JOURNAL OF FUNCTIONAL ANALYSIS 6, 412–440 (1970)

Unitary Operators on a Space with an Indefinite Inner Product

J. WILLIAM HELTON

Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York

Communicated by Ralph Phillips

Received July 25, 1969

INTRODUCTION

This paper concerns a conjecture of R. S. Phillips on the existence of invariant subspaces for certain groups of operators on a Hilbert space with an indefinite inner product. The bilinear form $Q(,)$ on a complex Hilbert space is called an *indefinite inner product* on H provided that H is the direct sum of two orthogonal subspaces H_+, H_- with respect to which $Q(,)$ has the representation

$$Q(x, y) = (E_+x, E_+y) - (E_-x, E_-y) \tag{I.1}$$

where $E_\pm$ are the orthogonal projections of H onto $H_\pm$ along $H_\mp$, and x, y are any two vectors in H. If γ is a bounded operator on H the bounded operator μ which satisfies $Q(\gamma x, y) = (x, \mu y)$ is called the *Q-adjoint* or *o-adjoint* of γ and is denoted by γ^0. A closed subspace P of H which contains only vectors p for which $Q(p, p) \geqslant 0$ is called *positive*. A *maximal* positive subspace is positive and not properly contained in any positive subspace of H. If a bounded operator γ on H satisfies $\gamma^0 = \gamma^{-1}$ $(\gamma^0 = \gamma)$, then γ is said to be *Q-unitary* (*Q-self-adjoint*).

QUESTION A. *Does every Q-unitary operator γ have an invariant maximal positive subspace?*

QUESTION B. *Does every commutating group of Q-unitary operators have an invariant maximal positive subspace?*

* The material presented here is contained in the author's doctoral dissertation written at Stanford University. The author was supported by a NSF graduate fellowship.

Historically, Phillips in [4] posed a question which seemed more general than Question B; however in [5] he was able to show that it and Question B are equivalent. Questions A and B are far from being answered. The main result of this paper is a generalization of Naimark's work on question B. It would be awkward to state our main result just now. However one consequence of it is Corollary 1, proved in Section 3, which we state after introducing some notation.

If γ is any bounded operator on H, then it can be written

$$\gamma = (E_+ + E_-)\,\gamma(E_+ + E_-) = E_+\gamma E_- + E_+\gamma E_- + E_-\gamma E_+ + E_-\gamma E_-$$

which we express in matrix form with respect to the "basis" H_+, H_- by

$$\gamma = \begin{pmatrix} E_+\gamma E_+ & E_+\gamma E_- \\ E_-\gamma E_+ & E_-\gamma E_- \end{pmatrix}.$$

We let $\sigma(A)$ denote the spectrum of the bounded operator A.

COROLLARY 1. *If Γ, a group of commuting Q-unitary operators on H, contains an operator M which has the form*

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + K$$

where K is compact and $\sigma(A)$ does not intersect $\sigma(D)$, then Γ has an invariant maximal positive subspace.

Before further describing the contents of this paper we shall list two results in the area, which give some perspective on Corollary 1.

Result A. M. G. Krein [2] has shown that if M is a single Q-unitary operator of the form

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + K \text{ w.r.t. } H_+ , H_-$$

where K is compact, then M has an invariant maximal positive subspace.

Result B. Naimark [3] has shown that if H_+ is finite dimensional, then the answer to Question B is yes.

Corollary 1 is stronger than Naimark's result, because

$$I = \begin{pmatrix} -E_+ & 0 \\ 0 & E \end{pmatrix} + 2E_+$$

w.r.t. H_+, H_- and if H_+ is finite dimensional, then E_+ is a compact operator. Corollary 1 falls short of generalizing Krein's result because of the requirement on $\sigma(A) \cap \sigma(D)$.

The problems we have been discussing can be posed in a completely different way. This second viewpoint is very important in this paper and it is in this second setting that our Theorems are best stated. Let $\mathscr{L}(H_+, H_-)$ denote the Banach spaces of operators which map H_+ into H_- and let $\mathscr{B}$ denote the unit ball in $\mathscr{L}(H_+, H_-)$. We define $\mathscr{G}$ to be the set of one-to-one bi-analytic maps of $\mathscr{B}$ onto itself. There is a certain subclass $\mathscr{G}_1$ of $\mathscr{G}$ whose members are called general symplectic maps. Krein [2] has shown that Question A is equivalent to asking if each map in $\mathscr{G}_1$ has a fixed point in $\mathscr{B}$. Phillips [6] has shown that if H_+ and H_- are unitarily equivalent (i.e., dimension H_+ = dimension H_-), then $\mathscr{G}_1$ is the principal component of $\mathscr{G}$. Thus $\mathscr{G}_1$ is a large and familiar class of maps. The question—does every commuting family of maps in $\mathscr{G}_1$ have a fixed point?—is more general than Question B. Our main theorem is a fixed point theorem about general symplectic maps.

Let $\mathscr{C}$ denote the set of compact operators in $\mathscr{L}(H_+, H_-)$. We show that if $\mathfrak{F}$ is a general symplectic map on $\mathscr{B}$, then $\mathfrak{F}$ induces a map $\tilde{\mathfrak{F}}$ on the unit ball $\tilde{\mathscr{B}}$ in $\mathscr{L}(H_+, H_-)/\mathscr{C}$. If $\mathfrak{F}$ has a fixed point in $\mathscr{B}$, then certainly $\tilde{\mathfrak{F}}$ has a fixed point in $\tilde{\mathscr{B}}$. Our main theorem and Krein's Result A seem to be along the lines of the converse to this. Result A put in this setting is

THEOREM 1. *If $\mathfrak{F} \in \mathscr{G}_1$ and $\tilde{\mathfrak{F}}$ has a fixed point in the interior of $\tilde{\mathscr{B}}$, then $\mathfrak{F}$ has a fixed point in $\mathscr{B}$.*

The main theorem in this paper is

THEOREM 2. *Suppose that $\mathscr{S}$ is a group of commuting maps in $\mathscr{G}_1$. Consider the set $\tilde{\mathscr{S}}$ of maps that the maps in $\mathscr{S}$ induce on $\tilde{\mathscr{B}}$. If there is an element $\tilde{J}_0$ in the interior of $\tilde{\mathscr{B}}$ which is the unique fixed point of some map $\tilde{\mathfrak{F}}_0 \in \tilde{\mathscr{S}}$, then $\mathscr{S}$ has a fixed point in $\mathscr{B}$.*

The paper begins (Section 1a) by recalling the relation between the fixed point problem for general symplectic maps and the positive invariant subspace problem for Q-unitary operators originally worked out by Krein. In (Section 1b) some simple basic properties of general symplectic maps are given and Theorem 1 is derived. Sections 2 and 3 are written from the geometric viewpoint and they are devoted to proving Theorem 2. The proof of Corollary 1 concludes Section 3 and this paper.

I certainly wish to thank M. G. Crandall and R. S. Phillips for many helpful discussions concerning these problems.

1a. Two Versions of the Same Problem

In this section we shall show how the two different viewpoints of the main question are related. Suppose that P is a positive subspace of H. If $p \in P$, then

$$0 \leqslant Q(p, p) = (E_+p, E_+p) - (E_-p, E_-p) = \| E_+p \|^2 - \| E_-p \|^2 \qquad (1.1)$$

or

$$\| E_+p \| \geqslant \| E_-p \| .$$

This implies that an operator $J : E_+P \to H_-$ is defined by

$$J(E_+x) = E_-x \qquad \text{for} \qquad x \in P$$

and that J is a contraction. We shall write $P \sim J$ and say that P corresponds to J. This correspondence between positive subspaces and contractions is one-to-one. Furthermore maximal positive subspaces correspond to contraction operators which are defined on all H_+ . These facts are well known and easy to prove (cf. [1], [4] or [5]). All remarks hold for negative subspaces where negative subspaces correspond to contractions from H_- to H_+ .

Suppose that U is a Q-unitary operator on H. If P is a maximal positive subspace of H it is easily seen that UP is a maximal positive subspace of H. Therefore the map $P \to UP$ induces a map $\mathfrak{F}$ of $\mathscr{B}$ onto itself. M. G. Krein in [2] derived the following explicit representation for the map $\mathfrak{F}$: If U is Q-unitary,

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ w.r.t. } H_+ , H_- , \qquad (1.2)$$

then define a map $\mathfrak{F} : \mathscr{B} \to \mathscr{B}$, for each $J \in \mathscr{B}$, by

$$\mathfrak{F}(J) = (C + DJ)(A + BJ)^{-1}. \qquad (1.3)$$

(The operator $A + BJ$ is invertible; cf. Remark 1.1). The map $\mathfrak{F}$ will have the property that

$$\begin{aligned} &\text{if } P \text{ is a maximal positive subspace,} \\ &\text{then } P \sim J \qquad \text{if and only if} \qquad UP \sim \mathfrak{F}(J). \end{aligned} \tag{1.4}$$

Any map of the form (1.3) which arises from a Q-unitary operator (1.2) is called *general symplectic*; the general symplectic maps form a group which we shall denote by $\mathscr{G}_1$. If U is given by (1.2) we let $\mathfrak{F}_U$ denote the map $\mathfrak{F}$ defined by (1.3). We call $\mathfrak{F}_U$ the general symplectic map associated with U.

If the maximal positive subspace P contains UP, then naturally $UP = P$ since UP is maximal positive. Therefore from (1.4) it follows that U has an invariant maximal positive subspace if and only if $\mathfrak{F}_U$ has a fixed point in $\mathscr{B}$. If S is a subspace of H let S' denote the Q-orthogonal complement of S.

LEMMA 1.1. *If U and V are Q-unitary operators and if $UP = VP$ for every maximal positive subspace P of H, then there is a scalar α with $|\alpha| = 1$ such that $U = \alpha V$.*

Proof. Suppose that $Y = U^0 V$. If P is a maximal positive subspace of H, $YP = P$. Furthermore Y is Q-unitary. We shall prove that $Y = \alpha I$ for some complex number α with $|\alpha| = 1$. If z is a positive vector in H, let $\langle z \rangle$ denote the linear span of z, let

$$S_z = \{P : P \text{ is a maximal positive subspace and } z \in P\},$$

and let

$$R_z = \bigcap_{P \in S_z} P.$$

If P_1 and P_2 are two maximal positive subspaces, then

$$Y(P_1 \cap P_2) = (YP_1) \cap (YP_2) = P_1 \cap P_2 .$$

Thus $YR_z = R_z$. Now $R_z = \langle z \rangle$, for if $z_1 \in H$ and $z_1 \notin \langle z \rangle$, then one can construct a contraction operator $J \in \mathscr{L}(H_+, H_-)$ such that $z \in [I + J](H_+)$, but $z_1 \notin [I + J](H_+)$. Therefore, $Y\langle z \rangle = \langle z \rangle$.

If N is a maximal negative subspace, then $YN' = N'$, which implies that $YN = N$. Consequently, in addition to working for any positive vector z, the above argument holds for negative vectors and we may conclude that $Y\langle x \rangle = \langle x \rangle$ if x belongs to H. This implies that Y is a scalar multiple of the identity. For let $\{x_n\}_{n=1}^{\infty}$ be a complete orthonormal system in H and set $z = \sum_{n=1}^{\infty} (\frac{1}{2})^n x_n$. Then there exist numbers α_i and α such that $Yx_i = \alpha_i x_i$ and $Yz = \alpha z$. These last two equations imply $\sum_{n=1}^{\infty} (\frac{1}{2})^n \alpha_n x_n = \sum_{n=1}^{\infty} (\frac{1}{2})^n \alpha x_n$; consequently $\alpha = \alpha_n$, that is $Y = \alpha I$. Since Y is Q-unitary, $|\alpha| = 1$.

If $\mathfrak{F}$ is a general symplectic map, then there are several Q-unitary operators related to $\mathfrak{F}$ as in (1.4). For any general symplectic map $\mathfrak{F}$ define

$$\tilde{U}_{\mathfrak{F}} = \{U : \text{for maximal positive } P,\ P \sim J \text{ if and only if } \mathfrak{F}(J) \sim UP\}.$$

By Lemma 1.1 two operators in the set $\tilde{U}_{\mathfrak{F}}$ differ only by a scalar multiple of absolute value 1.

Now suppose that $\mathscr{S}$ is a commutative subgroup of $\mathscr{G}_1$. Let

$$\Gamma = \{U : U \in \tilde{U}_{\mathfrak{F}} \text{ for some } \mathfrak{F} \in \mathscr{S}\}.$$

Certainly Γ is a group. If U, $V \in \Gamma$, and $P \sim J$ is a maximal positive subspace, then

$$UVP \sim \mathfrak{F}_U[\mathfrak{F}_V(J)], \qquad VUP \sim \mathfrak{F}_V[\mathfrak{F}_U(J)],$$

and since $\mathfrak{F}_U$ and $\mathfrak{F}_V$ commute,

$$UVP = VUP.$$

By Lemma 1.1 we see that

$$\text{there is a scalar } \alpha,\ |\alpha| = 1 \text{ such that } UV = \alpha VU. \tag{1.5}$$

Any two operators U and V which satisfy (1.5) are said to *scalar commute*. A set whose operators scalar commute is called *scalar commutative*. Clearly Γ is scalar commutative.

It is evident that $\mathscr{S}$ has a fixed point if and only if Γ has an invariant maximal positive subspace. Since Γ is not commutative but is scalar commutative, the question—does $\mathscr{S}$ have a fixed point?—is more difficult than Question B.

1b. Basic Properties of General Symplectic Maps

If γ is a bounded operator on H, then for $x, y \in H$

$$Q(\gamma^0 x, y) = ([E_+ - E_-]\, \gamma^0 x, y) = ([E_+ - E_-]\, x, \gamma y) = (\gamma^*[E_+ - E_-]\, x, y).$$

Thus

$$\gamma^0 = [E_+ - E_-]\, \gamma^*[E_+ - E_-]. \tag{1.6}$$

Suppose that

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ w.r.t. } H_+, H_-$$

is Q-unitary. Then $UU^0 = I$ or

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I.$$

This relation is equivalent to

$$AA^* - BB^* = I = DD^* - CC^* \quad \text{and} \quad AC^* = BD^*. \tag{1.7a}$$

In addition, $U^0U = I$ and this is equivalent to

$$A^*A - C^*C = I = D^*D - B^*B \quad \text{and} \quad A^*B = C^*D. \tag{1.7b}$$

Remark 1.1 (M. G. Krein [2]). If $J \in \mathscr{B}$, then $A + BJ$ is invertible. The proof is as follows:

A is invertible since $AA^* = I + BB^*$ and $A^*A = I + C^*C$. Furthermore,

$$\begin{aligned} \| A^{-1}B \|^2 &= \| B^*(AA^*)^{-1} B \| = \| B^*(I + BB^*)^{-1} B \| \\ &= \| B^*B(I + B^*B)^{-1} \| < 1. \end{aligned}$$

This implies that if $\| J \| \leqslant 1$, then $\| A^{-1}BJ \| < 1$ and consequently $A + BJ = A(I + A^{-1}BJ)$ is invertible.

Some basic properties of general symplectic maps are given in the next six lemmas. Each of the lemmas involves a map $\mathfrak{F}$ which belongs to $\mathscr{G}_1$. In the proofs of these lemmas we always assume that $\mathfrak{F}$ has the form (1.3), i.e.

$$\mathfrak{F}(J) = (C + DJ)(A + BJ)^{-1} \quad \text{for} \quad J \in \mathscr{B},$$

where the operator $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ which we denote by $U_{\mathfrak{F}}$ is Q-unitary. Of course, A, B, C and D satisfy (1.7).

LEMMA 1.2. *If $\mathfrak{F} \in \mathscr{G}_1$, M and $J \in \mathscr{B}$, and $M - J$ is a compact operator, then $\mathfrak{F}(M) - \mathfrak{F}(J)$ is compact.*

Proof. In this proof K will stand for a compact operator and any compact operator which arises in the course of this proof will be denoted by K. Recall that the sum of two compact operators is compact and that the product of any bounded operator with a compact operator is compact.

$$
\begin{aligned}
\mathfrak{F}(M) - \mathfrak{F}(J) &= \mathfrak{F}(J+K) - \mathfrak{F}(J) \\
&= [C + D(J+K)]\,[A + B(J+K)]^{-1} \\
&\quad - [C + DJ]\,[A + BJ]^{-1} \\
&= [C + DJ]\,[I + (A+BJ)^{-1}K]^{-1}(A+BJ)^{-1} + K \\
&\quad - [C + DJ]\,[A + BJ]^{-1} \\
&= [C + DJ]\,[I + K]^{-1}(A+BJ)^{-1} \\
&\quad - [C + DJ]\,[A + BJ]^{-1} + K \\
&= [C + DJ]\,[A + BJ]^{-1} + K - [C + DJ]\,[A + BJ]^{-1} + K \\
&= K.
\end{aligned}
$$

LEMMA 1.3. *If $\mathfrak{F} \in \mathscr{G}_1$ and M is a compact operator, with $\mathfrak{F}(M)$ compact, then $\mathfrak{F}$ maps the set of compact operators into itself. Furthermore, B and C are compact operators if and only if $\mathfrak{F}$ maps the set of compact operators into itself.*

Proof. If $J \in \mathscr{B}$ is compact, then $J - M$ is compact, so by Lemma 1.2 $\mathfrak{F}(J) - \mathfrak{F}(M)$ is compact. Since $\mathfrak{F}(M)$ is compact, $\mathfrak{F}(J)$ is compact. Thus $\mathfrak{F}$ maps the set of compact operators into itself. In particular, $\mathfrak{F}(0)$ is compact. However, $\mathfrak{F}(0) = CA^{-1}$, which implies that C is compact. Consequently C^*D is compact, but by (1.7b) $A^*B = C^*D$ and A^* is invertible, so B is compact. Conversely, if C is compact, $\mathfrak{F}(0)$ is compact, and so $\mathfrak{F}$ must map the set of compact operators into itself.

LEMMA 1.4. *If $\mathfrak{F} \in \mathscr{G}_1$ and $\mathfrak{F}(0) = 0$, then $\mathfrak{F}$ is a linear map on $\mathscr{B}$.*

Proof. $\mathfrak{F}(0) = CA^{-1} = 0 \Rightarrow C = 0 \Rightarrow C^*D = 0$. By (1.7b) $A^*B = 0$ and $B = 0$. Thus $\mathfrak{F}(J) = DJA^{-1}$.

LEMMA 1.5. *If $J \in \mathscr{B}$, and $\| J \| < 1$, then there is a general symplectic map which maps 0 into J.*

Proof. The proof is due to Phillips (Lemma 1.1 of [5]). He simply constructs the required map and it is

$$T_J(K) = (C + DK)(A + BK)^{-1} \tag{1.8}$$

where

$$U_J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (1 - J^*J)^{-\frac{1}{2}} & J^*(1 - JJ^*)^{-\frac{1}{2}} \\ J(1 - J^*J)^{-\frac{1}{2}} & (1 - JJ^*)^{-\frac{1}{2}} \end{pmatrix}. \tag{1.9}$$

It is straightforward to check that T_J has the necessary properties.
Henceforth T_J will always be the map defined by (1.8) and (1.9).

LEMMA 1.6. *If $\mathfrak{F} \in \mathscr{G}_1$ and $\mathfrak{F}(0)$ is compact, then $\mathfrak{F}$ is continuous in the weak operator topology on $\mathscr{B}$.*

Proof. This Lemma is due to M. G. Krein (cf. [2]).

Remark 1.2. If $\mathfrak{F}$ is continuous in the weak operator topology on $\mathscr{B}$, then $\mathfrak{F}(0)$ is compact. The proof of this appears in the author's doctoral thesis.

Let $\mathscr{C}$ denote the compact operators in $\mathscr{L}(H_+, H_-)$, and let $\mathscr{L}(H_+, H_-)/\mathscr{C}$ denote the Banach space of bounded operators from H_+ to H_- modulo the subspace of compact operators. If $J \in \mathscr{L}(H_+, H_-)$ we let $\tilde{J}$ denote the equivalence class in $\mathscr{L}(H_+, H_-)/\mathscr{C}$ which contains J. $\mathscr{L}(H_+, H_-)/\mathscr{C}$ is a Banach space in the usual quotient norm which we denote by $|\ |$. The norm $|\ |$ has the property that for any operator $\tilde{J} \in \mathscr{L}(H_+, H_-)/\mathscr{C}$ and any $\epsilon > 0$ there is an operator $J \in \tilde{J}$ such that $|\tilde{J}| \leqslant \|J\| \leqslant |\tilde{J}| + \epsilon$. Let $\tilde{\mathscr{B}} = \{\tilde{J} : \|\tilde{J}\| \leqslant 1\}$.

LEMMA 1.7. *If $\mathfrak{F} \in \mathscr{G}_1$, then $\mathfrak{F}$ canonically induces a map $\tilde{\mathfrak{F}}$ on $\tilde{\mathscr{B}}$.*

Proof. The map $\mathfrak{F}$ can be written in the form

$$\mathfrak{F}(J) = (C + DJ)(1 + A^{-1}BJ)^{-1} A^{-1}$$

where $\|A^{-1}B\| < 1$. Thus $\mathfrak{F}$ is defined not only on $\mathscr{B}$, but also in the interior of $\{J : J \in \mathscr{L}(H_+, H_-) \text{ and } \|J\| \leqslant 1/\|A^{-1}B\|\}$ a bigger set. We define a map $\tilde{\mathfrak{F}} : \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ on the element $\tilde{J}$ of $\tilde{\mathscr{B}}$ by

$$\tilde{\mathfrak{F}}(\tilde{J}) = \widetilde{\mathfrak{F}(J)} \tag{1.10}$$

where J is some representative of $\tilde{J}$ with

$$\|J\| < \frac{1}{\|A^{-1}B\|}.$$

By Lemma 1.2 the value of $\widetilde{\mathfrak{F}(J)}$ is independent of which representative J of $\tilde{J}$ is chosen.

Theorem 1 is a restatement of Result A in the introduction to this paper in terms of general symplectic maps.

Proof of Theorem 1. Suppose that $\tilde{\mathfrak{F}}(\tilde{J}) = \tilde{J}$ where $\tilde{J} \in \tilde{\mathscr{B}}$ and $|\tilde{J}| < 1$. There is a representative J of $\tilde{J}$ with $\| J \| < 1$. Let

$$f = T_J^{-1} \circ \mathfrak{F} \circ T_J$$

where T_J is the general symplectic map defined by (1.8). Since $T_J(0) = J$, $\mathfrak{F}(J) = J + K_1$ where K_1 is compact, and $T_J^{-1}(J) = 0$ it follows from Lemma 1.2 that

$$f(0) = T_J^{-1}[\mathfrak{F}(J)] = T_J^{-1}[J + K_1] = \text{a compact operator.}$$

If the Q-unitary operator U_f associated with f is expressed as a matrix $U_f = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ with respect to H_+, H_-, then B_1 and C_1 are compact operators by Lemma 1.3. Thus by Result A in the introduction U_f has an invariant maximal positive subspace, which is to say that f has a fixed point M in $\mathscr{B}$. Clearly $T_J(M)$ is a fixed point of $\mathfrak{F}$.

THEOREM 1.2. *Suppose that $\mathfrak{F}_0$, $\mathfrak{F}_1$,..., $\mathfrak{F}_N$ is a commuting family of general symplectic maps and $\tilde{\mathfrak{F}}_N(\tilde{0}) = \tilde{0}$. If the maps $\mathfrak{F}_0$,..., $\mathfrak{F}_{N-1}$ have a common fixed point J, if J is a compact operator, and if J is in the interior of $\mathscr{B}$, then the maps $\mathfrak{F}_0$,..., $\mathfrak{F}_N$ have a common fixed point.*

Proof. For $i = 0, 1,..., N$ let $f_i = T_J^{-1} \circ f \circ T_J$. If $i = 0,..., N - 1$,

$$f_i(0) = T_J^{-1}[\mathfrak{F}_i(J)] = T_J^{-1}(J) = 0$$

and so from Lemma 1.4 we see that f_i is a linear map on $\mathscr{B}$. This implies that the set

$$\mathscr{K} = \{M : f_i(M) = M \text{ for } i = 0,..., N - 1, \| M \| \leqslant 1\}$$

is a convex subset of $\mathscr{B}$ which is closed and therefore compact in the weak operator topology. Now f_0,..., f_N is a commutative family of maps, so if $M \in \mathscr{K}$

$$f_i[f_N(M)] = f_N f_i(M) = f_N(M)$$

for $i = 1,..., N - 1$. This means that $f_N(M) \in \mathscr{K}$. The maps T_J, T_J^{-1} and $\mathfrak{F}_N$ all map compact operators into compact operators, therefore f_N maps compact operators into compact operators. Consequently f_N is continuous in the weak operator topology. We can now apply the Schauder-Tychonoff Theorem to f_N on $\mathscr{K}$, and conclude that f_N has a fixed point M in $\mathscr{K}$. The operator M, in addition to satisfying

$f_N(M) = M$, satisfies $f_i(M) = M$ for $i = 0, 1, \ldots, N-1$ by the definition of $\mathscr{K}$. Clearly $T_J(M)$ is a fixed point of $\mathfrak{F}_0, \ldots, \mathfrak{F}_N$.

Remark 1.3. If $\dim H_+ = \dim H_-$, then the work of R. S. Phillips in [6] applies and from it one can show that our study of general symplectic maps carries over to one-to-one bi-analytic maps of $\mathscr{B}$ onto $\mathscr{B}$. The hypothesis $\mathfrak{F} \in \mathscr{G}_1$ in Lemmas 1.2 through 1.7 and in Theorems 1 and 1.2 can be replaced by the hypothesis $\mathfrak{F} \in \mathscr{G}$.

2a. Geometric Background

The way that we defined the indefinite inner product $Q(,)$ on H is deceptive. The definition relied heavily on a particular decomposition H_+, H_- of H. We shall now state things more generally. A *canonical decomposition* P, N of H is a pair of positive and negative subspaces of H which are Q-orthogonal such that $H = P + N$. Suppose that P, P' is a canonical decomposition of H and that $\phi_+(\phi_-)$ is the projection of H onto P along P' (P' along P). Then it is easy (cf. Section 2 [4]) to find an inner product $(,)'$ on H which gives a norm topologically equivalent to $\| \ \|$ on H such that

$$Q(x, y) = \phi_+x, \phi_+y)' - (\phi_-x, \phi_-y)' \tag{2.1}$$

We shall call $(,)'$ *the inner product associated with* P, P'. The subspaces P and P' are orthogonal in the $(,)'$ inner product. Thus any canonical decomposition of H could have been used to define $Q(,)$ as in (I.1) provided that we are willing to renorm H. Renorming H makes no difference as far as our invariant subspace problems are concerned (Questions A and B), whether or not an operator is Q-unitary and a vector is positive depends on $Q(,)$ alone and not on a particular canonical decomposition of H used to represent $Q(,)$. A correspondence exists between positive subspaces R of H and contraction (in the $'$ norm) operators $J : P \to P'$ which is similar to the one described for H_+, H_- in Section 1a. Henceforth this correspondence will be denoted by

$$R \sim J(\text{w.r.t. } P, P'). \tag{2.2}$$

Also, a general symplectic map on $\mathscr{L}(P, P')$ can be associated with a Q-unitary operator U, in a way analogous to (1.2) and (1.3), by using the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for U with respect to P, P'.

Remark 2.1. One necessary and sufficient condition (cf. Section 1, Section 2 in [4]) for a bilinear form $B(x, y)$ to be an indefinite inner product on a Hilbert space $\mathscr{H}$ is that $B(x, y)$ be regular on $\mathscr{H}$. A *regular bilinear form* $B(x, y)$ is one which can be written $B(x, y) = (Bx, y)$ for some bounded, invertible, and self-adjoint operator B on $\mathscr{H}$.

We shall assume in this section, as we did in Section 1, that H is a Hilbert space, that Q is a regular bilinear form on H, that H_+, H_- is a canonical decomposition of H, that $(,)$ is the inner product on H associated with H_+, H_-, and that $E_\pm$ are the orthogonal projections of H onto $H_\pm$; then $Q(,)$ has the form (I.1). We say that a closed subspace S of H is *Q-regular* if $Q(,)$ restricted to S is a regular bilinear form with respect to the inner product on S inherited from H. Alternatively, the Q-regular subspaces of H are the ones on which $Q(,)$ is an indefinite inner product. For example, a positive subspace P of H is Q-regular if and only if the pseudonorm $\| x \|' = \sqrt{Q(x, x)}$ for x in P is a norm on P and is equivalent to the Hilbert space norm on P.

Whereas the last section concerned general symplectic maps on $\mathscr{L}(H_+, H_-)$, this section will be written from a geometric viewpoint. We define a *null vector* $n \in H$ to be a vector for which $Q(n, n) = 0$. A *strictly positive* (*negative*) *subspace* is a positive (negative) subspace of H which contains no null vectors. We shall now review some basic facts about positive subspaces (which also hold for negative subspaces). Suppose that P is a positive subspace of H and that $P \sim J$ (w.r.t. H_+, H_-), then

F1. P is regular if and only if $\| J \| < 1$ (same proof as Remark 2.1 [5]).

F2. P is maximal positive if and only if P' is maximal negative (Lemma 2.3 [5]).

F3. If P is maximal positive, then $P' \sim J^*$ (w.r.t. H_-, H_+) (J^* is a map from H_- to H_+; Section 6, 3.8° [1]).

F4. If P is maximal positive, then $P + P' = H$ if and only if $\| J \| < 1$, i.e. P is Q-regular (cf. Lemma 6.3 [4] or Remark 2.1 [5]).

F5. If P is strictly positive, then

$$P' = P_+' \oplus P_-'$$

where

$$P_+' - P' \cap H_+ = H_+ \ominus E_+P$$

and

$$P_-' = (P + P_+')' = P' \cap [H_+ \ominus E_+P]^\perp.$$

The subspace P_-' is maximal negative; P_+' and P_-' are strictly positive and strictly negative subspaces which are Q-orthogonal and orthogonal (cf. Theorem 3.1 [5]).

F6. A subspace S of H is Q-regular if and only if there are two Q-orthogonal positive and negative subspaces S_+ and S_- of S such that

(a) $S = S_+ + S_-$

(b) S_+ and S_- are both Q-regular.

(This follows from Remark 2.1 and some straight forward calculation.)

The rest of this section will be devoted to proving some more specialized geometric facts which are necessary to the forthcoming theorems. Although most statements will be made in terms of positive subspaces they will hold for negative subspaces as well. Suppose that R, R' is a canonical decomposition H. We shall say that a positive (negative) subspace P of H is *compactly positive* (*negative*) w.r.t. R, R' if $P \sim J$ (w.r.t. R, R') and J is a compact operator. An operator U which has the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with respect to R, R' is said to have *compact off diagonal terms* (w.r.t.) R, R' if B and C are compact operators.

LEMMA 2.1. *If P is a positive Q-regular subspace of H, then P' is a Q-regular subspace of H and $P + P' = H$.*

Proof. If $P \sim J$ w.r.t. H_+, H_-, then $\| J \| < 1$ by F1. Define $J_+ : H_+ \to H_-$ by

$$J_+ x = \begin{cases} Jx & \text{if} \quad x \in E_+ P \\ 0 & \text{if} \quad x \in H_+ \ominus E_+ P. \end{cases} \tag{2.3}$$

The subspace $P_+' + P$ defined in F5 corresponds to J_+ and the subspace $P_-' = (P_+' + P)'$ corresponds to J_+^* (cf. F3). Now $\| J \| < 1 \Rightarrow \| J_+ \| < 1 \Rightarrow \| J_+^* \| < 1 \Rightarrow P_-'$ is Q-regular. By definition (F5) P_+' is Q-regular. Since $P' = P_+' + P_-'$ we may conclude from F6 that P' is Q-regular. Furthermore,

$$P + P' = P + (P_+' + P_-') = (P + P_+') + P_-'.$$

Thus it follows from F4 applied to $P + P_+'$ that $P + P' = H$.

LEMMA 2.2. *Suppose that P is a positive subspace of H, that $P \sim J$ (w.r.t. H_+, H_-), and that J is a compact operator. Let N denote* *the set of null vectors in P. The set N is a finite dimensional subspace of P, and N is trivial if and only if $\| J \| < 1$, i.e. P is Q-regular.*

Proof. The proof that N is a subspace appears in several places (eg. proof of Theorem 3.1 [5]) and consists of using Schwartz's Lemma for $Q(,)$ on P to show that $y \in N$ if and only if $y \in P$ and $Q(y, P) = 0$. Now from (1.1) it follows that a necessary and sufficient condition for a vector n to belong to N is that $\| E_+ n \| = \| J E_+ n \|$. This says that J restricted to $E_+ N$ is an isometry of $E_+ N$ into $E_- N$. Since J is a compact operator $E_+ N$ must be finite dimensional. Consequently N is finite dimensional. Now N is non-trivial if and only if there is an $x \in H_+$ such that $\| Jx \| = \| x \|$. Since J is a compact operator this condition is equivalent to the condition $\| J \| = 1$. Thus N is trivial if and only if $\| J \| < 1$.

LEMMA 2.3. *Suppose that the strictly positive subspace P corresponds to J* (w.r.t. H_+, H_-), *and that J is a compact operator. Then $P' + (P_-')$ is compactly positive (negative)* w.r.t. H_+, H_-. *If R is a positive (negative) subspace of P' and*

$$R \sim K \qquad \text{w.r.t.} \qquad P_+', P_-',$$

$$R \sim v \qquad \text{w.r.t.} \qquad H_+, H_-,$$

and

$$P + R \sim L \qquad \text{w.r.t.} \qquad H_+, H_-$$

when R is positive, then K is compact if and only if v is compact if and only if L is compact.

Proof. By Lemma 2.2, $\| J \|$ is less than one, so by Lemma 2.1, P' is Q-regular. Furthermore, P_+', P_-' defined in F5 is a canonical decompositon of P' in the form $Q(,)$ restricted to P'. Thus if R is a positive or negative subspace of P' it is legitimate to discuss the operator K, which corresponds to R w.r.t. P_+', P_-'. Let $Q_+ (Q_-)$ be the projection of P' onto P_+' along P_-' (P_-' along P_+'). Suppose that R is positive and that $r \in R$, then

$$E_+ r + v E_+ r = Q_+ r + K Q_+ r$$

and

$$K Q_+ r = v E_+ r + E_+ r - Q_+ r.$$

Since

$$Q_+ r \in P_+' \subset H_+, \qquad E_- K Q_+ r = E_- v E_+ r. \tag{2.4}$$

If J_+ is the operator defined by (2.3), then $P_-' \sim J_+^*$ (w.r.t. H_+, H_-) and since $KQ_+r \in P_-'$ equation (2.4) becomes

$$KQ_+r = [E_- + J_+^*E_-]\,KQ_+r = [E_- + J_+^*E_-]\,\nu E_+r$$

which implies, because J_+^* is compact, that K is compact if and only if ν is compact.

If R is a negative subspace of P' and $r \in R$, then

$$E_-r + \nu E_-r = Q_-r + KQ_-r. \tag{2.5}$$

Since $KQ_-r \in P_+' \subset H_+$ equation (2.5) can be written

$$KQ_-r = E_+KQ_-r = \nu E_-r - E_+Q_-r$$

which, because $E_+Q_-r = J_+^*E_-Q_-r$, becomes

$$KQ_-r = \nu E_-r - J_+^*E_-Q_-r.$$

Thus K is compact if and only if ν is compact.

Next we shall suppose that R is a positive subspace of P' and we shall consider $P + R$. Since P and R are Q-orthogonal each vector of $P + R$ is a positive vector, and from the fact that P is Q-regular one can deduce that $P + R$ is closed. Thus $P + R$ is a positive subspace and there is an operator L such that $P + R \sim L$(w.r.t. H_+, H_-). Let $X = E_+(P + R)$. The subspace X is closed because $P + R$ is a positive subspace, and X can be written $X = E_+P + E_+R$. Furthermore, the Q-orthogonality of P and R can be used to show that E_+P and E_+R have trivial intersection. Thus there exist projection oerators $Q_1(Q_2)$ of X onto E_+P along E_+R (E_+R along E_-P). If $z \in P + R$, then $z = z_1 + z_2$ where $z_1 \in P$ and $z_2 \in R$ and we write

$$z = E_+z_1 + JE_+z_1 + E_+z_2 + \nu E_+z_2 = E_+z + LE_+z.$$

Now $E_+z_1 = Q_1E_+z$ and $E_+z_2 = Q_2E_+z$, so $L = JQ_1 + \nu Q_2$. Thus L is a compact operator if and only if ν is a compact operator.

LEMMA 2.4. *Suppose that S is a closed subspace of H and that $S = S_+ + S_-$ where the strictly positive and strictly negative subspace S_+ and S_- are Q-orthogonal and correspond to K_+ and K_-* w.r.t. H_+, H_-. *If K_+ and K_- are compact operators, then*

(a) *S and S' are Q-regular; $H = S + S'$.*

(b) *$S' = S_+' + S_-'$ where S_+' and S_-' are strictly positive and* *strictly negative Q-orthogonal subspaces both of which are compactly positive* w.r.t. H_+, H_-.

Proof. Since S_+ is strictly positive, $(S_+)'$ can be written as

$$(S_+)' = V_+ \oplus V_-$$

according to F5. By Lemmas 2.1 and 2.2 S_+ and $(S_+)'$ are Q-regular. Thus $(S_+)'$ can be renormed so that $Q(,)$ has form (I.1) with respect to the canonical decomposition V_+, V_-. The negative subspace S_- is contained in $(S_+)'$ and consequently Lemma 2.3 implies that S_- is compactly positive w.r.t. V_+, V_-. Let T denote the Q-orthogonal complement of S_- in $(S_+)'$; since S_- is Q-regular, T is Q-regular (Lemma 2.1). If we apply F5 to the subspace T in $(S_+)'$ we get the usual decomposition

$$T = T_+ \oplus T_-$$

where T_+ and T_- are strictly positive and negative and Q-orthogonal subspaces of T. Furthermore, Lemma 2.3 applies to S_- in $(S_+)'$ and so we may conclude that $T_+(T_-)$ is compactly positive (negative) w.r.t. V_+, V_-. The second part of Lemma 2.3 implies that $T_+(T_-)$ is compactly positive (negative) w.r.t. H_+, H_-.

Now T is the Q-orthogonal complement of S_- in $(S_+)'$, i.e. $T = (S_+)' \cap (S_-)' = (S_+ + S_-)' = S'$. Thus if we let $S'_+ = T_+$ and $S'_- = T_-$ part (b) is proved. To complete this proof we need to show that $H = S + S'$. This follows from a double application of Lemma 2.1 which implies that

$$H = S_+ + (S_+)' = S_+ + S_- + T = S + T = S + S'.$$

LEMMA 2.5. *If P, P' is a canonical decomposition of H, $P \sim J$* (w.r.t. H_+, H_-) *a compact operator, and $\Phi_+(\Phi_-)$ denotes the projection of H onto P along P' (P' along P), then*

$$\Phi_\pm = E_\pm + \textit{a compact operator}.$$

Proof. Let T_J be the general symplectic map (1.8) and let U_J be the corresponding Q-unitary operator (1.9). The properties $T_J(0) = J$, $U_J H_+ = P$, and $U_J H_- = P'$ are fundemental to T_J and U_J and from them one obtains

$$U_J E_+ U_J^{-1} x = \begin{cases} x & \text{if} \quad x \in P \\ 0 & \text{if} \quad x \in P'. \end{cases} \tag{2.6}$$

The relation (2.6) can be stated more simply: $U_J E_+ U_J^{-1}$ is the projection of H onto P along P'. This and the similar calculation for E_- gives

$$\Phi_\pm = U_J E_\pm U_J^{-1}. \tag{2.7}$$

In (1.9) U_J is expressed explicitly in terms of J and since J is compact (1.9) reduces to

$$U_J = I + \text{a compact operator.}$$

Therefore

$$\Phi_\pm = U_J E_\pm U_J^{-1} = E_\pm + \text{a compact operator.}$$

LEMMA 2.6. *Suppose that $S = S_+ + S_-$ satisfies the hypothesis of Lemma 2.4, that U is a Q-unitary operator which has compact off diagonal terms* w.r.t. H_+, H_-, *and that $US = S$. Then $U|_S$ has compact off diagonal terms* w.r.t. S_+, S_-, *and $U|_{S'}$, has compact off diagonal terms* w.r.t. S'_+, S'_-.

Proof. Since U is Q-unitary and $US = S$, $US' = S'$. Let $R_+ = S_+ + S'_+$ and $R_- = S_- + S'_-$. The subspaces R_+ and R_- are Q-orthogonal, positive and negative respectively, and $R_+ + R_- = S + S' = H$; that is R_+, R_- is a canonical decomposition of H. In addition, one may note from Lemma 2.4 that S_+, S_-, (S'_+), and (S'_-) are compactly positive w.r.t. H_+, H_- which implies (Lemma 2.3) that R_+, R_- are compactly positive w.r.t. H_+, H_-. Let $\Phi_+(\Phi_-)$ be the projection of H onto R_+ along R_- (R_- along R_+). From Lemma 2.5 we may conclude that

$$\Phi_\pm U \Phi_\mp = E_\pm U E_\mp + \text{a compact operator.}$$

By hypothesis $E_\pm U E_\mp$ are compact, thus $\Phi_\pm U \Phi_\mp$ are compact operators.

The fact that $\Phi_\pm S$ is $S_\pm$ coupled with our hypothesis $US = S$ implies that $\Phi_\pm U \Phi_\pm S \subset S$ and that the operators $\Phi_\pm U \Phi_\mp|_S$ are the off diagonal terms w.r.t. S_+, S_- of $U|_S$. At the end of the preceeding paragraph we saw that the operators $\Phi_\pm U \Phi_\mp$ were compact, thus Lemma 2.6 is proved for $U|_S$. A similar argument works for $U|_{S'}$ since $\Phi_\pm S' = S'_\pm$.

2b. A Major Preliminary Theorem

A Q-unitary operator U_0 on H is said to satisfy *condition* A w.r.t. H_+, H_- provided that

(a) if P is a maximal positive subspace of H and $U_0P = P$, then P is compactly positive w.r.t. H_+, H_-.

(b) U_0 has compact off diagonal terms w.r.t. H_+, H_-.

If $\mathfrak{F}_0$ denotes the general symplectic map associated with U_0 (cf. (1.2) (1.3) (1.4), then the statement in terms of $\mathfrak{F}_0$ which is equivalent to condition A w.r.t. H_+, H_- is

(a) if $\mathfrak{F}_0(J) = J$, then J is a compact operator.

(b) $\mathfrak{F}_0(0)$ is compact, or equivalently, $\mathfrak{F}_0(K)$ is compact if K is compact (cf. Lemma 1.3).

Translating condition A into terms of $\tilde{\mathfrak{F}}$ on $\mathscr{L}(H_+, H_-)/\mathscr{C}$ is akward. However a succinct condition, (2.7) $\tilde{\mathfrak{F}}_0$ has only one fixed point in $\tilde{\mathscr{B}}$ and that point is $\tilde{0}$, implies condition A w.r.t. H_+, H_-.

Remark 2.2. If U is Q-unitary, U_0 satisfies condition A, and U scalar commutes with U_0, then U has compact off diagonal terms w.r.t. H_+, H_-. The proof of this is as follows. By Result A in the Introduction U_0 has an invariant maximal positive subspace P. Since U and U_0 scalar commute, $U_0UP = UU_0P = UP$. Now part (a) of Condition A implies that both P and UP are compactly positive w.r.t. H_+, H_-, which combined with Lemma 1.3 implies that U has compact off diagonal terms.

Lemma 2.7. *If U_0 satisfies condition A* w.r.t. *H_+ or H_- and R is a positive or negative subspace such that $U_0R = R$, then R is compactly positive or negative* w.r.t. H_+, H_-.

Proof. We shall prove that R is compactly positive w.r.t. H_+, H_- by showing that R is contained in a subspace which is compactly positive w.r.t. H_+, H_-. Suppose that R is positive and that $R \sim J$ (w.r.t. H_+, H_-). Let $\mathscr{K} = \{K \in \mathscr{B} : Kx = Jx \text{ for all } x \in E_+R\}$. The set $\mathscr{K}$ is convex and is compact in the weak operator topology. If a maximal positive subspace P contains R, then U_0P contains R since we have assumed that $U_0R = R$. This is equivalent to saying that $\mathfrak{F}_0$ maps $\mathscr{K}$ into itself. By Lemma 1.3 $\mathfrak{F}_0(0)$ is a compact operator, by Lemma 1.6 $\mathfrak{F}_0$ is continuous in the weak operator topology, and by the Schauder-Tychonoff Theorem $\mathfrak{F}_0$ has a fixed point M in $\mathscr{K}$. The maximal positive subspace S which corresponds to M (w.r.t. H_+, H_-) is invariant under U_0, contains R, and is compactly positive w.r.t. H_+, H_-.

Condition A (a) implies that if N is a maximal negative subspace of H for which $U_0N = N$, then N is compactly negative w.r.t. H_+, H_-. This is true because

$$U_0N = N \Rightarrow U_0N' = N' \Rightarrow N' \text{ is compactly positive}$$
$$\text{w.r.t. } H_+, H_- \Rightarrow N \text{ is compactly negative w.r.t. } H_+, H_-.$$

This subsection is devoted to proving,

THEOREM 2.1. *Suppose that U_0, $U_1 \cdots U_N$ are scalar commuting Q-unitary operators on H, that U_0 satisfies condition A* w.r.t. H_+, H_-, *and that there is a maximal positive subspace P invariant under U_0, U_1,..., U_{N-1}. Then there is a non-trivial positive subspace R of H which is invariant under U_0,..., U_N, U_0^{-1},..., U_N^{-1}. That is, $U_iR = R$ for $i = 0$,..., N.*

Proof. Let $\mathfrak{F}_i$ be the general symplectic map associated with U_i (cf. (1.2) (1.3) (1.4)) for $i = 0$,..., N. Since the U_i scalar commute, the maps $\mathfrak{F}_i$ commute (cf. Section 1a). Furthermore, if $P \sim J$ (w.r.t. H_+, H_-) then $\mathfrak{F}_i(J) = J$ for $i = 0$,..., $N - 1$. From these facts we see that

$$\mathfrak{F}_N(J) = \mathfrak{F}_N[\mathfrak{F}_0(J)] = \mathfrak{F}_0[\mathfrak{F}_N(J)]$$

which implies that $U_0S = S$ where $S \sim \mathfrak{F}_N(J)$ (w.r.t. H_+, H_-). Since U_0 satisfies condition A, $\mathfrak{F}_N(J)$ is a compact operator. The operator J is also compact because $U_0P = P$. Thus by Lemma 1.3 we have that $\mathfrak{F}_N$ maps the set of compact operators into itself, i.e. $\tilde{\mathfrak{F}}_N(0) = 0$. Now we consider two cases.

Case 1. If J is in the interior of $\mathscr{B}$, then Theorem 1.2 says that $\mathfrak{F}_0$,..., $\mathfrak{F}_N$ must have a fixed point K. The maximal positive subspace R which corresponds to K satisfies the conclusion of Theorem 2.1.

Case 2. Suppose that $\| J \| = 1$. Since J is compact, Lemma 2.2 implies that the set M of null vectors in P is a non-trivial finite dimensional subspace. Any Q-unitary operator maps null vectors into null vectors, so M is invariant under $U_0 \cdots U_{N-1}$, U_0^{-1},..., U_{N-1}^{-1}. The following is a generalization of Naimark's ideas in [3]. Let Γ_K be the group generated by U_0,..., U_K for $K = N - 1$ or $K = N$. Let

$\mathscr{M}$ be a maximal abelian subgroup of Γ_{N-1} which contains U_0. Since M is finite dimensional and invariant under the commutative family $\mathscr{M}$ of operators, there is an $x \in M$ which is an eigenvector of each $m \in \mathscr{M}$. Let $\lambda(m)$ denote the corresponding eigenvalue; $mx = \lambda(m)\,x$. Let

$$T = \{x \in H : mx = \lambda(m)\,x \text{ if } m \in \mathscr{M}\}.$$

We need three lemmas.

LEMMA 2.8. *If $\mu, \gamma \in \Gamma_N$, then either $\mu T = \gamma T$ or $Q(\mu T, \gamma T) = 0$.*

Proof. We wish to show that either $\gamma^{-1}\mu T = T$ or $Q(\gamma^{-1}\mu T, T) = 0$. Suppose that $\gamma^{-1}\mu$ commutes with $\mathscr{M}$. Then

$$m(\gamma^{-1}\mu)\,x = (\gamma^{-1}\mu)\,mx = \lambda(m)\,(\gamma^{-1}\mu)\,x,$$

for $x \in T$ and $m \in \mathscr{M}$. Thus if $x \in T$, $\gamma^{-1}\mu x \in T$. Since $\mathscr{M}$ is a group, $(\gamma^{-1}\mu)^{-1}$ commutes with $\mathscr{M}$ and so we get $\gamma^{-1}\mu T = T$.

Suppose that there is an operator $m \in \mathscr{M}$ which does not commute with $\gamma^{-1}\mu$. Since Γ_N is scalar commutative, $m(\gamma^{-1}\mu) = \alpha(\gamma^{-1}\mu)\,m$ for some α, $|\alpha| = 1$, $\alpha \neq 1$. If $x \in T$ and $y \in \gamma^{-1}\mu T$, then $y = \gamma^{-1}\mu z$ for some $z \in T$ and

$$\begin{aligned} Q(x, y) &= Q(x, \gamma^{-1}\mu z) = Q(mx, m\gamma^{-1}\mu z) = \bar{\alpha} Q(mx, \gamma^{-1}\mu m z) \\ &= \bar{\alpha}\Omega(\lambda(m)\,x, \gamma^{-1}\mu\lambda(m)\,z) = \bar{\alpha}\,|\,\lambda(m)|^2\, Q(x, y), \end{aligned}$$

that is

$$(1 - \bar{\alpha}\,|\,\lambda(m)|^2)\,Q(x, y) = 0$$

and so $Q(x, y) = 0$.

LEMMA 2.9. *If $b \in \Gamma_{N-1}$ and $bT \subset T$, then $b \in \mathscr{M}$.*

Proof. Since Γ_{N-1} is scalar commutative there is for each $g \in \Gamma_{N-1}$ a scalar $\alpha(g)$, $|\alpha(g)| = 1$, such that $bg = \alpha(g)\,gb$. If $x \in T$, then $mx = \lambda(m)\,x$ and so $bmx = \lambda(m)\,bx$ or equivalently $mbx = [\lambda(m)/\alpha(m)]\,bx$. However, the hypothesis $bx \in T$ says that $mbx = \lambda(m)\,bx$ for each $m \in \mathscr{M}$. Thus $\alpha(m) = 1$ for $m \in \mathscr{M}$ and b must commute with $\mathscr{M}$. Since $\mathscr{M}$ is maximal abelian, $b \in \mathscr{M}$.

LEMMA 2.10. *The subspace T can be decomposed into Q-orthogonal and orthogonal strictly positive, strictly negative, and null subspaces*

$$T = T_+ + T_- + T_0$$

where $T_0 = T \cap T'$. Furthermore T_+, T_-, and T_0 all correspond to compact contractions w.r.t. *H_+, H_-, and T is Q-regular if T_0 is trivial.*

Proof. The decomposition of T into T_+, T_- and T_0 is obtained by considering $Q(x, y) = (Bx, y)$ on T and splitting the self-adjoint operator B into positive, negative, and null parts. This technique is the one behind Remark 2.1. An argument in Section 4 [4] proves that $T_0 = T \cap T'$ (take $N = T_+ + T_0$, $P = T_- + T_0$ in Section 4 [4]). Since T is an eigenspace of U_0, the subspaces T_+, T_- and T_0 are invariant under U_0. Thus by Lemma 2.7 they are compactly positive and negative w.r.t. H_+, H_-. Lemma 2.4 implies that T is Q-regular if T_0 is trivial.

The proof of Theorem 2.1 will involve three special cases. In each case we shall define a certain subspace S of H. In each case it will be clear that S is invariant under Γ_N; all difficulties lie in showing that S is a positive subspace. The symbol $\bigvee$ will denote the closed linear span of subspaces written after it.

Case (a). *T_0 is a non-trivial subspace.*

$$\text{Define } S = \bigvee_{\gamma \in \Gamma_N} \gamma T_0 .$$

Case (b). *T_0 is trivial. In addition, $U_N{}^r T \neq \gamma T$ for each positive integer r and each $\gamma \in \Gamma_{N-1}$.*

Let $z \in T \cap M$, and denote the linear span of z by $\langle z \rangle$.

$$\text{Define } S = \bigvee_{\gamma \in \Gamma_N} \gamma \langle z \rangle.$$

Case (c). *T_0 is trivial. In addition, there is an integer v and a $y \in \Gamma_{N-1}$ such that $U_N{}^v T = \gamma T$.*

Let t be the smallest such positive integer, then $U_N{}^t T = hT$, for some $h \in \Gamma_{N-1}$. Since h^{-1} and $U_N{}^t$ commute to within a scalar, $h^{-1}T = U_N^{-t} T$ and we have lost nothing by requiring t to be positive. Now by Remark 2.2 $h^{-1}U_N{}^t$ has compact off diagonal terms w.r.t. H_+, H_-; also T_+ and T_- are compactly positive and negative w.r.t. H_+, H_-. Therefore, by Lemma 2.4 T is Q-regular and by Lemma 2.6 $h^{-1}U_N{}^t|_T$ has compact off diagonal terms w.r.t. T_+, T_-. Result A in the Introduction is applicable, so there exists a maximal positive subspace P in T for which

$$h^{-1}U_N{}^t P = P.$$

$$\text{Define } S = \bigvee_{\gamma \in \Gamma_N} \gamma P.$$

We now prove that in each case S is a positive subspace.

Case (a). Since T_0 is a null space, if γ is a Q-unitary operator, then γT_0 is a null space. The sum of Q-orthogonal null spaces is a null space, and null spaces are positive.

Thus it suffices to show that either $\mu T_0 = \gamma T_0$ or $Q(\mu T_0, \gamma T_0) = 0$ for each $\gamma, \mu \in \Gamma_N$. This is easy. If $\mu, \gamma \in \Gamma_N$ then Lemma 2.8 implies that either $\mu T = \gamma T$ or $Q(\mu T_0, \gamma T_0) = 0$, since $\gamma T_0 \subset \gamma T$ and $\mu T_0 \subset \mu T$. Suppose that $\mu T = \gamma T$. Then $\gamma^{-1}\mu T = T$ and $\gamma^{-1}\mu T' = T'$. This implies, since $T_0 = T \cap T'$, that $\gamma^{-1}\mu T_0 = T_0$ and so $\mu T_0 = \gamma T_0$.

Case (b). $\langle z \rangle$ is a null space. We need only show that either $\mu\langle z \rangle = \gamma\langle z \rangle$ or $Q(\mu\langle z \rangle, \gamma\langle z \rangle) = 0$ for each $\mu, \gamma \in T_N$. Now if $\mu, \gamma \in \Gamma_N$, then Lemma 2.8 implies that either $\mu T = \gamma T$ or $Q(\mu\langle z \rangle, \gamma\langle z \rangle) = 0$. Suppose that $\gamma^{-1}\mu T = T$. Since $\gamma^{-1}\mu \in \Gamma_N$, Γ_N is generated by Γ_{N-1} and U_N, and Γ_N is scalar commutative, we see that there is an integer r, a scalar c, and an operator $b \in \Gamma_{N-1}$ such that

$$\gamma^{-1}\mu = cbU_N{}^r, \tag{2.8}$$

Equation (2.8) implies that $U_N{}^r T = b^{-1}T$. By assumption $r = 0$, so $\gamma^{-1}\mu = cb$ and $bT = T$. Thus $b \in \mathscr{M}$ by Lemma 2.9, and $\gamma^{-1}\mu z = cbz = c\lambda(b)\, z$. In other words, $\mu\langle z \rangle = \gamma\langle z \rangle$.

Case (c). The sum of Q-orthogonal positive subspaces is a positive subspace. We need only show that either $\mu P = \gamma P$ or $Q(\mu P, \gamma P) = 0$ for each, $\mu, \gamma \in \Gamma_N$. Now if $\mu, \gamma \in \Gamma_N$, then Lemma 2.8 implies that either $\mu T = \gamma T$ or $Q(\mu P, \gamma P) = 0$. Suppose that $\gamma^{-1}\mu T = T$. Since $\gamma^{-1}\mu$ belongs to Γ_N it satisfies (2.8), which implies that $U_N{}^r T = b^{-1}T$. By assumption $0 < t \leqslant |r|$, so there is an integer j with $|j| < t$ and an integer k which satisfy $r = kt + j$. Recall that $U_N{}^t T = hT$; certainly $(U_N{}^t)^k\, T = h^k T$. Now

$$b^{-1}T = U_N{}^r T = (U_N{}^t)^k\, U_N{}^j T = h^k U_N{}^j T$$

$$h^{-k} b^{-1} T = U_N{}^j T.$$

However $h^{-k}b^{-1} \in \Gamma_{N-1}$ which, by the definition of t, implies that either $|j| \geqslant t$ or $j = 0$. Since $|j| < t$, $j = 0$. Consequently $r = kt$ and $bh^k T = T$. Thus $bh^k \in \mathscr{M}$ by Lemma 2.9, and by the definition of T we see that $bh^k x = \lambda(bh^k)\, x$, if $x \in T$. The subspace P is contained in T so $h^k P = b^{-1}P$; because h^k and b scalar commute $h^{-k}P = bP$. It follows from these calculations that

$$\gamma^{-1}\mu P = bU_N{}^r P = U_N{}^r bP = U_N{}^r h^{-k} P = h^{-k} U_N^{tk} P = (h^{-1}U_N{}^t)^k\, P.$$

Now P was defined to satisfy $(h^{-1}U_N{}^t)\,P = P$, thus $\gamma^{-1}\mu P = P$ and $\mu P = \gamma P$.

3. The Main Results

Before proceeding with the proof of the main theorem of this paper we need two Lemmas.

Lemma 3.1. *If P is a strictly positive subspace of H, U_0 is a Q-unitary operator on $\mathscr{H}$ which satisfies condition A* w.r.t. H_+, H_-, *and $U_0P = P$, then*

(1) *P' is Q-regular and P_+', P_-'* (as in Section 2a F5) *is a canonical decomposition for P'. Thus if the inner product associated with P_+', P_-' is placed on P', $Q(,)$ restricted to P' can be written in form* (I.1) *with respect to P_+', P_-'.*

(2) *If $V_0 = U_0\,|_{P'}$, then V_0 is a Q-unitary operator on P' and V_0 satisfies condition A* w.r.t. P_+', P_-'.

Proof. (1) The hypothesis and Lemma 2.7 imply that P is compactly positive w.r.t. H_+, H_-. Since P is strictly positive (Lemma 2.2) P is Q-regular, so P' is Q-regular, and the decomposition of P' given in F5 is a canonical decomposition of P'.

(2) Condition A w.r.t. P_+', P_-', part (a): Suppose that R is a maximal positive subspace of P' and that $V_0R = R$. The subspace R is Q-orthogonal to P so $P + R$ is a positive subspace of H, and $U_0(P + R) = U_0P + U_0R = P + R$. Since U_0 satisfies condition A w.r.t. H_+, H_-, the subspace $P + R$ is compactly positive w.r.t. H_+, H_-. By Lemma 2.3 the subspace R is compactly positive w.r.t. P_+', P_-'.

Part (b) is a direct consequence of Lemma 2.6 where we let the subspace S of Lemma 2.4 equal P.

Lemma 3.2. *If S_0 is a null subspace of H, U_0 is a Q-unitary operator on H which satisfies condition A* w.r.t. H_+, H_-, *and $U_0S_0 = S_0$, then*

(1) *The subspace $\hat{H} = H \ominus [E_+S_0 \oplus E_-S_0]$ of H is Q-regular and the subspaces $\hat{H}_\pm = H_\pm \ominus E_\pm S_0$ are a canonical decomposition of $\hat{H}$. Furthermore, $Q(,)$ restricted to $\hat{H}$ has the form* (I.1) *with respect to $\hat{H}_+$, $\hat{H}_-$ in the inner product on $\hat{H}$ given by restricting $(,)$ to $\hat{H}$.*

(2) *$U_0[\hat{H} \oplus S_0] = \hat{H} \oplus S_0$. Let β be the orthogonal projection of*

$\hat{H} \oplus S_0$ onto $\hat{H}$, and let $\hat{U}_0 = \beta U_0 \,|_{\hat{H}}$. Then $\hat{U}_0$ is a Q-unitary operator on $\hat{H}$ and satisfies condition A w.r.t. $\hat{H}_+$, $\hat{H}_-$.

Proof. (1) Is evident.

(2) The following is patterned after arguments found in Section 4 [4] and Section 4 [5]. A more comprehensive presentation than our rather sketchy one can be found in these places.

Since S_0 is a null space $S_0 \subset S_0'$. The vector x belongs to $S_0' \ominus S_0$ if and only if

$$0 = Q(x, S_0) = (x, E_+S_0) - (x, E_-S_0) \tag{3.1}$$

and

$$0 = (x, S_0) = (x, E_+S_0) + (x, E_-S_0). \tag{3.2}$$

The equations (3.1) and (3.2) are equivalent to

$$(x, E_+S_0) = 0 = (x, E_-S_0)$$

which says that $x \in H \ominus [E_+S_0 \oplus E_-S_0] = \hat{H}$. Thus $S_0' \ominus S_0 = \hat{H}$, that is $\hat{H} \oplus S_0 = S_0'$, and the statement $U_0[\hat{H} \oplus S_0] = \hat{H} \oplus S_0$, is equivalent to the statement $U_0S_0' = S_0'$. Now $U_0S_0' = S_0'$ because $U_0S_0 = S_0$.

The operator $\hat{U}_0$ is Q-unitary on $\hat{H}$: if $x, y \in \hat{H}$, then $U_0x, U_0y \in \hat{H} \oplus S_0$ and

$$\begin{aligned} Q(x, y) &= Q(U_0x, U_0y) = Q([1 - \beta + \beta]\, U_0x, [1 - \beta + \beta]\, U_0y) \\ &= Q([1 - \beta]\, U_0x, [1 - \beta]\, U_0y) + Q(\beta U_0x, [1 - \beta]\, U_0y) \\ &\quad + Q([1 - \beta]\, U_0x, \beta U_0y) + Q(\beta U_0x, \beta U_0y) \\ &= Q(\beta U_0x, \beta U_0y) = Q(\hat{U}_0x, \hat{U}_0y). \end{aligned} \tag{3.3}$$

Suppose that R is a maximal positive subspace of $\hat{H}$ such that $\hat{U}_0R = R$. The subspace $R \oplus S_0 = \beta^{-1}R$ is positive and

$$U_0[R \oplus S_0] = U_0R + U_0S_0 = \hat{U}_0R \oplus [1 - \beta]U_0R + S_0 = R \oplus S_0 \tag{3.4}$$

Thus $R \oplus S_0$ is compactly positive w.r.t. H_+, H_-. If $S_0 \sim J_0$ w.r.t. H_+, H_-, $R \sim \hat{K}$ (w.r.t. $\hat{H}_+$, $\hat{H}_-$), and $R \oplus S_0 \sim K$ (w.r.t. H_+, H_-), then

$$Kx = \begin{cases} J_0x & \text{if} \quad x \in E_+S_0 \\ \hat{K}x & \text{if} \quad x \in H_+ \ominus E_+S_0 . \end{cases}$$

Furthermore J_0 is compact, since $U_0 S_0 = S_0$. Consequently $\hat{K}$ is a compact operator and R is compactly positive w.r.t. $\hat{H}_+$, $\hat{H}_-$. It is clear that $\hat{U}_0$ has compact off diagonal terms w.r.t. $\hat{H}_+$, $\hat{H}_-$.

Induction Hypothesis for the Integer N. (I. H; N)

If (a) $\mathscr{H}$ is a Q-regular subspace of H, and $\mathscr{H}_+$, $\mathscr{H}_-$ is a canonical decomposition of $\mathscr{H}$,

(b) V_0 is a Q-unitary operator on $\mathscr{H}$ which satisfies condition A w.r.t. $\mathscr{H}_+$, $\mathscr{H}_-$,

(c) $V_1, \ldots, V_N$ are N Q-unitary operators on $\mathscr{H}$ such that V_0, $V_1, \ldots, V_N$ scalar commute,
then there is a maximal positive subspace of $\mathscr{H}$ invariant under $V_0, \ldots, V_N$.

THEOREM 3.1. *If N is an integer, the induction hypothesis is true for N.*

Proof. The proof is by induction.

For $N = 0$.

The operator V_0 has compact off diagonal terms w.r.t. $\mathscr{H}_+$, $\mathscr{H}_-$ and so Result A in the introduction to this paper implies that V_0 has an invariant maximal positive subspace.

For an arbitrary integer N.

We shall assume that the induction hypothesis is true for $N - 1$ and prove that it is true for N.

Let $\mathscr{H}$ be a subspace of H and let $U_0, \ldots, U_N$ be operators on $\mathscr{H}$ such that requirements (a), (b) and (c) of I. H;N are met. We shall take the inner product on $\mathscr{H}$ to be the inner product associated with $\mathscr{H}_+$, $\mathscr{H}_-$; $Q(,)$ is an indefinite inner product on $\mathscr{H}$ and has the form (I.1) with respect to $\mathscr{H}_+$, $\mathscr{H}_-$. Suppose that P is a positive subspace of $\mathscr{H}$, that $U_i P = P$ for $i = 0, 1, \ldots, N$, and that no other subspace of $\mathscr{H}$ which has this property properly contains P. Such a subspace exists by Zorn's lemma. We shall prove that P is a maximal positive subspace of $\mathscr{H}$. Note that P is compactly positive w.r.t. $\mathscr{H}_+$, $\mathscr{H}_-$, since it is invariant under U_0 and U_0 satisfies condition A w.r.t. $\mathscr{H}_+$, $\mathscr{H}_-$ (Lemma 2.7).

Case (a). *Suppose that P is strictly positive.* (Note: This includes the case where P is trivial.) Let P' be the Q-orthogonal complement of P in the subspace $\mathscr{H}$. The operators U_i satisfy $U_i P' = P'$ and so if we set $V_i = U_{i|P'}$ for $i = 0, 1, \ldots, N$, each V_i is an operator from P'

into itself. Now Lemma 3.1 implies that P' is Q-regular, that P'_+, P'_- as defined in F5 is a canonical decomposition of P', and that V_0 satisfies condition A for P'_+, P'_-. Thus P' and V_0 satisfy requirements (a) and (b) of I. H; N. Since the operators $U_0, \ldots, U_N$ scalar commute, the operators $V_0, \ldots, V_N$ scalar commute and requirement (c) of I. H; N is satisfied.

Since all requirements of I. H; N are met all requirements of I. H; $N - 1$ are met. We have assumed that I. H; $N - 1$ is true, and so we may conclude that there is a maximal positive subspace of P' which is invariant under $V_0, \ldots, V_{N-1}$. Now if P' is not strictly negative (i.e. $P_+' = \{0\}$) then Theorem 2.1 implies that there is a non-trivial positive subspace R of P' which satisfies $V_i R = R$ for $i = 0, \ldots, N$. Since P and R are Q-orthogonal, $P + R$ is a positive subspace of $\mathscr{H}$. Furthermore, $U_i[P + R] = U_i P + V_i R = P + R$ for $i = 0, \ldots, N$. Either $P + R$ properly contains P, or P' is a negative subspace of $\mathscr{H}$. The former can not be true by our definition of P and the latter is equivalent to saying that P is a maximal positive subspace of $\mathscr{H}$.

Case (b). *Suppose that P contains non-trivial null vectors* and let S_0 denote the subspace of null vectors in P. See Lemma 3.2 for the definitions of the terminology which follows. Lemma 4.1 [5] says that if R is a positive subspace of $\mathscr{H}$ which contains S_0, then R is contained in $\hat{\mathscr{H}} \oplus S_0$. Thus $\hat{R} = \beta R$ is defined and is a positive subspace of $\hat{\mathscr{H}}$. Certainly R is equal to $\hat{R} \oplus S_0$.

The operators U_i satisfy $U_i S_0 = S_0$ for $i = 0, \ldots, N$ because $U_i P = P$ for $i = 0, \ldots, N$. The calculation (3.3) applies to the operators $\hat{U}_i = \beta U_i \,|_{\hat{\mathscr{H}}}$ for $i = 1, \ldots, N$ as well as to the case $i = 0$ for which it was done and implies that each $\hat{U}_i$ is a Q-unitary operator on $\hat{\mathscr{H}}$. In addition, $\hat{U}_i^{-1} = \beta U_i^{-1} \,|_{\hat{\mathscr{H}}}$ and the positive subspace $\hat{P} = \beta P$ of $\hat{\mathscr{H}}$ inherit the property $\hat{U}_i \hat{P} = \hat{P}$ for $i = 0, \ldots, N$. It can be readily shown that the operators $U_0, \ldots, U_N$ scalar commute. By Lemma 3.2 $\hat{U}_0$ satisfies condition A w.r.t. $\hat{\mathscr{H}}_+$, $\hat{\mathscr{H}}_-$.

Suppose that there is a positive subspace $\hat{S}$ of $\hat{\mathscr{H}}$ which properly contains $\hat{P}$ and which satisfies $\hat{U}_i \hat{S} = \hat{S}$ for $i = 0, \ldots, N$. Then $\hat{S} \oplus S_0$ is a positive subspace of $\mathscr{H}$, and it contains $P = \hat{P} + S_0$ properly. Furthermore, the calculation (3.4) applies to U_i and $\hat{S}$ to give $U_i[\hat{S} \oplus S_0] = \hat{S} \oplus S_0$ for $i = 0, \ldots, N$. This is contrary to the definition of P. Therefore $\hat{P}$ is a positive subspace of $\hat{\mathscr{H}}$ such that $\hat{U}_i \hat{P} = \hat{P}$ for $i = 0, \ldots, N$ and which is maximal with respect to this property. Since S_0 contains all null vectors in P, the subspace $\hat{P}$ is strictly positive. Thus $\hat{P}$ in $\hat{\mathscr{H}}$ satisfies the same conditions that were on P and $\mathscr{H}$ in case (a) of this proof. Thus from case (a) we may conclude that $\hat{P}$ is maximal positive in $\hat{\mathscr{H}}$.

All that remains is for us to show that P is maximal positive in $\mathscr{H}$. If any positive subspace S of $\mathscr{H}$ properly contains P, then the positive subspace $\hat{S} = \beta S$ properly contains $\hat{P} = \beta P$, which contradicts the fact that $\hat{P}$ is maximal positive in $\hat{\mathscr{H}}$.

Proof of Theorem 2. The first step is to reduce the problem to the case where $J_0 = 0$. Define a map S on $\mathscr{G}_1$ by

$$S(\mathfrak{F}) = T_{J_0} \circ \mathfrak{F} \circ T_{J_0}^{-1}$$

for $\mathfrak{F} \in \mathscr{G}_1$. Set $\mathscr{S}' = \{f : f = S(\mathfrak{F}) \text{ for some } \mathfrak{F} \in \mathscr{S}\}$. Then, since $S(\mathfrak{F}_0)$ belongs to $\mathscr{S}'$, the group $\mathscr{S}'$ satisfies the hypothesis of Theoren 2 with $J_0 = 0$. Moreover, if $\mathscr{S}'$ has a fixed point M then $T_{J_0}^{-1}(M)$ is a fixed point of $\mathscr{S}$. Thus we may assume that $J_0 = 0$; in other words, $\mathfrak{F}_0$ satisfies condition (2.7).

As in 1a we associate a scalar commutative family of Q-unitary operators Γ with $\mathscr{S}$. The operator U_0 associated with $\mathfrak{F}_0$ satisfies condition A w.r.t. H_+, H_-, since $\mathfrak{F}_0$ satisfies (2.7). The following argument is due to Naimark [3]. If $\mathfrak{F} \in \mathscr{S}$, let $F_{\mathfrak{F}} = \{J \in \mathscr{B} : \mathfrak{F}(J) = J\}$. Since $\tilde{\mathfrak{F}}(0) = \tilde{\mathfrak{F}}[\tilde{\mathfrak{F}}_0(0)] = \tilde{\mathfrak{F}}_0[\tilde{\mathfrak{F}}(0)]$ condition (2.7) implies that $\tilde{\mathfrak{F}}(0) = 0$, and so by Lemma 1.6 $\mathfrak{F}$ is continuous in the weak operator topology. The fact that $\mathfrak{F}$ is continuous in the weak operator topology implies that $F_{\mathfrak{F}}$ is closed in the weak operator topology. The set $\mathscr{B}$ is compact in the weak operator topology which implies that the collection of $F_{\mathfrak{F}}$'s has the finite intersection property. Theorem 3.1 implies that (I.H.N.) is true for any N operators on H that belong to Γ since H is a Q-regular subspace of H and U_0 satisfies condition A w.r.t. H_+, H_-, however, this is just the geometric form of the statement that every finite collection of $F_{\mathfrak{F}}$'s has non-empty intersection.

Proof of Corollary 1. Suppose that M is a Q-unitary operator which has the form

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + K \text{ w.r.t. } H_+, H_- \tag{3.5}$$

where K is a compact operator and $\sigma(A)$ does not intersect $\sigma(D)$. Let $\mathfrak{F}_0$ denote the general symplectic map associated with M. We shall now prove that $\mathfrak{F}_0$ satisfies (2.7), in other words, $\mathfrak{F}_0$ satisfies the hypothesis of Theorem 1.

The equation $\tilde{\mathfrak{F}}_0(\tilde{J}) = \tilde{J}$ is equivalent to the equation $\tilde{J} = (\tilde{D}\tilde{J} + 0)(\tilde{A} + 0)^{-1}$ as one can see from (3.5) and (1.3). Thus

$$\tilde{\mathfrak{F}}_0(\tilde{J}) = \tilde{J} \quad \text{if and only if} \quad \tilde{J}\tilde{A} - \tilde{D}\tilde{J} = 0. \tag{3.6}$$

Define the map T: $\mathscr{L}(H_+, H_-)/\mathscr{C} \to \mathscr{L}(H_+, H_-)/\mathscr{C}$ by

$$T(\tilde{K}) = \tilde{K}\tilde{A} - \tilde{D}\tilde{K} \tag{3.7}$$

for each element $\tilde{K}$ of $\mathscr{L}(H_+, H_-)/\mathscr{C}$. M. Rosenblum [7] has shown that a map T of a Banach algebra into itself of the form (3.7) is invertible if and only if $\sigma(\tilde{A})$ and $\sigma(\tilde{D})$ are disjoint. Unfortunately $\mathscr{L}(H_+, H_-)/\mathscr{C}$ is not a Banach algebra and Rosenblum's Theorem [7] does not apply directly to the case at hand. This difficulty is easy to remedy. Define

$$A_1 = AE_+ + \lambda_A E_-$$

$$D_1 = DE_- + \lambda_D E_+$$

where $\lambda_A \in \sigma(A)$ and $\lambda_D \in \sigma(D)$. The operators A_1 and D_1 belong to the Banach algebra $\mathscr{L}(H, H)$ and their specturm is disjoint since $\sigma(A)$ and $\sigma(D)$ are disjoint. This implies that $\tilde{A}_1$ and $\tilde{D}_1$ have disjoint spectra as elements of the Banach algebra and consequently by Corollary 3.3 (ii) in [7] the map

$$T_1(\tilde{K}) = \tilde{K}\tilde{A}_1 - \tilde{D}_1\tilde{K}$$

is invertible. Thus $T_1(\tilde{K}) = 0 \Rightarrow \tilde{K} = 0$. Now suppose that $J \in \mathscr{B}$ and that $\mathfrak{F}_0(J) = J$. From (3.6) and (3.7) we see that $T(\tilde{J}) = 0$. However $\widetilde{JE_+}$ belongs to $\mathscr{L}(H, H)/\mathscr{C}$ and $T_1(\widetilde{JE_+}) = T(\tilde{J}) = 0$. Therefore, $\widetilde{JE_+} = 0$ and consequently $\tilde{J} = 0$. Thus, we have shown that $\mathfrak{F}_0$ satisfies (2.7).

References

1. Yu. P. Ginzburg and I. S. Iokhidov, The geometry of infinite dimensional spaces with a bilinear metric, *Russian Math. Surveys* **17**, No. 4 (1962), 1–51.
2. M. G. Krein, A new application of the fixed point principle in the theory of operators on a space with an indefinite metric, *Dokl. Akad. Nauk SSSR* **154** (1965), 1023–1026; *Soviet Math. Doklady* **5** (1964), 224–228.
3. M. A. Naimark, On commuting unitary operators in spaces with indefinite metric, *Acta Sci. Math.* **24** (1963), 177–189.

4. R. S. Phillips, The extension of dual subspaces invariant under an algebra, Proc. International Symp. on Linear Spaces, Israel (1960), 366–398.
5. R. S. Phillips, On dissipative operators, Technical Report Georgetown University Lecture Series in Differential Equations (1966), 1–64.
6. R. S. Phillips, On symplectic mappings of contraction operators, *Studia Math.* **31** (1968), 1–27.
7. M. Rosenblum, On the operator equation $BX - XA = Q$, *Duke Math. J.* **23** (1956), 263–270.