ALTERNATIVE APPROACHES IN FREQUENCY DOMAIN
DESIGN OF SINGLE LOOP FEEDBACK SYSTEMS WITH
PLANT UNCERTAINTY*

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SUMMARY

The frequency domain design of robust feedback control systems involves the fitting of a designable complex function of frequency to specification/uncertainty derived constraints. At present there are two basically different approaches to these problems: the gain versus frequency approaches and the gain-phase versus frequency approaches. Unfortunately, the commonalities, advantages and disadvantages of these approaches are not widely appreciated. In this paper we develop a common framework for examining these alternatives and use this framework to reveal some of their similarities, strengths and weaknesses.

1. INTRODUCTION

Some of the most effective techniques is the design of robust controllers for SISO plants are based on frequency domain design. The early work in this area was based on the loop gain shaping (LGS) ideas of Bode.¹ Later work based on the ideas of loop gain-phase shaping (LGPS) was published by Horowitz² (and then incorporated into his QFT approach³). Recent workers have developed more rigorous mathematical versions of both LGS and LGPS based on $H_\infty$ concepts. The work of Doyle, Francis and Tannenbaum⁴ describes an $H_\infty$ based approach to LGS while Helton and Merino⁵ describe an $H_\infty$ based approach to LGPS. Unfortunately, the similarities and differences between these several approaches are not well understood.

In this paper we develop a general description of the robust control system design problem using the $H_\infty$ optimization framework described by Helton and Merino.⁵ In this first section we introduce the robust control problem and then relate typical design specifications to a mixed sensitivity performance function $\hat{\Gamma}$. In Section 2 we introduce plant uncertainty, define the worst case performance function $\Gamma$ as the supremum of $\hat{\Gamma}$ over plant uncertainty and then formulate the robust control problem as an optimization of the worst case performance function $\Gamma$. In Sections

This paper was recommended for publication by editor M. J. Grimble

* An earlier version of this paper was presented at the American Control Conference, June 1994.
3 and 4 we use this formulation to compare the several alternative approaches to robust control and describe some strengths and weaknesses of each approach.

The basic design problem

We will assume that the plant \( P(s) \) is SISO and that the feedback system has the single loop two degree-of-freedom (TDF) structure shown in Figure 1. (Here \( r \) is the reference, \( d \) is a disturbance and \( n \) represents sensor noise.) It is easily shown that a TDF structure is the most general feedback structure for the case of linear control of SISO plants. Other TDF structures can be considered but all are equivalent from the viewpoint of performance.\(^2\) Note that while there may be multiple input and outputs in this feedback system, a scalar transfer function approach is maintained.

The following transfer functions arising in this problem will be used throughout:

<table>
<thead>
<tr>
<th>Transfer Function</th>
<th>( L(s) = G(s)P(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity:</td>
<td>( S(s) = [1 + L(s)]^{-1} )</td>
</tr>
<tr>
<td>Complementary sensitivity:</td>
<td>( T(s) = 1 - S(s) )</td>
</tr>
<tr>
<td>Tracking transfer function:</td>
<td>( T_t(s) = F(s)L(s)S(s) )</td>
</tr>
</tbody>
</table>

The output \( y(s) \) in Figure 1 can be expressed as

\[
y(s) = T(s)[F(s)r(s) - n(s)] + S(s)d(s)
\]

while the plant input \( u(s) \) can be expressed as

\[
u(s) = \frac{1}{P(s)} T(s)[F(s)r(s) - d(s) - n(s)]
\]

These relations suggest that the specifications on performance and relative stability of the entire closed loop system can be reduced to specifications on the two transfer functions \( T(s) \) and \( S(s) \).

Specifications and performance functions

The range of specifications encountered in a control system design problem can be quite broad. First it will be required that the feedback loop be stable. Some important additional specifications are:

1. Specifications on the closed loop step response.
2. Specifications on allowable variation in magnitude of the closed loop transfer function.
3. Specifications on allowable variation in phase of the closed loop transfer function.
4. Specifications on open loop gain and bandwidth.
5. Specifications on relative stability of the feedback loop.
6. Specifications on closed loop frequency response (gain and phase).
7. Specifications on closed loop disturbance response.
8. Specifications on the plant input transfer function (the transfer function from \( r \) to \( u \)).
9. Specifications on closed loop response to sensor noise.
10. Specifications on closed loop bandwidth.

There are approximative techniques (e.g., see Reference 6) for translating 1 into 2 so it will not be considered separately. Specifications 2 and 3 appear in designs of certain high performance...
tracking systems. Specifications 5 to 10 can be expressed (possibly with some approximation) as frequency dependent constraints on the magnitude of the closed loop sensitivity and/or complementary sensitivity defined in (1).

The above comments suggest that the specifications fall into two classes: those that can be translated into disk constraints on \(|S(j\omega)|\) and/or \(|T(j\omega)|\) and those that cannot. (As we will see below, this separation also applies to different approaches to robust design discussed in Section 4.) To maintain a common context for the discussion to follow we will focus on those specifications which can be reduced to constraints on \(|S(j\omega)|\) and/or \(|T(j\omega)|\) and thus can be expressed in terms of a frequency dependent, mixed sensitivity performance function of the form

\[
\hat{\Gamma}(\omega) = \hat{W}_1(\omega)|T(\omega) - G(j\omega)|^2 + \hat{W}_2(\omega)|S(\omega) - G(j\omega)|^2
\]

where \(\hat{W}_1(\omega), \hat{W}_2(\omega) \geq 0\) are real valued weight functions.

The designable parameters in Figure 1 are the transfer functions \(F(s)\) and \(G(s)\). However, \(F(s)\) is used mainly to adjust the nominal value of the tracking transfer function \(T_1(s)\) after \(G(s)\) is selected to meet the remaining specifications. The discussion here focuses on the choice of \(G(s)\), or more precisely, its frequency domain representation \(G(j\omega)\). Thus, in terms of the designable parameter \(G(j\omega)\) the performance function (2) becomes

\[
\hat{\Gamma}_G(\omega, G(j\omega); P(j\omega)) = \hat{W}_1(\omega)|T(\omega, G(j\omega), P(j\omega))|^2 + \hat{W}_2(\omega)|S(\omega, G(j\omega), P(j\omega))|^2
\]

As noted above, the control system design problem involves choosing \(G(j\omega)\) so that \(\hat{\Gamma}_G\) satisfies frequency dependent disk constraints. That is, given the specifications-determined, real valued functions \(R(\omega), \hat{W}_1(\omega), \hat{W}_2(\omega) \geq 0\) for all \(\omega\),

\[
\text{find } G(j\omega) \text{ such that } \hat{\Gamma}_G(\omega, G(j\omega); P(j\omega)) < R(\omega) \quad \forall \omega
\]

For further simplification the weights \(\hat{W}_1(\omega), \hat{W}_2(\omega)\) are normalized by \(R(\omega)\) and we obtain the basic design problem,

\[
\text{(OPT}_G) \quad \text{find } G(j\omega) \text{ such that } \hat{\Gamma}_G(\omega, G(j\omega); P(j\omega)) < 1 \quad \forall \omega
\]

where

\[
\hat{\Gamma}_G(\omega, G(j\omega); P(j\omega)) = W_1(\omega)|T(\omega, G(j\omega), P(j\omega))|^2 + W_2(\omega)|S(\omega, G(j\omega), P(j\omega))|^2
\]

with \(W_1 = \hat{W}_1/R\) and \(W_2 = \hat{W}_2/R\). Note that after some algebraic manipulation (4) can be rewritten as

\[
\hat{\Gamma}_G(\omega, G(j\omega); P(j\omega)) = \left(\frac{W_2(\omega)}{w(\omega)}\right) \left| \frac{G(j\omega)P(j\omega)}{1 + G(j\omega)P(j\omega)} - w(\omega) \right|^2 + W_1(\omega)w(\omega)
\]
where
\[ w(j\omega) = \frac{W_2(j\omega)}{W_1(j\omega) + W_2(j\omega)} \]

The plant model \( P(j\omega) \) is shown explicitly in (3), (4) and (5) to allow for the fact that there may be plant uncertainty. The question of how to handle the plant model \( P(j\omega) \) in solving \( \text{OPT}_G \) is addressed below.

**Uncertainty**

In the control system design problem described above there are two potential types of uncertainty; plant uncertainty and environmental uncertainty. The first is uncertainty in the actual plant model and/or its parameters while the second represents uncertainty in the disturbances and/or sensor noise. It will be assumed that environmental uncertainty can be handled through specifications on noise or disturbance response that are reflected in the weight functions \( W_1(j\omega) \) and \( W_2(j\omega) \). Thus the focus here is on plant uncertainty and the design of control systems to obtain desired performance in the presence of plant uncertainty.

### 2. ABSTRACT FORMULATION OF THE DESIGN PROBLEM

The problem of interest here is the problem of control system design in the presence of plant uncertainty. However, it is instructive to first consider the case of no uncertainty.

**The case of no plant uncertainty**

In the case where there is no plant uncertainty we assume that the plant model \( P(s) \) is fixed and represented by the known nominal transfer function \( P_o(s) \). Thus \( P(j\omega) \) can be eliminated from the problem description in (3), (4) and (5) and the basic design problem becomes,

\[ \text{(COPT}_G \text{)} \quad \text{find } G(j\omega) \text{ such that } \Gamma_G(\omega, G(j\omega)) = \tilde{\Gamma}_G(\omega, G(j\omega), P_o(j\omega)) < 1 \quad \forall \omega \]  \hspace{1cm} (6)

Conceptually, problem \( \text{COPT}_G \) is solved by finding a \( G^*(j\omega) \) that minimizes \( \Gamma_G(\omega, G(j\omega)) \) using numerical optimization and then checking to see if the constraint in (6) is satisfied. With an appropriate reparameterization (see below) \( \text{COPT}_G \) can be solved using \( H_\infty \) optimization tools (e.g., see Reference 5).

In the process of solving \( \text{COPT}_G \) it is helpful to examine (fixed frequency) level sets such as

\[ S_{G\omega}(\omega) = \{ G(j\omega) | \Gamma_G(\omega, G(j\omega)) < 1 \} \]  \hspace{1cm} (7)

When \( P(j\omega) = P_o(j\omega) \) is fixed in (5), the expression (7) describes level sets of the magnitude of a linear fractional transformation in \( G(j\omega) \). Thus in this case of no uncertainty the level sets \( S_{G\omega} \) are disks in the complex plane.

**The case of plant uncertainty**

Here we assume that the plant model \( P(s) \) is one member of a family \( \mathcal{P} \) of possible plants. In this case the design problem is more complex as \( \tilde{\Gamma}_G \) now depends on both \( G(j\omega) \) and \( P(j\omega) \). The basic design problem \( \text{OPT}_G \) now becomes,

\[ \text{(UNCOPT}_G \text{)} \quad \text{find } G(j\omega) \text{ such that } \tilde{\Gamma}_G(\omega, G(j\omega), P(j\omega)) < 1 \quad \forall \omega \text{ and } \forall P \in \mathcal{P} \]
The problem UNCOPT\(_G\) can be easily shown to be equivalent to a modified problem obtained by defining the worst case performance function,

\[
\Gamma_G(\omega, G(j\omega)) = \sup_{P \in \mathcal{P}} \hat{\Gamma}_G(\omega, G(j\omega); P(j\omega))
\]  

(UNC\(_G\))

and then solving the worst case design problem,

\[
\text{find } G(j\omega) \text{ such that } \Gamma_G(\omega, G(j\omega)) < 1 \quad \forall \omega
\]  

(WCOPT\(_G\))

This approach appears to reduce the problem with uncertainty to a form similar to COPT\(_G\). However, the sup in (8) destroys the simple linear fractional transformation structure found in \(\hat{\Gamma}_G\). Thus the level sets for the worst case problem are not necessarily disks and the optimization problem required to solve (9) may no longer be convex. In addition, the computation of the worst case performance function \(\Gamma_G(\omega, G(j\omega))\) may itself be quite difficult. Clearly WCOPT\(_G\) is considerably more difficult than COPT\(_G\).

**Alternative parametrizations of OPT\(_G\)**

While the above formulation of the design problem is conceptually clear, there are equivalent formulations that are more convenient for its solution. To obtain some of these, note that for a given \(P = P_o\), the relation

\[
S_o = 1 - T_o = (1 + L_o)^{-1} = (1 + GP_o)^{-1}
\]  

(10)

offers several possible reparametrizations of OPT\(_G\). That is, for a given \(P_o\) either the designable parameter or the performance function (or both) in OPT\(_G\) can be given alternative representations in terms of \(S_o\) or \(T_o\) using (10). Here we use the relation

\[
G = \frac{1}{P_o} \left( \frac{T_o}{1 - T_o} \right)
\]  

(11)

to obtain a representation of the problem with the nominal closed loop transfer function \(T_o\) as the designable parameter. This yields a reparametrization of the problem UNCOPT\(_G\) to the form,

\[
\text{find } T_o(j\omega) \text{ such that } \hat{\Gamma}_T(\omega, T_o(j\omega); P(j\omega)) < 1 \quad \forall \omega \text{ and } \forall P \in \mathcal{P}
\]  

(UNCOPT\(_T\))

and the associated worst case problem,

\[
\text{find } T_o(j\omega) \text{ such that } \Gamma_T(\omega, T_o(j\omega)) < 1 \quad \forall \omega
\]  

(WCOPT\(_T\))

where the worst case performance function is,

\[
\Gamma_T(\omega, T_o(j\omega)) = \sup_{P \in \mathcal{P}} \hat{\Gamma}_T(\omega, T_o(j\omega); P(j\omega))
\]  

(UNC\(_T\))

A second reparametrization of OPT\(_G\), using the relation

\[
G = \frac{1}{P_o} L_o
\]  

(13)

yields a version of the problem UNCOPT\(_G\) with the nominal open loop transfer function \(L_o\) as the designable parameter. The resulting form is,

\[
\text{find } L_o(j\omega) \text{ such that } \hat{\Gamma}_L(\omega, L_o(j\omega); P(j\omega)) < 1 \quad \forall \omega \text{ and } \forall P \in \mathcal{P}
\]  

(UNCOPT\(_L\))
The associated worst case problem,
\[ \text{(WCOPT}_L \text{)} \quad \text{find } L_o(j\omega) \text{ such that } \Gamma_L(\omega, L_o(j\omega)) < 1 \quad \forall \omega \]
where the worst case performance function is,
\[ \text{(UNC}_L \text{)} \quad \Gamma_T(\omega, L_o(j\omega)) = \sup_{P \in \mathcal{P}} \Gamma_L(\omega, L_o(j\omega); P(j\omega)) \quad (14) \]

**Plant uncertainty models**

In any plant model there are two basic forms of uncertainty: uncertainty in parameters included in the model structure and additional plant dynamics not included in the model structure. Examples of the former are uncertainties in the masses, spring constants, gains, time constants, etc. included in the plant model. These parameters may be uncertain due to manufacturing tolerances, aging, etc. Examples of the latter are resonances or high frequency dynamics intentionally or unintentionally omitted from the model. The main difference between these two types of uncertainty is the structural information available. In the first case, the uncertainty is parametric (sometimes termed structured). That is, it can be represented by uncertainty in the parameters of the chosen plant model. In the second case, the uncertainty is non-parametric (sometimes termed unstructured) and can be represented by uncertainty in the structure of the plant model or some representation of this uncertainty in terms of, say, uncertainty in the frequency response. In practice both forms of uncertainty generally exist but at any given frequency one may dominate so that the other can be ignored. When both are considered the problem is said to have a mixed uncertainty.

For further study of the control system design problem we need explicit models of the plant uncertainty. Here we will assume that the plant can be represented as
\[ P(s) = P_o(s)Q(s) \text{ with } Q(s) \in \mathcal{Q} \quad (15) \]
where \( P_o(s) \) is the nominal plant and \( \mathcal{Q} \) is a bounded set of possible plant variations from the nominal due to one or both types of plant uncertainty (i.e., parametric or non-parametric). Thus, given a nominal plant \( P_o(s) \), the plant family \( \mathcal{P} \) mentioned in Section 2 is the family
\[ \mathcal{P} = \{ P(s) | P(s) = P_o(s)Q(s) \quad \forall Q(s) \in \mathcal{Q} \} \]

The plant variations described by \( \mathcal{Q} \) can be represented in several different forms. One common but rather limited form is the non-parametric or unstructured plant uncertainty model popularized in Reference 7 where \( \mathcal{Q} \) is the set of transfer functions of the form \( [1 + \Delta_m(s)] \) and \( \Delta_m(s) \) is any stable transfer function with \(|\Delta_m(j\omega)| < r(j\omega)\). More generally \( \mathcal{Q} \) is a generalization of the plant uncertainty template introduced by Horowitz. That is, \( \mathcal{Q} \) is an explicit representation of all uncertainty induced deviations of \( P(s) \) from the nominal plant model \( P_o(s) \). The key difference here is that the uncertainty can have structure (e.g., parametric structure) that is explicitly represented in \( \mathcal{Q} \). We assume that at each frequency \( \omega \) the set \( \mathcal{Q}(\omega) \) is a compact subset of the complex plane \( \mathbb{C} \) containing the point \( 1 + j0 \). This subset of \( \mathbb{C} \) will be called the plant uncertainty template at frequency \( \omega \).

### 3. ALTERNATIVE SOLUTION STRATEGIES

Here we use the \( H_\infty \) optimization framework developed in Section 2 to describe several solution strategies that have been used for solving the robust control system design problem. While the use
of the $H_{\infty}$ optimization framework may make some familiar approaches seem strange or slightly awkward, the application of one common framework to the analysis of the several design approaches helps to reveal their similarities and differences and facilitates the comparison that follows in Section 4.

Open loop gain shaping

Perhaps the first approach to the robust design problem is the OLGS (open loop gain shaping) technique due to Bode\textsuperscript{1} and more recently popularized in Reference 7. In the context outlined above this approach can be described as follows.

1. The performance function used is (4) with $W_1 = 0$ and an assumed nominal plant model $P_0$. That is

$$\Gamma_G(\omega, G(j\omega)) = \bar{G}_G(\omega, G(j\omega), P_0(j\omega)) = W_2(j\omega)|S(j\omega; G(j\omega), P_0(j\omega))|^2$$

(16)

The uncertainty model is the non-parametric model described in Section 3. It uses only the magnitude of the plant uncertainty template. That is, only $r(j\omega)$ is given for all $\omega$ and

$$|Q(j\omega)| = |1 + \Delta_m(j\omega)| \text{ with } |\Delta_m(j\omega)| \leq r(j\omega)$$

(17)

2. Given specifications on performance and the non-parametric plant uncertainty model (17), calculate $\chi_2(j\omega)$ by an appropriate procedure (see comments below).

3. Reparametrize (16) using (13) to obtain the performance function

$$\Gamma_L(\omega, L_o(j\omega)) = W_2(j\omega)|S(j\omega; L_o(j\omega), P_0(j\omega))|^2 = W_2(j\omega)|1 + L_o(j\omega)|^2$$

(18)

4. Using the approximation $|1 + L_o| \approx |L_o|$ when $|L_o| \gg 1$ write the design constraint as

$$\Gamma_L(\omega, L_o(j\omega)) \approx W_2(j\omega)|L_o(j\omega)|^{-2} < 1 \quad \forall \omega$$

or

$$|L_o(j\omega)| > \sqrt{W_2(j\omega)} \quad \forall \omega$$

(19)

5. Add an appropriate stability condition.

6. Choose a rational function $L_o(s)$ such that $T(s)$ is stable, (19) is satisfied and $L_o(j\omega)$ has minimum bandwidth.

7. Find $G(s)$ using (13). That is

$$G(s) = \frac{L_o(s)}{P_o(s)}$$

(20)

Comments

(1) The basic steps in this approach were outlined in Reference 1. In the classic Bode approach sensitivity arguments were used to determine $W_2(j\omega)$ in (18) as

$$W_2^{-1}(j\omega) = \left| \frac{T(j\omega) - T_o(j\omega)}{T_o(j\omega) P(j\omega) - P_o(j\omega)} \right|^2 \left| \frac{\Delta T(j\omega)}{T_o(j\omega)} \right| = \left| \frac{\Delta T(j\omega)}{\Delta_m(j\omega)} \right|^2$$

(21)
where $|\Delta T|$, the allowable variation in $|T|$ around the nominal $|T_o|$ was specified. This approach does not guarantee robust stability or performance and works well only when $|\Delta P/P_o|$ and $|\Delta T/T_o|$ are small.

(2) A separate stability test is required because the designer is working only with the magnitude of $L_o(j\omega)$ in Step 6.

(3) The control of loop bandwidth is left to the designer in Step 6.

(4) Modifications in Reference 7 added a robust stability test based on the small gain theorem which required that

$$ \left| \frac{L_o(j\omega)}{1 + L_o(j\omega)} \right| < |\Delta m(j\omega)|^{-1} \quad \forall \omega \geq 0 $$

This was used to introduce a second bound on $|L_o|$ in Step 4 (normally used as a lower bound applicable when $|L_o| \ll 1$) and an additional test in Step 5.

(5) Further modifications described in Reference 4 obtain bounds on $|L_o|$ that guaranteed both robust performance and robust stability or relative stability.

(6) In all of this OLGS work the non-parametric uncertainty model based only on $|Q(j\omega) - 1|$ is used.

Open loop gain-phase shaping

The open loop gain-phase shaping (OLGPS) approach was first proposed by Horowitz as an improved approach to robust (then called insensitive) control.² It has more recently been presented under the rubric of QFT (e.g., see Reference 3). In the context developed here this approach can be described as follows.

1. Commonly the basic performance function is (5) with $W_1 = 0$ but plant uncertainty is explicitly considered. That is,

$$ \tilde{\Gamma}_L(\omega, L_o(j\omega); P_o(j\omega)) = W_2(j\omega)|S(j\omega; L_o(j\omega), P_o(j\omega))|^2 $$

Other performance functions can be used (see comments below). The uncertainty model is the complete plant uncertainty template $\mathcal{P}$.

2. Given allowable specifications on performance and relative stability, calculate $W_2(j\omega)$ by some technique such as direct specification of bounds on $|S(j\omega)|^2$.

3. Given $P_o(j\omega)$; parameterize (23) using (13) to get

$$ \tilde{\Gamma}_L(\omega, L_o(j\omega); P_o(j\omega)) = W_2(j\omega)|S(j\omega; L_o(j\omega), P_o(j\omega))|^2 $$

$$ = W_2(j\omega)|S(j\omega; L_o(j\omega), P_o(j\omega)Q(j\omega))|^2 $$

4. Calculate the worst case performance function

$$ \Gamma_L(\omega, L_o(j\omega)) = \max_{Q \in \mathcal{P}} \tilde{\Gamma}_L(\omega; L_o(j\omega), P_o(j\omega)Q(j\omega)) $$

5. Choose a minimum bandwidth nominal loop transfer function $L_o(s)$ such that

$$ \Gamma_L(\omega, L_o(j\omega)) < 1 \quad \forall \omega $$

6. Find $G(s)$ using (20).
**Comments**

1. Here we are working with both gain and phase of \( L_o(j\omega) \) in (25) so both robust relative stability and robust performance are obtained simultaneously; no external stability test is required.

2. The difficult steps in OLGPS are the calculation of the worst case performance function (Step 4) and the selection of an appropriate nominal loop transfer function (Step 5). To date the only work reported on the OLGPS problem is in the QFT literature which offers a clever solution to Step 4 but only limited help with Step 5.

3. The QFT procedure for Step 4 proceeds as follows.\(^*\) (The reader should note that the terminology and notation used in the QFT literature differs significantly from that used here.) First, one chooses a finite set \( \Omega \) of design frequencies where

\[ \Omega = \{ \omega_i \text{ for } i = 1, \ldots, w \} \]

Then one computes *nominal* level sets (note that these are compliments of the level sets defined in (7))

\[ \mathcal{F}_{L_o}(\omega_i) = \{ L(j\omega) | \tilde{G}_L(\omega_i, L(j\omega); P_o(j\omega)) > 1 \} \quad \forall \omega_i \in \Omega \] (26)

using the *nominal* performance function \( \tilde{G}_L(\omega_i, L(j\omega); P_o(j\omega)) \) (i.e., with no uncertainty so \( Q = 1 \)). As argued in Section 2, when the nominal performance function is mixed sensitivity these nominal level sets are disks in the complex \( z \)-plane (and \( M \)-contours in the \( \mathbb{N} \)-plane).

In QFT these nominal level sets are commonly displayed in the \( \mathbb{N} \)-plane: the complex plane of \( \log(z) = \log|z| + j\arg\{z\} \) (with the imaginary axis traditionally displayed horizontally). Using the fact that

\[ \log L = \log[GP_o] = \log[GP_o] + \log Q = \log L_o + \log Q \]

one notes that for each frequency \( \omega_i \) and for a specific uncertainty realization \( Q(j\omega_i) \in \mathcal{Q} \)

\[ \log L(j\omega_i) \in \log \mathcal{F}_{L_o}(\omega_i) \iff \log L(j\omega_i) \in \log \mathcal{F}_{L_o}(\omega_i) - \log Q(j\omega_i) \]

where \( \log \mathcal{F}_{L_o}(\omega_i) \) indicates the \( \mathbb{N} \)-plane image of the nominal level set \( \mathcal{F}_{L_o}(\omega_i) \) defined in (26). Given the set of uncertainty models \( \mathcal{Q} \), one uses the \( \mathbb{N} \)-plane image of this set at frequency \( \omega_i \), here denoted \( \{ \log \mathcal{Q}(\omega_i) \} \), to define the \( \mathbb{N} \)-plane worst case level sets

\[ B_L(\omega_i; \mathcal{Q}) = \log \mathcal{F}_{L_o}(\omega_i) - \{ \log \mathcal{Q}(\omega_i) \} \] (27)

by adding every point in the set \( \{ \log \mathcal{Q}(\omega_i) \} \) to every point in the set \( \log \mathcal{F}_{L_o}(\omega_i) \). The QFT version of Step 5 is then, (QFT) find a minimum bandwidth \( L_o(j\omega) \) such that \( L_o(j\omega) \notin B_L(\omega_i; \mathcal{Q}) \quad \forall \omega_i \in \Omega \) (28)

Careful analysis of the above procedure shows that (28) is equivalent to the condition that

\[ \Gamma_L(\omega_i, L_o(j\omega)) < 1 \quad \forall \omega_i \in \Omega \]

In QFT it is assumed that the choice of the design frequency set \( \Omega \) is such that an \( L_o(j\omega) \) satisfying (28) also satisfies the stronger condition (25) of Step 5.

\(^*\) In early versions of OLGPS,\(^2\) Step 4 was accomplished by a rather tedious graphical procedure in the complex plane. This step was significantly simplified in Reference 8 by working in the log complex plane—the Nichols plane or \( \mathbb{N} \)-plane.
(4) In QFT terminology, the set \( \{ \log \theta(\omega_i) \} \) is called a template at frequency \( \omega_i \) and the boundaries of the sets \( B_L(\omega_i; \theta) \) are called boundaries at frequency \( \omega_i \). Thus (28) states that at each \( \omega_i \in \Omega \) the nominal loop frequency function \( L_o(j\omega) \) is outside of and should not cross these boundaries. (With reasonable assumptions the boundaries can be shown to be closed and include the point \(-1 + j0\) in the \( \mathbb{C} \)-plane or the point \((0, -180^\circ)\) in the \( \mathbb{N} \)-plane. For details see Reference 9.)

(5) The first difficult step in the QFT approach to OLGPS is the QFT equivalent of Step 4: computation of the boundaries \( B_L(\omega_i; \theta) \) for given templates \( \theta(\omega_i) \) and given performance specifications. This step is traditionally accomplished in a graphical or semi-graphical procedure. For example, in many cases the template is approximately polyhedral and one can readily locate the \( \mathbb{N} \)-plane images of the disks (i.e., \( M \)-contours) corresponding to the vertices of this polyhedron and then use the shape of the template to achieve an approximate interpolation between these disk images. Software packages for generating templates and boundaries are described in Reference 10.

(6) The remaining difficult step in QFT, finding \( L_o(j\omega) \) in (28), is more-or-less equivalent to Step 5, in the original OLGPS procedure. This requires the fitting of a frequency function to the three dimensional surface \( B_L(\omega_i; \theta) \) in the space \( \mathbb{C} \times \mathbb{R}^+ \) of gain-phase versus frequency. To avoid this complexity QFT techniques plot level curves of this surface (i.e., \( M \)-contours) in the \( \mathbb{N} \)-plane. To date, the best fitting results are obtained using trial and error modifications of \( P_o(j\omega) \) with simple first or second order compensator terms. Some limited success with numerical solution of this process has been reported\(^{11,12}\). Since a solution may not always exist and available existence conditions are limited (e.g., see Reference 9) this step can become quite difficult.

(7) In traditional QFT the control of the bandwidth of \( L_o(j\omega) \) is left to the designer in Step 5.

(8) The OLGPS procedure can be easily extended to handle the case of mixed sensitivity in Step 1. In addition, more general specifications such as Type 2, 3 or 4 (see list in Section 1 above) are also easily included in the QFT formulation as there is no requirement that the nominal level sets in (26) be disks. For details see References 8 or 9. However, such modifications can further complicate the fitting procedure is Step 5.

Closed loop gain-phase shaping

The closed loop gain-phase shaping (CLGPS) approach, one form of what is commonly called \( H_\infty \) control, has been popularized by Helton and Merino\(^5\). This approach can be described as follows.

1. The basic performance function is mixed sensitivity. That is

\[
\hat{\Gamma}_g(\omega, G(j\omega); P(j\omega)) = W_1(j\omega) | T(j\omega; G(j\omega), P(j\omega)) |^2 + W_2(j\omega) | S(j\omega; G(j\omega), P(j\omega)) |^2
\]  
(29)

The uncertainty model is again the complete template \( \theta \).

2. Given allowable specifications on performance and relative stability, calculate \( W_1(j\omega) \) and \( W_2(j\omega) \) by some technique (Some suggestions are given in Reference 13).

3. Given \( P_o(j\omega) \), reparametrize (29) using (11) to get

\[
\hat{\Gamma}_T(\omega, T_o(j\omega); P_o(j\omega) Q(j\omega)) = W_1(j\omega) | T(j\omega; T_o(j\omega), P_o(j\omega) Q(j\omega)) |^2 + W_2(j\omega) | S(j\omega; T_o(j\omega), P_o(j\omega) Q(j\omega)) |^2
\]  
(30)
4. Calculate the worst case performance function

$$\Gamma_T(\omega, T_o(j\omega)) = \max_{Q \in \mathbb{Z}} \tilde{\Gamma}_T(\omega; T_o(j\omega), P_o(j\omega)Q(j\omega))$$  (31)

5. Find a closed loop transfer function $T_o(s)$ such that

$$\Gamma_T(\omega, T_o(j\omega)) < 1 \quad \forall \omega$$  (32)

6. Find $G(s)$ using (11). That is

$$G(s) = \frac{1}{P_o(s)} \left( \frac{T_o(s)}{1 - T_o(s)} \right)$$  (33)

Comments

(1) Although they were developed independently, CLGPS directly parallels OLGPS with major difference lying in the choice of parametrization: the use of $T_o$ rather than $L_o$ as the designable parameter. However, this similarity is hidden by the formalism of QFT.

(2) The main advantage of CLGPS lies in the fact that owing to stability requirements $T_o$ is guaranteed to be analytic in the right half plane and thus $H_o$ optimization techniques can be applied in Step 5. In this process, interpolation conditions can also be added to guarantee internal stability of the resulting design when the plant has poles and/or zeros in the RHP.

(3) The difficult steps in CLGPS are again Steps 4 and 5. Conceptually, Step 5 again requires the fitting of a rational function to a three-dimensional surface in the space $\mathbb{C} \times \mathbb{R}^+$ of gain-phase versus frequency. Algorithms for fitting a function of frequency to this surface are available (e.g., see Reference 5) but convergence problems may be encountered when the level sets of $\Gamma_T$ are not convex. Moreover, algorithms by Helton and Merino work with what basically are polynomials of large degree (after a change of co-ordinates), so a rational function approximation of the result is required to complete Step 5. The procedure in Step 4 also remains difficult. Results for some special cases of parametric uncertainty are reported in Reference 14.

4. COMPARISON OF THE THREE ALTERNATIVES

The above discussion reveals that in all of the approaches considered there are three common steps: (1) representation of plant uncertainty, (2) formation of design constraints and (3) fitting of a rational function to these design constraints. Since these steps are common but each alternative takes a different approach to one or more of these three steps, the three alternative approaches can be compared on this basis.

Uncertainty representation

Here the three approaches can be compared in terms of the shape of the plant uncertainty templates used. In OLGS the templates are not used explicitly. However, the use of only magnitude information in the uncertainty representation in Step 1 (and the use of the small again theorem in the more recent versions) indicates that the implied templates are complex plane disks. In OLGPS and CLGPS, both gain and phase information are used so the templates are arbitrary shapes in the complex plane and thus can represent more complete information about plant uncertainty.
Formation of design constraints

In OLGS, constraints are developed only on the gain of the desired rational function. In the Bode procedure these constraints were developed from a simple approximation of the sensitivity function. In the modern approaches (e.g., see Reference 4) a more accurate representation of the specifications is obtained but again only the gain of the desired function is specified. Thus, in OLGS the constraints are two-dimensional gain versus frequency plots and an external stability test is required. In both OLGPS and CLGPS, constraints are developed on both the gain and phase of the desired rational function. Thus, in these LGPS approaches the constraints describe a three-dimensional surface of gain and phase versus frequency, so performance and stability can be handled simultaneously.

Fitting the constraints

In OLGS the constraints are most commonly fit by trial and error, although mathematical techniques are available. In OLGPS, trial error approaches are most commonly used and no generally applicable mathematical technique is available. In CLGPS, mathematical optimization tools are available but convergence of the iterations becomes problematical when the level sets of the performance functions become highly non-convex.

Comments. All of these approaches have strengths and weaknesses. The following comments are meant to be suggestive of some of these but others may be apparent to the reader.

OLGS has the advantage of simplicity, convenience and clarity. Since practitioners are generally more comfortable working with the loop transfer function as the designable parameter, it seems familiar. Moreover, since the working data (constraints, etc.) can be completely represented in simple and familiar two-dimensional plots (typically Bode plots) the approach feels comfortable and standard CACSD tools can be applied. However, these conveniences are not obtained without some costs. The designer must accept the fact that (1) the approach can fail even when a relatively simple solution exists and (2) the resulting solutions may require excessive gain-bandwidth (Examples of these two situations are given in Reference 15.) In addition, the use of the small gain theorem (22) to test stability seriously limits the uncertainty that can be considered at low frequencies where \( |L_o| \) is large.

OLGPS has the advantage of avoiding the weaknesses of OLGS noted above (this fact is demonstrated in Reference 15). In addition, it can handle a wider range of specifications. However, these features come at the cost of additional complexity. First, Step 4 can be difficult and second, to accomplish Step 5, the designer must learn to become comfortable working in a three-dimensional design space. The QFT procedures offer some help with these steps but do not eliminate this added complexity.

CLGPS also avoids the weaknesses of OLGS and at the same time provides some mathematical rigour and tools missing in OLGPS. However, difficulties in both Steps 4 and 5 can again make this approach quite formidable.

5. CONCLUSIONS

In this paper we have provided an overview of the three major competing techniques in the area of frequency domain design of robust controllers for SISO plants. The comparison presented suggests that the user must decide between efficiency (in the use of available gain and bandwidth)
and convenience in the design process. The steps of modern OLGS are much more convenient that those of either form of LGPS. However, in many cases OLGS can fail or require excessive bandwidth where the LGPS approaches can avoid these pitfalls. The major weaknesses of the two LGPS alternatives are in the area of fitting a rational function to the design constraints. Here is where considerable potentially fruitful future work remains to be accomplished.

ACKNOWLEDGEMENTS

This research was supported in part by the Air Force Office of Scientific Research and the National Science Foundation.

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