat{b}e_2(x)\hat{b}^T + k(x,z) \le 0$ holds in as large a neighborhood of $(0,0) \in \mathscr{X} \times \mathscr{X}$ as possible. Note that $\hat{b}$ vanishes for $x = z$. Hence, given that the necessary condition $k(x,z) \le 0$ holds for all $x, z$, $H^{*W} \le 0$ is automatic for any choice of $b(z)$ on the diagonal $z = x$. We would like to choose $b(z)$ in such a way to make $\hat{b} = 0$ at all $x$ and $z$; in the linear case indeed this is possible. However, in the general nonlinear case obtaining $\hat{b} = 0$ is only possible if one allows $b$ to depend on both $x$ and $z$; unfortunately, we are allowed only to let $b$ be a function of $z$. The idea then is, for each fixed $z$, to choose $b(z)$ so that $\hat{b}$ vanishes to maximum possible order (order two) at $x = z$ as a function of $x$. In this way, we expect $H^{*W} \le 0$ to remain true on a large neighborhood surrounding the origin. This leads directly to the formula for $b(z)$ in Step 3) which we obtain by differentiating $\hat{b}^T$ in (39) with respect to $x$ and evaluating at $x = z$.

*Remark 3:* The inequality $IAX(x) \le 0$ and related equation $IAX(x) = 0$ are nonlinear generalizations of Riccati equations well known in classical mechanics as Hamilton–Jacobi equations (or inequalities). They also appear in various forms in nonlinear optimal control and game theory. There are a number of methods of solution; for a discussion, especially the connection between solutions of Hamilton–Jacobi equations and Lagrangian invariant manifolds of Hamiltonian vector fields, see [33].

Next, we give sufficient conditions for the compensator constructed via our RECIPE to solve $\epsilon$-*DISFBK* and to obtain asymptotic stability in the $H^\infty$-problem.

*Theorem 5.3:* Let $\epsilon$ be as in C3), $a^*(z), b(z), c^*(z)$ as in the RECIPE, $X(x) = (1/2)\nabla_x \epsilon(x,x)$, and $Y_I(x) = (1/2)\nabla_x \epsilon(x,0)$. Define $\Omega_{diss} = \{(x,z)\colon x, z$ in state space satisfying (38) $\le 0\}$. Assume that $X(x)$ and $Y_I(x)$ satisfy the Hamiltonian–Jacobi inequalities $IAX(x) \le 0$ and $IAYI(x) \le 0$ [see (26) and (27)] such that each of $X(x)$, $Y_I(x)$, and $Y_I(x) - X(x)$ is the gradient of a nonnegative function.

a) Assume in addition that $IAX(x) < 0$ for all $x$. Then the RECIPE produces a solution to the region ($\epsilon$-*DISFBK*) problem on the set $\Omega_{diss}$.

b) Assume there exists $\rho > 0$ so that $\mathscr{S}_\rho = \{(x,z) \in \mathscr{X} \times \mathscr{X}\colon \epsilon(x,z) \le \rho\} \subset \Omega_{diss}$.
   1) Then if $(x(t), z(t))$ is a trajectory of the closed-loop system subject to zero input signal $W(t) = 0$ for $t \ge 0$, then $(z(t), z(t)) \in \mathscr{S}_\rho$ whenever $(z(0), z(0)) \in \mathscr{S}_\rho$.
   2) Assume in addition:
      i) The DGKF simplifying assumptions $D_{12}$ $d(x)^T C_1(x) = 0$ and $B_1(x)D_{21}(x)^T = 0$.
      ii) $(C_1, A)$ is detectable, i.e., $\dot{x}(t) = A(x(t))$ and $C_1(x(t)) = 0$ for all $t \ge 0$ implies $x(t) \to 0$ at $t \to \infty$.
      iii) The system $\dot{z} = A(z) + B_1(z)B_1(z)^T X(z) - b(z)C_2(z)$ is asymptotically stable.
      iv) $\epsilon$ is proper and strictly positive.

Then whenever $(x(t), z(t))$ is a trajectory of the closed-loop system subject to zero input signal $W(t) = 0$ and $(x(0), z(0)) \in \mathscr{S}_\rho$, then $(x(t), z(t)) \to 0$ as $t \to \infty$.

*Proof:* a) $\Omega_{diss}$ contains an open set around the diagonal $x = z$, since $k(x,x) = IAX(x) < 0$. Since $X(x) = (1/2)\nabla\eta_1(x)$, and $Y_I(x) - X(x) = (1/2)\nabla\eta_2(x)$ are the gradients of strictly positive functions, $\eta_1$ and $\eta_2$ are positive. By assumption, $r$ has nonnegative values. Hence, $\epsilon(x,z)$ is strictly positive and the regional ($\epsilon$ − *DISFBK*) problem is satisfied on the region ($\Omega_{diss}$). b) This is similar to [29, theorem 3.1], and one should see that paper for a proof which converts easily to this situation. The use of detectability predates this and is in [12], [32], [33].

*Remark 4:* In Theorem 5.3-b) we need assume that the detectability assumption ii) and the asymptotic stability assumption iii) hold only regionally, i.e., only for the case where $(x(0), z(0)) \in \mathscr{S}_\rho$. Also, no particular form for $b(z)$ was required.

*Remark 5:* An analog of Theorem 5.3 holds if one replaces the compensator $a^*(z), b(z), c^*(z)$ in the RECIPE by the linear compensator which solves the $H^\infty$-problem for the linearization of the plant $(F, G_1, G_2)$ at the origin. The article [29] mentions this as does [33]. We expect but do not know how to prove that the region $(\Omega_{diss})$ for the compensator $(a^*(z), b(z), c^*(z))$ in the RECIPE is usually much larger than the corresponding region associated with the linear compensator which solves the linearized problem.

## VI. W-Input Affine Systems

We often consider a special class of systems (7) called WIA systems. These are affine in $W$ but not necessarily in $U$. We make the same assumptions as for IA systems in this more general case with the additional assumption A3) to simplify notation. In fact, many of the results are identical in form, but the resulting equations are not as explicit as for the IA case. Expressions for $W^*$ and $H^{*W}$ for WIA systems were given by (19) and (20) respectively. We have the following analog of IA theorems.

*Lemma 6.1:* Fix $x, z$ and assume

$$c^*(x) = \arg\min_c H^{*W}_{0,0,c}(x, x)$$

exists. Then

$$\inf_{a,b,c} H^{*W}_{a,b,c}(x, z) = \begin{cases} -\infty, & x \neq z \\ \min_c H^{*W} = WIAX(x), & z = x \end{cases}, \quad z \neq 0 \quad (45)$$

$$\inf_b H^{*W}_{0,b,0}(x, z) = \min_b H^{*W}_{0,b,0}(x, z) = WIAYI(x), \quad z = 0. \quad (46)$$

Here, we define

$$WIAX(x) = \|C_1(x, c^*(z))\|^2 + X(x)^T B_1(x) B_1(x)^T X(x) + X(x)^T AB(x, c^*(x)) + AB(x, c^*(x))^T X(x)$$

and

$$\begin{aligned} WIAYIx = {} & \left[AB(x,0)^T - C_2^T(x) e_2(x)^{-1} D_{21}(x) B_1(x)^T\right] \\ & \cdot Y_I(x) + Y_I(x)^T \\ & \cdot\left[AB(x,0) - B_1(x) D_{21}(x)^T e_2(x)^{-1} C_2(x)\right] \\ & + \|C_1(x,0)\|^2 - C_2(x)^T e_2(x)^{-1} C_2(x) \\ & + Y_I(x)^T\left[B_1(x) B_1(x)^T\right. \\ & \left. - B_1(x) D_{21}(x)^T e_2(x)^{-1} D_{21}(x) B_1(x)^T\right] Y_I(x) \end{aligned}$$

where $X(x)$ and $Y_I(x)$ were defined by (28) with $\varphi(x) = x$.

The minimizing $c$ when $x = z$ is defined implicitly by

$$2C_1(z, c^*(z))^T D_U C_1(z, c^*(z)) + \nabla_x \epsilon(z, z)^T D_U AB(z, c^*(z)) = 0. \quad (47)$$

The minimizing $b$ when $z = 0$ is the same as for IA systems and is given by (30).

*Proof:* Parallels the proof of Lemma 4.1. ■

*Theorem 6.2:* The obvious analog of Theorem 4.2.

*Theorem 6.3: Separation Principle for WIA Systems.* Assume that an energy function $\epsilon(x, z)$ satisfies the assumptions A1)–A4) and also $WIAX(x) = 0$ for all $x$ (where it is assumed that the critical $c^*$ exists for each $z$). Then $c(z)$ is given by the solution $c^*(z)$ to (47) and $a(z)$ is given by

$$\begin{aligned} a^*(z) &= F(z, W^*(z, z, b(z)), c^*(z)) \\ &\quad + b(z)[-G_2(z, W^*(z, z, b(z)))] \\ &= AB(z, c^*(z)) - b(z) C_2(z) \\ &\quad + (B_1(z) - b(z) D_{21}(z)) B_1(z)^T X(z). \quad (48) \end{aligned}$$

*Proof:* Identical to Theorem 4.4. ■

*Theorem 6.4:* Under the hypotheses of Theorem 6.3 a necessary condition for the system $a^*(z), b(z), c^*(z)$ with $a^*, c^*$ given by Theorem 6.3 to solve *$\epsilon$-DISFBK* is $k(x, z) \leq 0$ for all $x, z$ where $k$ is

$$\begin{aligned} k(x, z) := {} & AB(x, c^*(z))^T \nabla_x \epsilon(x, z) \\ & + AB(z, c^*(z))^T \nabla_z \epsilon(x, z) + \|C_1(x, c^*(z))^T\|^2 \\ & - \left[C_2(x) - C_2(z) + \tfrac{1}{2} D_{21}(x) B_1(x)^T \nabla_x \epsilon(x, z)\right. \\ & \left. - D_{21}(z) B_1(z)^T X(z)\right]^T e_2(x)^{-1} \\ & \cdot\left[C_2(x) - C_2(z) + \tfrac{1}{2} D_{21}(x) B_1(x)^T \nabla_x \epsilon(x, z)\right. \\ & \left. - D_{21}(z) B_1(z)^T X(z)\right] \\ & + \tfrac{1}{4} \nabla_x \epsilon(x, z)^T B_1(x) B_1(x)^T \nabla_x \epsilon(x, z) \\ & + \nabla_x \epsilon(x, z)^T B_1(z) B_1(z)^T X(z) \end{aligned}$$

which is independent of $b$.

*Proof:* The proof is identical to Theorem 4.5, except that there is no explicit form given for $c^*(z)$, and $a^*(z)$ is now given by (48). The expression for $\hat{b}$ is the same as given by (39) in the proof of Theorem 4.5. ■

## VII. Separation Principle

This section runs at at a higher level of generality than the previous sections; specifically, in this section we do not assume that our systems are affine in the inputs. Consider the general feedback system (1) and (2) defined in the introduction with $G_1$ independent of $W$ and $G_2$ independent of $U$. We want to find a feedback system $\dot{z} = f(z, Y), U = g(z)$ which makes the closed-loop system dissipative. Fix a candidate smooth storage function $\epsilon(x,z)$ and define

$$h_{f,U}(W, x, z) = \nabla_x \epsilon(x, z) \cdot F(x, W, U) + \nabla_z \epsilon(x, z) \cdot f(z, G_2(x, W)) + \|G_1(x, U)\|^2 - \|W\|^2.$$

Note that here $f$ is a function while $U$ is a variable so that, for example, $h_{f,g(z)}(W, x, z) = \mathscr{H}_{f,g}(W, x, z)$, which for IA systems is $H_{a(z), b(z), c(z), 0}(W, x, z)$.

*Theorem 7.1: Separation Principle.* Assume $W = W^*_{f,U}(x,z)$ solves

$$0 = D_W h_{f,U}(W,x,z).$$

We will abbreviate $W^*_{f,U}(x,z)$ to $W^*(x,z)$. Define

$$h^{*W}_{f,U}(x,z) = h_{f,U}(W^*(x,z),x,z).$$

Assume there exists a function $f^*(z,Y)$ and a twice continuously differentiable energy function $\epsilon(x,z)$ such that

1) $h^{*W}_{f^*,g^*(x)}(x,x) = \min_U h^{*W}_{f^*,U}(x,x)$ has a differentiable solution $g^*$.
2) $D_z h^{*W}_{f^*,U}(x,z)|_{z=x, U=g^*(x)} = 0.$
3) The function pair $(f^*,g^*)$ satisfies $h^{*W}_{f^*,g^*(z)}(x,z) \le 0$ for all $x, z$.
4) $\nabla_z \epsilon(x,z)|_{z=x} = 0$ as usual.
5) $D_z(\nabla_z \epsilon(x,z))|_{z=x}$ has full rank.

If the feedback $U$ is given by $g^*(z)$, then $f^*$ has the form

$$f^*(z,Y) = F(z,W^*(z,z),g^*(z)) + b(z,Y - G_2(z,W^*(z,z))) \quad (49)$$

for some function $b(z,Y)$ satisfying $b(z,0) = 0$.

*Corollary 7.2:* A condition which guarantees Theorem 7.1-2) when the other conditions are met is $h^{*W}_{f^*,g^*(z)}(z,z) = 0$ for all $z$. For IA systems this is equivalent to $IAX(x) = 0$ for all $x$.

*Remark 1:* Theorem 7.1 can be interpreted as a nonlinear extension of the so-called *separation principle* for the central solution of the linear $H^\infty$-control problem (see [15] and [31]). In this context, the separation principle amounts to the interpretation of the formula for the compensator $(f^*,g^*)$ as follows. If we assume that $g^*$ is independent of $Y$, the state feedback map $g^*$ is simply the solution of the state-feedback problem under the assumption that the compensator state $z$ is the same as the plant state $x$. The first term in the compensator dynamics $f^*$ is the same as the plant dynamics would be under the assumption that the compensator state $z$ is the same as the plant state $x$ and that the worst choice $W^*(z,z)$ of $W$ is fed in as input. The $b$ function term in $f^*$ is used to make adjustments for the fact that $z \ne x$. In the linear case, the dynamics $f^*$ is the solution of the observer-based state estimation problem with the appropriate choice of $b$; in the nonlinear case unlike the state feedback problem this latter problem has no simple solution. In this way, we see the output feedback problem as reducing to the solution of two separate less complicated problems, the state feedback problem and the state estimation problem. A problem having some elements in common with the $H^\infty$-problem is the so-called output regulation problem, where one seeks a feedback which guarantees that an error signal asymptotically approaches zero in the presence of a disturbance but where there is no quantitative measure of performance. A principle for this problem analogous to the separation principle for the *LQG* and $H^\infty$-problem, called the *internal model principle*, has been extended to a nonlinear setting in [30].

*Remark 2:* There are other interpretations of the separation principle which lead to different formulas even in the linear case. One was mentioned earlier and it yields the maximum entropy solution. It takes $IAX(x) = 0$ which makes the $D_z = 0$ condition 2) correspond to $x = z$ being a maximum of $H^{*W}_{a(z),b(z),c^*(z)}(x,z)$. However, it is intriguing to pursue the strategy of taking $z = x$ to be a minimum. That is, instead of picking $\epsilon$ to make $IAX$ as big as possible (least dissipative) subject to the constraint $IAX(x) \le 0$, pick $IAX$ to be very small. The objective is to make $H^{*W}$ very negative on the diagonal $z = x$, indeed to make it a minimum there. Thus, $D_z = 0$ there so condition 2) holds; this intuition forces the formula (49) for $f^*$ to hold.

*Proof of Theorem 7.1:* By 1) we have

$$\text{a)}\quad D_U h^{*W}_{f^*,U}(x,z)\Big|_{z=x, U=g^*(x)} = 0.$$

By the chain rule

$$D_z\{h^{*W}_{f^*,g(z)}(x,z)\} = D_U h^{*W}_{f^*,U}(x,z)|_{U=g(z)} D_z g(z) + D_z h^{*W}_{f^*,U}(x,z)|_{U=g(z)}.$$

Letting $z = x$ and $g(z) = g^*(z)$, by a) combined with 2) we have

$$\text{b)}\quad D_z\{h^{*W}_{f^*,g^*(z)}(x,z)\}|_{z=x} = 0.$$

Define $\hat{F}(x,z) = F(x,W^*(x,z),g^*(z)) - f^*(z,G_2(x,W^*(x,z)))$.

*Lemma 7.3:* $f^*$ has the form claimed in the separation principle iff $\hat{F}(z,z) = 0$.

*Proof:* Set $b(z,\hat{Y}) = f^*(z,\hat{Y} + G_2(z,W^*(z,z))) - F(z,W^*(z,z),g^*(z))$. Then we recover $f^*$ as

$$f^*(z,Y) = F(z,W^*(z,z),g^*(z)) + b(z,Y - G_2(z,W^*(z,z)))$$

and $b(z,0) = 0$ is equivalent to $\hat{F}(z,z) = 0$. ■

*Lemma 7.4:* $\hat{F}(z,z) = 0$ is equivalent to

$$D_z\{\nabla_z \epsilon(x,z) \cdot \hat{F}(x,z)\}|_{z=x} = 0. \quad (50)$$

*Proof:*

$$D_z\{\nabla_z \epsilon(x,z) \cdot \hat{F}(x,z)\}|_{z=x} = \hat{F}(x,x)^T D_z(\nabla_z \epsilon(x,z))|_{z=x} + \nabla_z \epsilon(x,x) \cdot D_z \hat{F}(x,z)|_{z=x}.$$

Assumption 4) gives the last term is 0, and then assumption 5) provides the equivalence. ■

*Conclusion of Proof:* From b) we have

$$D_z\{\nabla_x \epsilon(x,z) \cdot F(x,W^*(x,z),g^*(z)) + \nabla_z \epsilon(x,z) \cdot f^*(z,G_2(x,W^*(x,z))) + \|G_1(x,g^*(z))\|^2 - \|W^*(x,z)\|^2\}|_{z=x} = 0$$

while (50) is equivalent to

$$D_z\{\nabla_z \epsilon(x,z) \cdot [F(x, W^*(x,z), g^*(z)) - f^*(z, G_2(x, W^*(x,z)))]\}|_{z=x} = 0.$$

When these expressions are added, the terms involving $f^*$ cancel and we are left with

$$D_z\{(\nabla_x \epsilon(x,z) + \nabla_z \epsilon(x,z)) \cdot F(x, W^*(x,z), g^*(z)) + \|G_1(x, g^*(z))\|^2 - \|W^*(x,z)\|^2\}|_{z=x} = 0. \quad (51)$$

Hence, by Lemma 7.4, the Separation Principle follows once we show (51). By the product rule, the left-hand side of (51) is equal to

$$F(x, W^*(x,x), g^*(x))^T D_z\{\nabla_x \epsilon(x,z) + \nabla_z \epsilon(x,z)\}|_{z=x} + \{\nabla_x \epsilon(x,x) + \nabla_z \epsilon(x,x)\} \cdot D_z\{F(x, W^*(x,z), g^*(z))\}|_{z=x} + D_z\{\|G_1(x, g^*(z))\|^2\}|_{z=x} - D_z\{\|W^*(x,z)\|^2\}|_{z=x}.$$

Since $\epsilon$ is twice continuously differentiable, $D_z\{\nabla_x \epsilon(x,z) + \nabla_z \epsilon(x,z)\}|_{z=x} = [\{D_x + D_z\}\nabla_z \epsilon(x,z)|_{z=x}]^T = 0$. Since the function $\nabla_z \epsilon(x,z)$ is identically zero on the set $z = x$ [assumption 4], the first term above vanishes.

Again using assumption 4), the remaining terms are:

$$\left\{\left[\nabla_x \epsilon(x,x) \cdot D_W F(x, W^*(x,z), g^*(z)) - 2W^*(x,z)^T\right] D_z W^*(x,z)\right\}\Big|_{z=x} + \left\{\left[\nabla_x \epsilon(x,x) \cdot D_U F(x, W^*(x,z), U) + 2G_1(x,U)^T D_U G_1(x,U)\right] D_z g^*(z)\right\}\Big|_{\substack{z=x \\ U=g^*(x)}}.$$

The first term vanishes by the defining equation for $W^*$. The second term vanishes by the defining property 1) for $g^*$. ■

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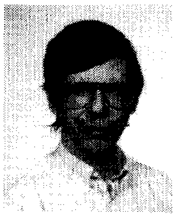
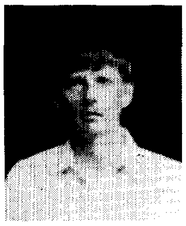


**Joseph A. Ball** (M'90) was born in Washington, DC, on June 4, 1947. He received the B.S. degree in mathematics from Georgetown University, Washington, DC, in 1969, and the M.S. and Ph.D. degrees in mathematics from the University of Virginia, Charlottesville, in 1970 and 1973, respectively.

He joined the faculty at Virginia Polytechnic Institute and State University, Blacksburg, in 1973, where he is currently Professor of Mathematics. Short term visits elsewhere include the University of California at San Diego, the Weizmann Institute of Science in Israel, the University of Maryland in College Park, and the Vrije Universiteit in Amsterdam. His general area of interest is operator and system theory, particularly the theory of interpolation and factorization for matrix valued functions with applications to control problems and extensions of this theory to nonlinear settings.

Dr. Ball is an Editor of *Integral Equations and Operator Theory* and of *Systems and Control Letters*.



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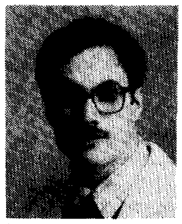
Dr. Helton was a recipient of a Guggenhiem Fellowship and the Outstanding Paper Award from the TAC in 1986. He has delivered pleniary addresses at the annual meeting of the AMS, at the EECTD, at several of the International Symposia on the Mathematical Theory of Networks and Systems, as well as at other large meetings. His articles concern circuit theory, distributed systems, and aspects of the theory of operators on Hilbert space which come from circuits, systems, differential and integral equations. Also, recent work focuses on a systematic methodology for frequency domain design most akin to a branch of several complex variables, computer algebra for linear systems, and nonlinear control.



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