Abstract. This article concerns the positive semidefinite matrices $M_+(G)$ with zero entries in prescribed locations; that is, matrices with given sparsity graph $G$. The issue here is the rank of the extremals of the cone $M_+(G)$. It was shown in [J. Agler, J. W. Helton, S. McCullough, and L. Rodman, Linear Algebra Appl., 107 (1988), pp. 101–149] that the key in constructing high rank extreme points resides in certain atomic graphs $G$ called blocks and superblocks. The $k$-superblocks are defined to be sparsity graphs $G$ that contain an extreme point of rank $k$ while containing (in an extremely strong sense) no graph with the same property. The goal of this article is to write down all graphs that are superblocks. The article succeeds completely for $k \leq 4$ and it lists necessary conditions in general as well as sufficient conditions. The subject is closely related to orthogonal representations of graphs as studied earlier in [L. Lovász, M. Saks, and A. Schrijver, Linear Algebra Appl., 114/115 (1989), pp. 439–454] and in the previously mentioned paper by Alger et al. Indeed, the paper is an extension of the findings of Alger et al.

Key words. extremal matrix, order of a graph, superblocks, orthogonal representation

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Introduction. Let $G$ be an undirected graph without multiple edges or loops. Let $V(G) (= \{1, \ldots, n\})$ denote the set of vertices of $G$ and $E(G) \subseteq V(G) \times V(G)$ the set of edges. Note that the absence of loops means that $(i, i) \notin E(G)$, $i = 1, \ldots, n$. Define $M_+(G)$ to be the closed cone of all positive semidefinite $n \times n$ real symmetric matrices whose $(i, j)$ entry is zero whenever $(i, j) \in E(G)$. (Note the difference in definition compared to preceding papers on the subject ([AHMR], [HPR]); in those papers the zero entries would correspond to edges in the complementary graph.)

A matrix $A$ in $M_+(G)$ is called an extremal when each additive decomposition of $A$ in $M_+(G)$ is a trivial one, i.e., $A$ is an extremal when $A = B + C$ with $B, C \in M_+(G)$ yields that $B, C$ span $\{A\}$. We say that $G$ has order $k$ if $k$ is the maximum of the ranks of extremals in $M_+(G)$.

We are interested in determining the order of a given graph. The graphs of order 1 have a very elegant characterization (see [AHMR], [PPS]) based on the main result in [GJSW]. The general case turns out to be very hard, and, therefore, some reductions must be made. Recall from [AHMR] the following definitions. A graph $G$ is called a $k$-block if $G$ has order $k$ but no induced subgraph has order $k$. (The graph $\hat{G}$ is an induced subgraph of $G$ if $V(\hat{G}) \subseteq V(G)$ and $E(\hat{G}) = E(G) \cap (V(\hat{G}) \times V(\hat{G}))$.) In terms of matrices this means that for a $k$-block $G$ any rank $k$ extremal in $M_+(G)$ does not have zero rows or columns. A full description of $k$-blocks, $k = 1, 2, \ldots$, would give a solution to our problem, since the order of a graph equals the maximal $k$ for which the graph has a $k$-block as an induced subgraph ([AHMR, Thm. 1.2]). Classifying $k$-blocks is based on the study of much better behaved objects called $k$-superblocks.
A graph $G$ is called a $k$-superblock when it is a $k$-block that does not properly contain another $k$-block. (In this paper “$G$ is contained in $G$” always means $V(G) \subseteq V(G)$ and $E(G) \subseteq E(G)$.) In terms of matrices, this means that as soon as you allow some of the zero entries prescribed by $G$ to be nonzero there are no extremals of rank $k$ anymore. It is true (see [AHMR]) that any $k$-block (or any order $k$ graph, for that matter) contains a $k$-superblock. However, to obtain all $k$-blocks assuming one knows how to characterize $k$-superblocks still requires work.

The following theorem gives necessary conditions for a sparsity pattern to be a $k$-superblock.

**THEOREM 0.1.** Let $G$ be a $k$-superblock. Then the following are true:

1. $|E(G)| \leq \frac{1}{2}(k+2)(k-1)$;
2. $G$ contains no $K_{p,q}$, $p+q \geq k+1$;
3. For all $i_1, \ldots, i_m \in V(G)$ with $1 \leq m < k$ we have that

\[
\begin{align*}
\# \{(i,j) \in E(G) & \mid i \text{ or } j \in \{i_1, \ldots, i_m\}\} \\
&< \frac{1}{2}(k+2)(k-1) - \frac{1}{2}(k-m+2)(k-m-1) = \frac{1}{2}m(2k+1-m)
\end{align*}
\]

Conversely, when $k = 1, 2, 3$, or $4$ these conditions imply that $G$ is a $k$-superblock.

Here $K_{p,q}$ denotes the bipartite graph described by

\[V(K_{p,q}) = \{1, \ldots, p+q\}, \quad E(K_{p,q}) = \{(i,j) \mid 1 \leq i \leq p, \; p+1 \leq j \leq p+q\} .\]

The necessary conditions (i) and (ii) were established earlier in [AHMR]. Condition (iii) is implied by (i) and (ii) when $k = 1, 2, 3$ but not when $k \geq 4$. For $k = 1, 2, 3$ the $k$-superblocks were described earlier in [AHMR], and indeed they are precisely the graphs which satisfy the necessary conditions in Theorem 0.1. For $k = 4$ this is also true (as stated in Theorem 0.1). This follows from the description of 4-superblocks given in the next theorem, which is the second main result in this paper.

**THEOREM 0.2.** Let $G$ be a graph with nine edges. The following are equivalent:

1. $G$ is a 4-superblock;
2. $G$ cannot be obtained from

\[
\begin{align*}
\text{or } & \quad \text{or } \\
\end{align*}
\]

by identifying vertices;¹
3. $G$ is a graph which, after identifying vertices, is one of the following 28 graphs:

\[
\begin{align*}
G_1 = & \quad G_2 = & \quad G_3 = \\
G_4 = & \quad G_5 = & \quad G_6 = \\
G_7 = & \quad G_8 = & \quad G_9 = \\
\end{align*}
\]

¹See the definition of a collapse of a graph in §2.
Note that (ii) in Theorem 0.2 is a restatement of the necessary conditions in Theorem 0.1 for \( k = 4 \).

In §2 we develop some results which give sufficient conditions for a graph to be a \( k \)-superblock. Unfortunately these conditions do not equal the necessary conditions in Theorem 0.1. In the case when \( k = 4 \), for instance, 17 of the 28 graphs in Theorem 0.2(iii) meet the necessary conditions of Theorem 0.1, but not our general sufficient conditions. To prove that these 17 graphs are 4-superblocks, we used a computer program employing Mathematica (using integer arithmetic). It is natural to ask whether this gap in the theory can be dissolved in the following way.

**SPECULATION 0.3.** Let \( G \) be a graph satisfying (i), (ii), and (iii) in Theorem 0.1. Then \( G \) is a \( k \)-superblock.

We shall point out in the end of §1 what remains to be done in order to prove this speculation. We used our Mathematica program to check some likely candidates for counterexamples (with \( k = 5 \) and 6), but so far we have been unsuccessful (partly because the program is very slow when \( k \) is large).

1. **Making extremals in** \( M_+(G) \). From [AHMR] one can deduce the following recipe for making all extremals in \( M_+(G) \) of rank \( k \).

Let \( G \) be a graph, and let \( k \leq \#V(G) \).
(Step I) Find an assignment \( f: V(G) = \{1, \ldots, n\} \to \mathbb{R}^k \) such that
(a) \( (f(i), f(j)) = 0, \ (i, j) \in E(G); \)
(b) \( \text{span} \{ f(j) \mid j \in V(G) \} = \mathbb{R}^k. \)

(Step II) Check if all \( M = MT \in \mathbb{R}^{k \times k} \) satisfying
\[
\langle Mf(i), f(j) \rangle = 0, \quad (i, j) \in E(G),
\]
are multipliers of the \( k \times k \) identity matrix \( I_k. \)

(Step III) If so, then
\[
\begin{pmatrix}
(f(1)^T \\
\vdots \\
f(n)^T
\end{pmatrix}
\begin{pmatrix}
f(1) \\
\vdots \\
f(n)
\end{pmatrix}
\]
is an extremal of rank \( k \) in \( M_+(G). \)

An assignment \( f: V(G) \to \mathbb{R}^k \) such that (a) in Step I holds is called an orthogonal representation of \( G. \) Such representations were introduced and studied independently in [LSS] and [AHMR] (for quite different reasons).

Example. Let
\[
G = \begin{array}{ccccc}
1 & \rightarrow & 2 & \rightarrow & 3 \\
6 & \rightarrow & 5 & \rightarrow & 4
\end{array}
\]
are extremals in \( M_+(G) \) of ranks 1 and 2, respectively. There are no extremals of rank \( \geq 3 \) because of dimension counting. Indeed, if \( k = 3, \) either \( f(5) = 0 \) or span \( \{ f(2), f(3), f(6) \} \) has dimension of at most 2. This results in at most four constraints, appearing in (1.1), on \( M. \) This can never force \( M \) to be a scalar multiple of \( I_3. \)

Furthermore, when \( k > 3, \) the five constraints (1.1) on \( M \) are too few to force \( M \) to be a scalar multiple of \( I_k. \)

The type of arguments used in the example led in [AHMR] to the following result.

Let \( G \) be a graph. If order \( G = k, \) then \#\( E(G) \geq \frac{1}{2} (k + 2)(k - 1). \) Furthermore, if order \( G = k \) and \#\( E(G) = \frac{1}{2} (k + 2)(k - 1), \) then \( G \) contains no \( K_{p,q}, \) \( p+q \geq k + 1. \)

Let us now prove that a \( k \)-superblock must satisfy the conditions (i), (ii), and (iii) in Theorem 0.1.

Proof of the necessary part of Theorem 0.1. Let \( G \) be a \( k \)-superblock. In particular, order \( G = k, \) and thus the recipe works for a representation \( f, \) say. Let \( i_1, \ldots, i_k \in V(G) \) be such that \( f(i_j), \ j = 1, \ldots, i_k, \) span \( \mathbb{R}^k. \) Furthermore, choose
distinct \(i_{k+1}, \ldots, i_l\) such that each edge in \(E(G)\) has an endpoint in \(\{i_1, \ldots, i_l\}\) and write

\[
f(i_p) = \sum_{j=1}^{k} \lambda_j^{(p)} f(i_j), \quad p = k + 1, \ldots, l.
\]

Let \(M\) be a \(k \times k\) matrix, and put \(w_j = MF(i_j), j = 1, \ldots, k\). Let \(u_1^{(j)}, \ldots, u_{p_j}^{(j)}\) denote the vertices adjacent to \(i_j\) that are not in the set \(i_1, \ldots, i_{j-1}\). Put

\[
U_j = [f(u_1^{(j)}) \cdots f(u_{p_j}^{(j)})], \quad j = 1, \ldots, l.
\]

Put

\[
W = \begin{bmatrix}
U_1 & \lambda_1^{(k+1)} U_{k+1} & \cdots & \lambda_1^{(l)} U_l \\
U_2 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \lambda_k^{(k+1)} U_{k+1} & \cdots & \lambda_k^{(l)} U_l \\
U_k & \lambda_1^{(k+1)} U_{k+1} & \cdots & \lambda_1^{(l)} U_l \\
\end{bmatrix}
\]

and

\[
\Sigma = \begin{bmatrix}
-f(i_2) & -f(i_3) & 0 & \cdots & -f(i_k) & 0 & \cdots & 0 \\
-f(i_1) & 0 & -f(i_3) & \cdots & 0 & -f(i_k) & \cdots & 0 \\
0 & f(i_1) & f(i_2) & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & f(i_1) & f(i_2) & \cdots & f(i_{k-1})
\end{bmatrix}.
\]

That is to say, if \(\frac{1}{2} j(j-1) < p \leq \frac{1}{2} (j+1)j\) then column \(p\) in \(\Sigma\) has on the \(p-\frac{1}{2} j(j-1)\) row the entry \(-f(i_{j+1})\) and on the \((j+1)\) row the entry \(f(i_{p-j(j-1)/2})\) \((j = 1, \ldots, k-1)\).

Note that \(W\) is of size \(k \times \#E(G)\) and \(\Sigma\) of size \(k \times \frac{1}{2} k(k-1)\). The equations (1.1) are equivalent to

\[
\text{col}(w_i)_{i=1}^k \in \text{cokernel } W
\]

and the symmetry of \(M\) is ensured by

\[
\text{col}(w_i)_{i=1}^k \in \text{cokernel } \Sigma.
\]

Checking Step II in the recipe now comes down to checking that

\[
(1.2) \quad \text{cokernel } [W, \Sigma] = \text{span} \{\text{col}(f(i_j))_{j=1}^k\}.
\]

Note that the inclusion \(\supset\) in (1.2) is always fulfilled since \(f\) is an orthogonal representation. We assume that the recipe works, and therefore (1.2) holds. Suppose that \(\#E(G) > \frac{1}{2} (k + 2)(k - 1)\), then the number of columns in \([W, \Sigma]\) is \(\geq k^2\). Since the columns in \(\Sigma\) are linearly independent, and because of (1.2), we can remove a column in \(W\) without changing the cokernel of \([W, \Sigma]\). But removing a column in \(W\) corresponds to removing an edge in \(G\), yielding that \(G\) properly contains a \(k\)-block. Thus \(\#E(G) = \frac{1}{2} (k + 2)(k - 1)\).

Now it follows from the quoted result before the proof that (ii) holds. It remains to prove (iii). First note that if the recipe works for \(f\), it also works for an orthogonal representation \(\tilde{f}\) with the property that \(\|f(i) - \tilde{f}(i)\|\) is small enough. Indeed, such a perturbation will not destroy the invertibility of a \((k^2 - 1) \times (k^2 - 1)\) invertible submatrix.
of \([W, \Sigma]\). Since the graph does not contain any \(K_{p,q}\)'s, \(p + q \geq k + 1\), we know from [LSS] that in any neighborhood of \(f\) we can find an orthogonal representation \(\tilde{f}\) that has the property that any set of \(k\) representing vectors are linearly independent (in the terms of [LSS]: \(f\) is in general position). Thus without loss of generality we may assume that \(f\) has the latter property. Choose now \(i_1, \ldots, i_m \in V(G), m < k\), arbitrary. Then \(f(i_1), \ldots, f(i_m)\) are linearly independent. Choosing the \(i_1, \ldots, i_m\) as the first \(m\) vertices in \(\{i_1, \ldots, i_k, i_{k+1}, i_1\}\) we can set up the matrix \(W\) and \(\Sigma\) as before. After permutation of columns of the matrix \([W, \Sigma]\) we obtain the matrix

\[
(1.3) \quad \begin{bmatrix}
\Lambda_{11} & \Lambda_{12} \\
0 & \Lambda_{22}
\end{bmatrix},
\]

where

\[
\Lambda_{11} = \begin{bmatrix}
U_1 & -f(i_2) & \cdots & -f(i_m) & 0 & \cdots & 0 \\
U_2 & f(i_1) & 0 & f(i_m) & 0 \\
\vdots & 0 & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
U_m & 0 & f(i_1) & f(i_2) & \cdots & f(i_{m-1})
\end{bmatrix}.
\]

Note that

\[
\text{col}(f(i_j))_{j=1}^{m} \in \text{coker} \Lambda_{11}.
\]

But then, since the cokernel of (1.3) should have dimension exactly equal to 1, the matrix \(\Lambda_{22}\) should have at least as many columns as rows. Since \(\Lambda_{22}\) is of size

\[
(k^2 - mk) \times \left( (k^2 - 1) - \left( \#\{(i,j) \in E(G) \mid i \text{ or } j \in \{i_1, \ldots, i_m\} \} + \frac{1}{2} m(m-1) \right) \right),
\]

this inequality precisely yields (0.1).

In order to prove Speculation 0.3, it remains to prove that for a graph satisfying (i), (ii), and (iii) there is an orthogonal representation \(f\) such that the matrix \([W, \Sigma]\), constructed in the proof of Theorem 0.1, has a one-dimensional cokernel.

2. A sufficiency result. For a vertex \(v \in V(G)\) the degree is defined to be the number of adjacent vertices.

**Theorem 2.1.** Let \(P\) be a graph with an induced subgraph \(G\) that satisfies

(i) \(\#E(G) \geq \frac{1}{2} (k + 2)(k - 1)\);

(ii) \(G\) contains no \(K_{p,q}\)'s, \(p + q \geq k + 1\);

(iii) for any \(\{i_1, \ldots, i_m\} \subset V(G)\) with \(1 \leq m \leq k - 1\),

\[
\begin{align*}
\#\{(i,j) \in E(G) \mid i \text{ or } j \in \{i_1, \ldots, i_m\} \} & < \#E(G) - \frac{1}{2} (k - m + 2)(k - m - 1) ;
\end{align*}
\]

(iv) \(G\) has \(k-1\) vertices \(\{u_1, \ldots, u_{k-1}\}\) of degree \(k-1\) such that one of the vertices \(u_j\) (\(1 \leq j \leq k - 1\)) is not adjacent to any \(u_i, j \neq i \in \{1, \ldots, k - 1\}\).

Then order \(P \geq k\).

The conclusion remains true when (iv) is replaced by

(v) \(G\) has at least \(k\) vertices of degree \(k-1\), and, after deleting \(m\) of these vertices, the remaining graph always contains a \(K_{p,q}\), \(p + q \geq k - m + 1, m = 1, \ldots, k - 2\).
It should be noted that it follows from the last part of the proof of Theorem 0.1 that (iii) is a necessary condition for $G$ to be of order $\geq k$.

Before proving the theorem, let us make some remarks. Condition (iv) is a very stringent one since it requires the subgraph $G$ to be fairly condensed. However, combined with the next result from [AHMR] the theorem shows that a substantial number of graphs have order $\geq k$.

We introduce the following partial ordering on graphs. We say that $G \leq_e \tilde{G}$ ($G$ is a collapse of $\tilde{G}$) if $G$ can be obtained from $\tilde{G}$ by identifying vertices without identifying edges. For example,

$$
\begin{align*}
\begin{array}{c}
\text{PROPOSITION 2.2. Let } G \text{ and } \tilde{G} \text{ be graphs satisfying } G \leq_e \tilde{G}. \text{ Then } \\
\text{order } G \leq \text{ order } \tilde{G}.
\end{array}
\end{align*}
$$

This is a restatement of Theorem 4.7 in [AHMR].

*Example.* Theorem 2.1 and Proposition 2.2 together prove that the order of

$$
\begin{align*}
\begin{array}{c}
\text{(2.3)}
\end{array}
\end{align*}
$$

is $\geq 4$. Indeed, the graph

$$
\begin{align*}
\begin{array}{c}
\text{(2.4)}
\end{array}
\end{align*}
$$

has order $\geq 4$ by Theorem 2.1, and (2.4) is a collapse of (2.3). Therefore, (2.3) has order $\geq 4$. In fact, (2.3) has precisely order 4 since, by Theorem 0.1, a graph of order $\geq 5$ should at least have 14 edges, the number of edges of a 5-superblock. \(\square\)

In order to prove Theorem 2.1, we need some auxiliary results. Recall that an orthogonal representation $f : V(G) \rightarrow \mathbb{R}^k$ is in general position if any set of $k$ representing vectors is linearly independent.

**Lemma 2.3.** Let $G$ be a graph that does not contain $K_{p,q}$'s $p + q \geq k + 1$, and let $f : V(G) \rightarrow \mathbb{R}^k$ be an orthogonal representation in general position. If $M = M^T \in \mathbb{R}^{k \times k}$ satisfies (1.1), then for any $v \in V(G)$ with degree equal to $k - 1$ the vector $f(v)$ is an eigenvector of $M$.

**Proof.** Both $f(v)$ and $Mf(v)$ belong to the orthogonal complement of span $\{f(u) \mid (u, v) \in E(G)\}$. Since $\deg v = k - 1$, the vectors $Mf(v)$ and $f(v)$ both belong to a one-dimensional space. \(\square\)
Recall from [LSS] that an orthogonal representation $f$ of a graph $G$ is called \textit{faithful} if $\langle f(u), f(v) \rangle = 0$ if and only if $(u, v) \in E(G)$.

\textbf{Lemma 2.4.} Let $G$ be a graph that contains no $K_{p,q}$'s, $p + q \geq k + 1$. Further, suppose that $u$ is a vertex of degree $k - 1$ that has $r$ nonadjacent vertices of degree $k - 1$. Then for any faithful orthogonal representation $f$ of $G$ in $\mathbb{R}^k$ in general position the set of symmetric matrices $M$ satisfying (1.1) is either span $\{I_k\}$ or contains an element of rank $< k - r$.

\textit{Proof.} Let $M$ be symmetric such that (1.1) holds. Since $f(u)$ is an eigenvector of $M$ (at $\lambda_0$, say), which is not orthogonal to $r$ other eigenvectors, the dimension of the eigenspace of $M$ at $\lambda_0$ is at least the dimension of the span of $f(u)$ and these other $r$ eigenvectors. Since $f$ is in general position we obtain that

$$\text{rank } (M - \lambda_0 I) \leq \max \{0, k - (r + 1)\}.$$ 

Since $M - \lambda_0 I$ is symmetric and satisfies (1.1), the lemma is proved. \hfill \Box

\textbf{Proposition 2.5.} Let $G$ be a graph that contains no $K_{p,q}$'s, $p + q \geq k + 1$. Then for every orthogonal representation $f : V(G) \to \mathbb{R}^k$ there exists a symmetric matrix $M$ of rank 1 satisfying (1.1) if and only if there is a set $V$ of at most $k - 1$ vertices in $G$ such that any edge in $G$ has an endpoint in $V$.

\textit{Proof.} Suppose such a set $V$ exists. Choose $0 \neq w \in \mathbb{R}^k$ such that $\langle w, f(v) \rangle = 0$ for any $v \in V$. It is easy to check that $M := ww^T$ satisfies (1.1).

In order to prove the \textit{only if} part, let $f$ be an orthogonal representation in general position (such an $f$ exists: Theorem 1.1 in [LSS]). Also let $M = ww^T$ with $w \neq 0$ satisfy (1.1). Then for all edges $(i,j) \in E(G)$

$$\langle w, f(i) \rangle \langle w, f(j) \rangle = 0,$$

thus $w$ is orthogonal to one of the endpoints of each edge in $G$. Since $w$ can be orthogonal to at most $k - 1$ linearly independent vectors we obtain the proposition above. \hfill \Box

We are now ready to prove Theorem 2.1.

\textit{Proof of Theorem 2.1.} Suppose (i)-(iv) hold. We have to prove that order $G \geq k$.

Let $f : V(G) \to \mathbb{R}^k$ be in general position and faithful (existence is assured by Theorem 1.1 and (the proof of) Corollary 1.4 in [LSS]). Lemma 2.4 yields, because $G$ satisfies (iv), that either the set of symmetric matrices $M$ satisfying (1.1) is span $\{I_k\}$ or has an element of rank 1. Since $G$ satisfies condition (iii) in the theorem, the latter possibility is ruled out. (Since, if $M$ is symmetric of rank 1 satisfying (1.1), then Proposition 2.5 yields that (2.1) is violated (for $m = k - 1$).)

Now suppose that (i), (ii), (iii), and (v) hold. Let $f : V(G) \to \mathbb{R}^k$ be an orthogonal representation in general position. Further suppose that $M$ is positive semidefinite, satisfies (1.1) and has rank $d$, with $1 < d < k$. (We can always assume that $M$ is positive semidefinite, since if $M$ satisfies (1.1) then any linear combination of $M$ and $I_k$ satisfies (1.1) also.) Since $G$ has $k$ vertices of degree $k - 1$, the representing vectors corresponding to these vertices are a basis of eigenvectors of $M$. But then $k - d$ of these vertices represent the kernel. Delete those vertices. Then from (1.1) it easily follows that $M^{1/2} f$, defined by

$$(M^{1/2} f)(v) = M^{1/2} (f(v)),$$

is an orthogonal representation in general position of the remaining graph. Since (v) holds, this is impossible by Theorem 1.1 in [LSS]. As before, a symmetric $M$ satisfying
(1.1) cannot have rank 1. But then all symmetric $M$ satisfying (1.1) must be in span \( \{ I_k \} \).

\[ \square \]

3. 4-superblocks. In this section we prove Theorem 0.2.

Proof of Theorem 0.2. The implication (i) \( \Rightarrow \) (ii) is merely a restatement of the necessary conditions in Theorem 0.1 in this special case.

For the proof of (ii) \( \Rightarrow \) (iii) we determine the \( \leq_c \)-minimal elements in the set of graphs \( \mathcal{G} \) described under (ii) in Theorem 0.1, i.e., \( \mathcal{G} \) is the set of graphs with nine edges that are not a collapse of one of the three graphs under (ii). The result is given in the following proposition.

Proposition 3.1. The \( \leq_c \)-minimal elements in \( \mathcal{G} \) are the graphs \( G_i, i = 1, \ldots, 28 \), defined in Theorem 0.1(iii).

The proof requires a lemma.

Lemma 3.2. Let \( G \in \mathcal{G} \) be \( \leq_c \)-minimal. Then \( G \) does not have a vertex \( u \) of degree 1 and a vertex \( v \) of degree \( \leq 2 \) such that the distance \( d(u, v) \) between \( u \) and \( v \) is larger than 2.

Proof. Suppose that \( G \) has vertices \( u \) and \( v \) with \( \deg u = 1 \), \( \deg v \leq 2 \) and \( d(u, v) > 2 \). By identifying \( u \) and \( v \) we do not create a degree-4 node. Suppose we create a \( K_{2,3} \). Then we must have had

\[
\text{(1)} \quad \text{or} \quad \text{(2)}
\]

with three other edges. In case (1) these three other edges do not form a subgraph \( K_{1,3} \); otherwise \( G \leq_c 3 \times K_{1,3} \), which contradicts \( G \in \mathcal{G} \). Here \( 3 \times K_{1,3} \) denotes the graph on the right-hand side of (2.2). But then \( G \) must be disconnected. This yields that we can identify one of the vertices in a connected component of \( G \) not containing \( u \) and \( v \) and obtain a graph in \( G \) that is \( \leq_c \)-smaller than \( G \). For the second possibility (2), the reasoning is similar.

Suppose that by identifying \( u \) and \( v \) we create a graph that is \( \leq_c \)-smaller than \( 3 \times K_{1,3} \). Since \( G \) has no vertices of degree 4, \( G \) must, in this case, have vertices of degree \( \leq 2 \) besides \( u \) and \( v \). But then \( u \) may be identified with one of these other vertices and stay in the class \( \mathcal{G} \).

Proof of Proposition 3.1. In the reasoning to follow we shall quite frequently use the fact that a nine-edge graph satisfies

\[
\sum_{u \in V(G)} \deg(u) = 18.
\]

Let us now determine the \( \leq_c \)-minimal elements \( G \) in \( \mathcal{G} \).

Case 1. \( G \) has a vertex \( u \) of degree 1.

Let \( v \) denote the vertex adjacent to \( u \). When \( \deg v = 1 \) the remaining vertices should have degree 3 (Lemma 3.2). This is impossible by (3.1). Consider now the case when \( \deg v = 2 \), and let \( w \neq u \) be the other neighbor of \( v \). The cases, \( \deg w = 1 \) and \( \deg w = 2 \), are quickly disregarded again by using Lemma 3.2 and (3.1). When \( \deg w = 3 \), the only graph one obtains by requiring that all other vertices have degree 3 (which must be the case because of Lemma 3.2) contains a \( K_{2,3} \) and is therefore not in \( \mathcal{G} \). Consequently, we are left with the case that \( \deg v = 3 \). Let \( \{u, w_1, w_2\} \) denote the adjacency set of \( v \). When \( \deg w_1 = \deg w_2 = 1 \) we obtain \( G_2 \) as the only possibility. The cases \( \{\deg w_1 = 1, \deg w_2 = 2\}, \{\deg w_1 = \deg w_2 = 2\} \), and
\{\deg w_1 = \deg w_2 = 3\} are quickly discarded, leaving the case \{\deg w_1 = 2, \deg w_2 = 3\}. From this we obtain \(G_1\) as the only possibility.

Case 2. All vertices of \(G\) have degree 2.

Then \(G\) must have nine vertices and consist of circuits. The only possibilities are \(G_3, G_4, G_5,\) and \(G_6\).

We are left with the cases that \(G\) has some vertices of degree 3 and some of degree 2. Because of (3.1), the number of degree-3 vertices must be even.

Case 3. \(G\) has two vertices of degree 3 and six of degree 2.

First we consider the case when the vertices of degree 3 are adjacent. Disconnected graphs with these requirements are easily recognized to be \(G_7\) and \(G_8\). Considering the possible paths between the vertices of degree-3 nodes, one obtains the graphs \(G_9, G_{10},\) and \(G_{11}\) in case there are three different paths; when there is only one path, the graphs \(G_{12}\) and \(G_{13}\) are obtained (note that two paths between the degree-3 nodes is impossible). In a similar way, one obtains the graphs \(G_{14}-G_{18}\) for the case when vertices of the degree 3 have distance 2 or 3.

Case 4. \(G\) has four vertices of degree 3 and three of degree 2.

Since \(G \not\subseteq 3 \times K_{1,3}\), there should be no three vertices of degree 3 that are all nonadjacent. Let \(u_1, u_2, u_3,\) and \(u_4\) denote the vertices of degree 3. When \(u_1, \ldots, u_4\) are all adjacent to one another, one obtains the only possibility, \(G_{26}\). When they form

\[
\begin{array}{c}
u_1 \\ | \\ v_2 \\ | \\ v_3 \\ | \\ v_4
\end{array}
\]

one obtains \(G_{25}\). When \(u_1, \ldots, u_4\) form a square \((K_{2,2})\), one obtains \(G_{23}\). When \(u_1, \ldots, u_4\) form

\[
\begin{array}{c}
u_4 \\ | \\ v_3 \\ | \\ v_2 \\ | \\ v_1
\end{array}
\]

one obtains \(G_{19}\) and \(G_{21}\). The case when \(u_1, \ldots, u_4\) form a connected line gives possibilities \(G_{20}, G_{22},\) and \(G_{28}\). The last case is when \(u_1, \ldots, u_4\) form a line of three vertices and an isolated vertex. This gives as the only possibility, \(G_{24}\).

Case 5. \(G\) only has vertices of degree 3.

Here \(G_{27}\) is the only possibility.

This proves Proposition 3.1. \(\Box\)

This concludes the proof of (ii) \(\implies\) (iii).

To prove (iii) \(\Rightarrow\) (i) we need to show that for the graphs \(G_1-G_{28}\) we can find a rank-4 extremal in \(M_+(G)\). Then Proposition 2.2 yields that all graphs under (iii) have a rank-4 extremal. The graphs \(G_1, G_2, G_{19}, G_{20}, G_{21}, G_{22}, G_{24}, G_{27},\) and \(G_{28}\) have a vertex of degree 3 that is not adjacent to two other vertices of degree 3. Therefore, by Theorem 2.1 (i)-(iv), the order of these graphs is \(\geq 4\). But then, since the numbers of edges is smaller than 14 we obtain that the order is at most 4, giving equality. The graph \(G_{26}\) has as its complement the graph \(K_{3,4}\). Consequently, \(G_{26}\) has order 4, by Theorem 6.1 in [HPR]. The graphs \(G_{23}\) and \(G_{28}\) are recognized to have order \(\geq 4\) by Theorem 2.1 using (i)-(iii) and (v).
The remaining graphs $G_3 - G_{18}$ and $G_{25}$ are dealt with by "brute force." A program using Mathematica (using integer arithmetic) produced for us the following rank-4 extremals in $M_+(G)$ for $G$ in $G_3 - G_{18}, G_{25}$. For each $G_i$ this rank-4 extremal is given by $A_i^T A_i$, where $A_i, i = 3, \ldots, 18, 25,$ is given by

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -2 & \frac{9}{22} & \frac{37}{4} & -\frac{32}{45} \\ 0 & 2 & -2 & -\frac{5}{2} & -\frac{9}{2} & -3 & \frac{8}{11} & -3 & -\frac{386}{135} \\ 0 & -3 & 3 & -2 & -1 & -3 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 & -3 & -2 & -3 & -2 & 2 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 2 & -3 & 0 & 0 & -3 & -5 & -6 & \frac{11}{7} \\ 0 & 1 & 3 & 2 & 4 & 3 & -5 & -1 & \frac{59}{7} \\ 0 & 3 & 2 & 2 & -1 & 3 & 3 & -3 & -3 \\ 0 & 2 & -3 & -3 & 2 & -3 & 3 & 3 & 2 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 1 & 3 & \frac{1}{3} & -\frac{2}{3} & -12 & 0 & 0 & 6 & \frac{13}{12} \\ 0 & 1 & 2 & -1 & 2 & -2 & \frac{7}{2} & 3 & \frac{1}{2} \\ 0 & -2 & 3 & -2 & 3 & 3 & 2 & -3 & 3 \\ 0 & 1 & 3 & -1 & -3 & 1 & 1 & -1 & -1 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 1 & 0 & -2 & 1 & \frac{1}{2} & 0 & -3 & -6 & \frac{14}{5} \\ 0 & -2 & -1 & -3 & 1 & 1 & -9 & -2 & -\frac{47}{5} \\ 0 & -3 & 3 & -1 & 2 & 2 & 5 & -1 & 3 \\ 0 & 3 & 2 & 1 & -2 & 1 & -1 & -1 & -1 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 1 & 2 & \frac{3}{2} & 0 & 0 & \frac{3}{2} & -1 & -1 & -1 \\ 0 & -1 & 1 & 3 & \frac{2}{3} & -1 & \frac{1}{2} & -\frac{9}{11} & \frac{9}{11} \\ 0 & 1 & -1 & -2 & 2 & -2 & -2 & \frac{8}{11} & \frac{8}{11} \\ 0 & 1 & -1 & 1 & 2 & -2 & 1 & 1 & 1 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & -14 & 16 & -\frac{9}{50} & -\frac{9}{50} \\ 0 & -3 & -1 & 3 & -\frac{8}{3} & -9 & 2 & -\frac{103}{50} & -\frac{103}{50} \\ 0 & -3 & -1 & 2 & 2 & 3 & 3 & 3 & 1 \\ 0 & -2 & 3 & 2 & 1 & -2 & 2 & 2 & 2 \end{pmatrix}$$

$$A_9 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -\frac{15}{2} & -\frac{2}{5} & \frac{97}{8} & \frac{97}{8} \\ 0 & 2 & 6 & -\frac{11}{2} & \frac{5}{2} & -1 & 3 & -\frac{1}{20} & -\frac{1}{20} \\ 0 & -3 & 3 & -3 & -3 & -2 & 1 & 2 & 2 \\ 0 & -1 & 3 & -2 & -2 & 2 & 1 & 3 & 3 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 2 & 7 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & -1 & -3 & -2 & \frac{4}{3} & 0 & -3 & -\frac{35}{6} & -\frac{35}{6} \\ 0 & 1 & 1 & -1 & 1 & 1 & -1 & 2 & -1 \\ 0 & -1 & 3 & 1 & 1 & -3 & -2 & -1 & -1 \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & -2 & -\frac{59}{6} & -\frac{10}{3} & \frac{6}{59} & \frac{1062}{277} & \frac{1062}{277} \\ 0 & -3 & 1 & -\frac{5}{3} & 7 & -2 & 1 & -\frac{2324}{277} & \frac{2324}{277} \\ 0 & -2 & -1 & 1 & 1 & -1 & -3 & -2 & -2 \\ 0 & 1 & -2 & -3 & 3 & 3 & -1 & -2 & -2 \end{pmatrix}$$
\[
A_{12} = 
\begin{pmatrix}
1 & 0 & 1 & -3 & 0 & 0 & 9 & \frac{5}{6} \\
0 & -1 & 5 & 2 & -2 & 4 & -2 & \frac{3}{4} \\
0 & -3 & -3 & -2 & 2 & 1 & 1 & -3 \\
0 & 2 & -2 & -2 & -2 & -3 & -2 & 2
\end{pmatrix}
\]

\[
A_{13} = 
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & -2 & -1 & \frac{18}{11} \\
0 & 3 & 3 & 1 & 4 & \frac{2}{3} & \frac{1}{2} & \frac{75}{22} \\
0 & 2 & -3 & 3 & -3 & 1 & -\frac{1}{2} & 3 \\
0 & -2 & 3 & -1 & 3 & 2 & -1 & -1
\end{pmatrix}
\]

\[
A_{14} = 
\begin{pmatrix}
1 & 0 & 0 & -1 & 3 & 2 & 7 & \frac{16}{11} \\
0 & 2 & -3 & -4 & 0 & \frac{4}{3} & 1 & -\frac{57}{11} \\
0 & 2 & -1 & 3 & -1 & -2 & 3 & -1 \\
0 & -2 & 1 & -1 & -1 & 2 & -2 & 1
\end{pmatrix}
\]

\[
A_{15} = 
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & -\frac{27}{4} & 2 & -\frac{48}{223} \\
0 & 2 & 2 & 2 & -\frac{3}{2} & \frac{11}{2} & \frac{5}{2} & -\frac{140}{223} \\
0 & -3 & -1 & -2 & 3 & 3 & 2 & 2 \\
0 & 1 & 3 & -3 & -3 & -2 & -1 & 2
\end{pmatrix}
\]

\[
A_{16} = 
\begin{pmatrix}
1 & 0 & 2 & 0 & 0 & \frac{25}{2} & \frac{47}{2} & \frac{8}{11} \\
0 & -1 & 12 & -3 & -5 & -3 & -3 & \frac{48}{11} \\
0 & -2 & -3 & 3 & -3 & -3 & 3 & -2 \\
0 & 3 & 2 & -2 & 3 & 1 & -1 & -2
\end{pmatrix}
\]

\[
A_{17} = 
\begin{pmatrix}
1 & 0 & -2 & -\frac{5}{2} & 0 & 0 & -\frac{6}{5} & -\frac{80}{61} \\
0 & 2 & -3 & 3 & -3 & -\frac{5}{3} & -1 & \frac{96}{61} \\
0 & -3 & -3 & -2 & 2 & 2 & -3 & 2 \\
0 & 3 & -1 & 2 & -3 & 3 & -3 & -2
\end{pmatrix}
\]

\[
A_{18} = 
\begin{pmatrix}
1 & 0 & 0 & 0 & -3 & -1 & 3 & \frac{294}{151} \\
0 & -1 & -3 & 1 & 12 & 10 & -\frac{5}{3} & \frac{99}{302} \\
0 & 3 & 1 & 3 & -2 & 2 & 1 & -\frac{117}{302} \\
0 & -2 & -3 & 2 & -3 & -2 & 2 & -3
\end{pmatrix}
\]

\[
A_{25} = 
\begin{pmatrix}
1 & 0 & -2 & 0 & \frac{2}{3} & \frac{30}{7} & -\frac{81}{20} \\
0 & 3 & -\frac{4}{3} & -1 & 2 & -\frac{27}{14} & -9 \\
0 & -1 & 2 & -3 & 3 & 1 & 2 \\
0 & 3 & 2 & 3 & -1 & 2 & -1
\end{pmatrix}
\]
Here the numbering of the nodes is basically from top to bottom and from left to right, or, more explicitly, given by the following.

\[
G_3 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
\end{array}
\]

\[
G_4 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
\end{array}
\]

\[
G_5 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
\end{array}
\]

\[
G_6 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
\end{array}
\]

\[
G_7 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
\end{array}
\]

\[
G_8 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
\end{array}
\]

It is easy to check by hand that the representations of \(G_3\)–\(G_{18}\) and \(G_{25}\), indicated by \(A_3\)–\(A_{18}\) and \(A_{25}\), respectively, satisfy the conditions in Step I of the recipe. In order to check that Step II is satisfied, one has to go through more elaborate calculations. These checks were made using the Mathematica program.

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REFERENCES


