POLYNOMIALLY CONTRACTIBLE SETS

J. William Helton* and D. F. Schwartz**
University of California, San Diego
La Jolla, California 92093

I. Introduction.

The objective of this article is to introduce the notion of an n-contractible set (a generalization of the idea of starlike set), to suggest some problems concerning n-contractibility, to give some rudimentary results, and to describe an application to a type of optimization problem.

Definition. A nonempty connected planar set $S$ will be called n-contractible if there exists a polynomial $p(z)$ of degree $n$ such that $p(z)$ points inward on $\partial S$, the boundary of $S$. More precisely, if $n(z)$ is the interior normal to $\partial S$ then $|\arg p(z) - \arg n(z)| < \frac{\pi}{2}$ for $z$ on $\partial S$. The polynomial $p$ will be called an n-field for $S$.

As an example, we note that a set which is starlike or radial about a point $z_0$ is contractible by $p(z) = z_0 - z$ and so is 1-contractible.

We note that Theorem II.2 of [HSW] (disregarding the parameter $\zeta$) implies the following:

Theorem I.1. Let $D$ be a closed domain whose boundary curve is twice differentiable. Then $D$ is n-contractible for some $n$.

---

*Supported in part by the Office of Naval Research, Air Force Office of Scientific Research and the National Science Foundation.
**Supported by the Office of Naval Research and the Air Force Office of Scientific Research.
We pose the following questions, for a given domain what is its minimal order of contractibility and what is a polynomial of that degree which contracts it. In this article we obtain answers to these questions for some special cases. In particular we provide some guidelines for determining if a given domain is contractible by a specific polynomial. Then using these guidelines we prove:

(1) All lunes are quadratically contractible.

(2) There exists a simply connected region (reproduced to scale in Figure I.B) that is not contractible by a polynomial of degree $\leq 6$.

However, there are less extreme regions (examples are Figures I.C and I.D) which are contractible by polynomials of degree 5.

While our primary motivation is simply to investigate a natural generalization of a starlike set we also mention that these concepts apply to optimization
problems. In particular, for $\Gamma(\zeta,w)$ a positive, real-valued function for $|\zeta| = 1$ and complex $w$ we consider the optimization problem of finding

$$\gamma_0 = \inf_{h \in H^\infty} \sup_{|\zeta| = 1} \Gamma(\zeta, h(\zeta)) .$$

This problem occurs in circuit and systems design when stability is a serious constraint (see [H]). To link n-contractibility with (OPT) let $S_{\zeta}(c)$ denote the sub-level set

$$S_{\zeta}(c) = \{z : \Gamma(\zeta, z) \leq c\} .$$

of $\Gamma$. One might hope that if the $S_{\zeta}(c)$'s are all low order contractible, then (OPT) is easier to solve than if they are not. Indeed in §III we propose a computer algorithm for solving (OPT) based on n-contractibility. Namely, we assume that we have $p_{\zeta}(z)$ a smoothly varying n-field for each $S_{\zeta}(c)$ and we propose a descent method for solving (OPT) in terms of $p_{\zeta}(z)$.

II. Determining the degree of contractibility.

We begin this section by presenting some guidelines applicable to determining if a given domain is contractible by a specific polynomial. Then we proceed to prove the assertions made about contractibility in Figure I.


We begin with four principles to be used in this section to determine if a region is contractible by a given polynomial $p(z)$. The zero principle requires that the region contain one zero of $p(z)$. The direction condition exhibits a simple analytical expression which can be evaluated to determine if a function points inward at a boundary point. The symmetry requirement says that for a
contractible region symmetric with respect to both axes there exists a polynomial whose zeroes mirror this symmetry that contracts the region. And the *degree condition* adds that if such a region is contractible by a polynomial of degree $2n > 0$ then it is contractible by a polynomial of degree $< 2n$.

(A.) **Zero condition principle:** A necessary condition for a bounded region $\Omega$ to be contractible by an analytic function $f(z)$ is that $f(z)$ has a unique zero in $\Omega$.

*Proof.* Consider the function $F(z) = z + \epsilon f(z)$ which maps $\Omega$ strictly into itself when $\epsilon > 0$ is suitably small. By the fixed point theorem of Earle and Hamilton [E-H] $F$ has a unique fixed point in $\Omega$. Thus $f$ has exactly one zero in $\Omega$. $\square$

(B.) **Direction condition:** Let $C$ be a simple closed curve with interior normal $n(z)$. Then $f(z)$ points inward at $z$ on $C$ if and only if the direction function

$$\text{Re} \left\{ \frac{f(z)}{n(z)} \right\}$$

is strictly positive.

*Proof.* The condition that $f(z)$ points inward at $z$ is precisely that $|\arg f(z) - \arg n(z)| < \frac{\pi}{2}$, or $|\arg \frac{f(z)}{n(z)}| < \frac{\pi}{2}$. But the condition $|\arg w| < \frac{\pi}{2}$ is equivalent to $w$ being the right half plane or $\text{Re } w > 0$. $\square$

(C.) **Symmetry requirement:** Let $C$ be a simple closed curve symmetric with respect to both axes such that at each point the magnitude of the angle from the interior normal to either of the tangents is $\leq \frac{\pi}{2}$. Suppose there exists a polynomial $p_1(z)$ which points
inward on $C$. Then there exists a polynomial $P(z)$ of no higher degree with zeroes symmetric with respect to both axes which also points in on $C$.

**Proof.** Given $p_1(z)$ which points in on $C$ we construct $P(z)$ whose zeroes are symmetric with respect to both axes and which also points in on $C$.

We begin by constructing a polynomial $p(z)$ with zeroes symmetric across the real axis. To do this we note that our hypotheses imply that at each point of $C$ the following hold:

(i) If $v_1$ and $v_2$ point in then $t_1v_1 + t_2v_2$ points in for any positive scalars $t_1$ and $t_2$.

(ii) If $v$ points in at $z$ then $\bar{v}$ points in at $\bar{z}$.

Now if $p_1(z) = \sum_{k=0}^{m} c_k z^k$ let $p_2(z) = \sum_{k=0}^{m} \bar{c}_k z^k$. By (ii) $p_2(z)$ points in on $C$ since $p_2(z) = \overline{p_1(z)}$ and $p_1(z)$ points in at $z$. Thus by (i) $p(z) = p_1(z) + p_2(z)$ points in on $C$ and its zeroes are symmetric across the real axis since it has real coefficients. We note that $p(z)$ is of no higher degree than $p_1(z)$.

Using this procedure we proceed to symmetrize the zeroes of $p(z)$ across the imaginary axis. To do this let $C' = \{iz : z \in C\}$ and note that $q(z) = ip(-iz)$ points in on $C'$. By the previous argument $Q(z) = q(z) + \overline{q(z)}$ has zeroes symmetric across the real axis and points in on $C'$. Now rotating back to a polynomial which points in on $C$ we define

$$P(z) = -i Q(iz) = p_1(z) - p_1(-z) + \overline{p_1(z)} - \overline{p_1(-z)}$$

$$= 2 \sum_{k \text{ odd}}^{m} (c_k + \bar{c}_k) z^k.$$  \hspace{1cm} (II.1.2)
Since $P(-z) = -P(z)$ and $P(z) = \overline{P(z)}$ our assertions are verified.

(D.) **Degree condition:** Let $C$ be a curve satisfying the hypotheses of (C). If $p(z)$ is a polynomial of even degree $2n > 0$ which points in on $C$ then there exists a polynomial $P(z)$ of odd degree $< 2n$ which also points in on $C$.

**Proof.** This is immediate from the construction of $P(z)$ given by (II.1.2).

**Example.** If $C = \{z: |z| = 1\}$ then $p(z) = z(z - 2)$ points in on $C$. The symmetrization construction (II.1.2) gives $P(z) = -4z$ which also points in and is of lower order.

II.2. **Lunes are quadratically contractible.**

We now show that all lunes are quadratically contractible.

**Theorem II.1.** If $D_1$ and $D_2$ are two disks such that $D_1 \cap D_2^t$ is simply connected and $D_1^c \cap D_2$ has nonempty interior then $D_1 \cap D_2^c$ is contractible by a polynomial of degree 2.

The proof will be based on the following lemma.

\[\overline{D}^c\] is meant to be the complement of $D$. 
Lemma II.2 Let $D$ be a disk of the form $\{ |w - w_0| < r \}$. If $w_1$, $w_2$ and $w_0$ are colinear with $|w_1 - w_0| < r$ and $|w_2 - w_0| > r$ then $p(w) = \kappa (w_1 - w) (w - w_2)$ points inward on $\partial D$ when $\arg \kappa = - \arg (w_1 - w_2)$.

Proof. Suppose $D$ is the disk $|z| < r$ and $a < -r < b < r$. Then by (II.1.B) $q(z) = (b - z) (z - a)$ points in on $|z| = r$ as the following calculation shows.

$$\Re \left\{ \frac{P(e^{i\theta})}{-e^{i\theta}} \right\} = \Re \left\{ (r - be^{i\theta}) (re^{i\theta} - a) \right\} = r(|a| - b) + (r^2 - b|a|) \cos \theta$$

is strictly positive since $r(|a| - b) > r^2 - b|a|$. Indeed $r(|a| - b) > r^2 - b|a|$ is equivalent to $|a| > r$ since $|b| < r$.

Now consider $D = \{ |w - w_0| < r \}$ with $w_1$ and $w_2$ as in the hypotheses. For $\kappa$ chosen so that $\kappa (w_1 - w_2)$ is strictly positive we note that $q(z) = (b - z) (z - a)$ points in on the translated rotated image of $\partial D$ given by $z = \kappa (w - w_0)$ where $b = \kappa (w_1 - w_0)$ and $a = \kappa (w_2 - w_0)$ are real with $a < -r < b < r$. Thus $p(w) = \bar{\kappa} q(\kappa (w - w_0)) = \kappa (w_1 - w) (w - w_2)$ points in on $\partial D$.

Proof of Theorem II.1. Choose $w_1$ in $D_1 \cap D_2^C$ and $w_2$ in $D_1^C \cap D_2$ so they are colinear with the centers of $D_1$ and $D_2$. With $|\kappa| = 1$ chosen so that $\kappa (w_1 - w_2) > 0$ we have by Lemma II.2 that $p(w) = \kappa (w_1 - w) (w - w_2)$ points in on $\partial D_1$. Likewise, $-p(w) = -\kappa (w_2 - w) (w - w_1)$ points in on $\partial D_2$ or, equivalently, $p(w)$ points outward on $\partial D_2$. Thus $p(w)$ points inward on $\partial (D_1 \cap D_2^C)$.
II.3. A region not contractible of degree $\leq 6$.

**Theorem II.3.** The region depicted in Figure II.1 is not contractible by polynomials of degree $\leq 6$.

To prove this we employ the principles set forth in Section II.1, which specify that we only need consider polynomials of odd order with zeroes symmetric with respect to both axes. These plus some estimates reduce the problem to that of checking a finite collection of polynomials. These were checked on a VAX 11/780 and none were found to contract the domain.

**Proof.** Let $C$ be a simple closed curve forming the boundary of the intersection of the unit disk and the exterior of two circles centered at the points $(x_1,0)$ and $(-x_1,0)$ each with radius $r$ where $1 - x_1 < r < x_1$.

\[
\begin{align*}
x_0 &= .05 \\
x_1 &= .530581 \\
r &= .4805851 \\
x_c &= .99 \\
y_c &= .1410672
\end{align*}
\]

*Figure II.1*
bounded by $C$. Since this region is symmetric with respect to both axes, our symmetry requirement implies that no polynomials of degree 3 contract it. By our degree condition in II.1.C we can extend this result to include polynomials of degree 4.

These arguments illustrate the use of principles II.1.A,B,C,D and the nature of our methods sufficiently that we can be sketchy in the rest of the proof.

*Polynomials of degree 5.*

The symmetrization requirement of II.1.C allows us to consider only four arrangements of five zeroes placed symmetrically with respect to both axes. The zero principle of II.1.A requires that we have one zero interior to $C$ which must be located at the origin. We utilize three different methods to verify that strict positivity of the direction function, $\Re\left\{\frac{p(z)}{n(z)}\right\} > 0$, of II.1.B must fail for each of the four arrangements of zeroes. We sketch arguments and refer the reader to [S] for details.

**Case 1.** We begin with the arrangement of all zeroes along the imaginary axis.

The proof consists of an estimate of $\arg p(z)$ on an arc and from this estimate one obtains a contradiction to the strict positivity requirement on the direction function at a point on this arc.
Case 2. Here the same method will be used on two different arrangements:

(A.) All zeroes along the real axis, and (B) zeroes on both axes.

It is straightforward to calculate a bound on the location of one pair of zeroes from the requirement that $p(z)$ points in at a cusp point of $C$. Subject to this restriction we choose the zeroes of $p(z)$ to maximize the direction function on the arc of $\{|z - x_1| = r\} \cap C$ in the first quadrant. Even after this constrained maximization the direction function fails the positivity requirement at a point on this arc.

Case 3.

We now deal with the case where the zero at the origin is the only zero on the axes and the positions of the other zeroes are determined by that of the zero $a$ in the first quadrant by symmetry across the axes. That is,

$$p(z) = \kappa (z - a) (z - \bar{a}) (z + a) (z + \bar{a}).$$

The objective is to show that for each possible placement of the zero $a$ there exists a point on $C$ where the direction function (II.1.1) is negative and thus the polynomial fails to point in on $C$. This is done in three steps.

Step 1. Begin by deriving estimates which produce a region $R$ in which the zero $a$ must lie in order for $p$ to possibly have a positive direction function.

Step 2. The family of polynomials generated by $a$ in $R$ produces a family $F$ of direction functions which are equicontinuous in $a$. For a given error tolerance $\epsilon > 0$ one can place a grid on $R$ whose size is
calculated to be small enough so that two direction functions $f_{a_1}$ and $f_{a_2}$ in $F$ vary (over $C$) by less than $\epsilon$ when $a_1$ and $a_2$ are adjacent grid points.

**Step 3.** For each grid point $a$ we construct the corresponding polynomial $p(z)$ and then sample the direction function along $C$. If we do not find a point on $C$ where the direction function $< -2\epsilon$, then we decrease the value of $\epsilon$ and return to step 2.

This computation was done on a VAX 11/780. For our choice of curve $C$ it terminated when $\epsilon = .175$. Since any $f$ in $F$ is within (sup norm over $C$) $\epsilon$ of an $f_a$ in $F$ coming from an $a$ which is a grid point, we conclude that any $f$ in $F$ has some value on $C$ which is less than $- .35$. Machine accuracy is 6 decimal digits which allows one to prove that any $f$ in $F$ takes some negative value on $C$.

We have now eliminated all possible polynomials of degree $\leq 5$ with zeroes symmetric with respect to both axes from being able to contract the region depicted in Figure II.2. Now by the symmetry condition of II.1.C there is no polynomial of degree 5 that contracts the region and the degree condition of II.1.D extends this result to polynomials of degree 6. Theorem II.3 is now proved.

### III. An Optimization Algorithm

The purpose of this section is to present a computer implementable algorithm for solving the following problem: Given a positive function $\Gamma(\xi,z)$ on $T \times C$ find

$$\gamma_{\text{OPT}} = \inf_{f \in A} \sup_{|\xi|=1} \Gamma(\xi,f(\xi)) = \inf_{f \in A} \| \Gamma(\cdot,f(\cdot)) \|$$

(OPT)

where $A$ is the disk algebra of functions continuous on $|\xi| \leq 1$ and analytic in the interior.
Our algorithm was motivated by consideration of the $\zeta$-cross sections of the sublevel sets of $\Gamma$ which we denote as

$$S_\zeta(\gamma) \triangleq \{z : \Gamma(\zeta,z) \leq \gamma\} . \quad (\text{III}.1)$$

These sets are often defined in terms of a function $f \in A$ determining $\gamma$ as

$$\gamma(f) \triangleq \sup_{|z|=1} \Gamma(\zeta,f(z)) . \quad (\text{III}.2)$$

The idea behind our algorithm is the following: Fix $f \in A$ and let $\gamma(f)$ be determined as in (III.2). Assume there exists a polynomial in $z$, $p(\zeta,z) = \sum_{j=0}^{N} c_j(\zeta)z^j$ which points in on $\partial S_\zeta(\gamma(f))$ for each $\zeta \in \mathbb{T}$ and has coefficients $c_j(\cdot)$ continuous on $\mathbb{T}$. If $E \triangleq \{\zeta \in \mathbb{T} : \Gamma(\zeta,f(\zeta)) = \gamma(f)\}$ and the measure of $\mathbb{T} \setminus E > 0$ then we can approximate the coefficient functions $c_j$ by functions $b_j \in A$ so that the new polynomial $P(\zeta,z) = \sum_{j=0}^{N} b_j(\zeta)z^j$ points in on $\partial S_\zeta(\gamma(f))$ for $\zeta \in E$. Thus for small $\epsilon > 0$ we obtain a new function in $A$,

$$\tilde{f}(\zeta) = f(\zeta) + \epsilon \sum_{j=0}^{N} b_j(\zeta)f(\zeta)^j \quad (\text{III}.3)$$

which is in the interior of $S_\zeta(\gamma(f))$ so that $\gamma(\tilde{f}) < \gamma(f)$. Thus we have made an improvement on minimizing $\|\Gamma(\cdot,f(\cdot))\|_{\infty}$ over $A$.

Our algorithm for approximating a solution to OPT requires that we assume the sublevel sets of $\Gamma$ are contractible by a polynomial.

**Basic Assumption:** Assume that for each $\gamma > \gamma_{\text{OPT}}$ there exists a polynomial

$$P(\gamma,\zeta,z) = \sum_{j=0}^{\text{N}(\gamma)} c_j(\gamma,\zeta)z^j \text{ with coefficients } \quad (\text{III}.4)$$

continuous on $|\zeta| = 1$ and satisfying
The Algorithm: Given $f_n \in A$ our procedure for updating it to $f_{n+1}$ is the following:

1. Compute $\gamma(f_n)$ as defined in (III.2) and let

$$E_n \triangleq \{ \xi \in T : \Gamma(\gamma, f_n(\xi)) = \gamma(f_n) \}.$$  

(III.6)

If the measure of $T \setminus E_n > 0$ then proceed to (2).

2. Construct functions $b_j \in A$ for $0 < j < N(\gamma(f_n))$ to approximate the coefficient functions $c_j(\gamma(f_n), \xi)$ for $\xi \in E_n$ close enough that for $0 < t < \alpha$ we have

$$\text{dist}(f_n(\xi) + t \sum_{j=0}^{N} b_j(\xi)f_n(\xi)^j, S_{\xi}) \geq \frac{\nu}{2} t \text{ on } E_n.$$  

(III.7)

3. Let $f_{n+1} = f_n + t_n \sum_{j=1}^{N} b_j f_n$  

(III.8)

where $0 < t_n < \alpha$ is chosen to minimize $\sup_{|\xi|=1} \Gamma(\xi, f_{n+1}(\xi))$.

Example. Suppose we are given two functions

$$\Gamma_1(\xi, z) = |c(\xi) - z|^{-2} \quad \text{and} \quad \Gamma_2(\xi, z) = |z|^2$$

For a nonempty set $W$ we define $\text{dist}(z, W) \triangleq \inf \{|z-w| : w \in W\}$ and $W^c \triangleq \{ z \in C : z \notin W \}$.

A more general scheme is to numerically solve the differential equation $\frac{df}{dt} \Gamma(\xi, f)$ with $f^0 = f_n(\xi)$ to obtain $f_{n+1}$. One step of Euler's method amounts to the formula in (3).
where \( c \) is a given smooth function and we wish to minimize \( \Gamma_1(\zeta, f(\zeta)) \) with \( f \in A \) subject to the condition that \( \Gamma_2(\zeta, f(\zeta)) \leq 1 \) on \( T \). Thus we wish to find

\[
\gamma_{\text{OPT}} = \inf_{f \in A} \{ \| \Gamma_1(\cdot, f(\cdot)) \|_{\infty} : \| \Gamma_2(\cdot, f(\cdot)) \|_{\infty} \leq 1 \}.
\]

(OPT')

and make the technical assumption that \( c \) is such that

\[
\gamma_{\text{OPT}} < \max_{|\zeta|=1} \frac{1}{|1 - c(\zeta)|^2}.
\]

The relevant sublevel sets for this problem

\[
S_\zeta(\gamma) = \{ z : |c(\zeta) - z|^2 \geq \frac{1}{\gamma} \text{ and } |z|^2 \leq 1 \}
\]

are lunes provided that

\[
|1 - |c(\zeta)|| < \gamma^{-1/2} < 1 + |c(\zeta)|
\]

(which is why we inserted the technical assumption).

Lunes are 2-contractible by Theorem II.1 and its proof indicates that

\[
p(\zeta, z) = \kappa(\zeta)(a(\zeta) - z)(z - b(\zeta))
\]
contracts each of the $S_r(\gamma)$ to the point $a(\zeta)$ chosen in the interior of $S_r(\gamma)$. We would choose $a(\zeta) = \frac{-c(\zeta)}{|c(\zeta)|+\epsilon}$ and $b(\zeta) = \frac{c(\zeta)}{|c(\zeta)|-\epsilon}$ where $\epsilon > 0$ is small so that

(i) $|a(\zeta)| < 1, \; |a(\zeta) - c(\zeta)| > \gamma^{-1/2},$

(ii) $|b(\zeta)| > 1, \; |b(\zeta) - c(\zeta)| < \gamma^{-1/2},$

and $|\kappa| = 1$ is chosen so that $\kappa(a-b) > 0$.

We emphasize that this polynomial function $p(\zeta,w)$ works for the full optimization problem $(OPT')$ and not just for one sublevel set.

IV. Estimates

The goal of this section is to show

**Theorem IV.1.** If $f_n \in A$ is not a local minimum for $(OPT)$, then our algorithm produces an $f_{n+1}$ with $\gamma(f_{n+1}) < \gamma(f_n)$.

In the course of the proof we derive estimates which give an idea of which factors determine the amount of improvement each iteration makes.

Begin by observing that there is a tradeoff between the nearness to an optimum and the distance we can go in our descent direction. This is to be expected from a cursory analysis of our choice of update

$$\tilde{f}(\zeta) = f(\zeta) + tP(\zeta,f(\zeta))$$

for the following reason: As $\gamma(f) \to \gamma_{OPT}$ the peak set $E = \{\zeta \in T : \Gamma(\zeta,f(\zeta) = \gamma(f)\}$ becomes dense in most of $T$ since we expect $\Gamma(\zeta,f(\zeta))$ to be constant for a local optimum $f^\dagger$. The increase in size of the region $E$ on which the coefficients $b_n(\zeta)$ of $P(\zeta,z)$ must closely approximate

\[\text{See [HSW] and [HH] to corroborate our hunch.}\]
the \( c_n(\zeta) \) of \( p(\zeta,z) \) leads to wilder behavior of \( P(\zeta,z) \) off of \( E \). The values of \( \Gamma(\zeta,\hat{f}(\zeta)) \) for \( \zeta \notin E \) must be controlled by only allowing small values for \( t \). Note that while \( t \) is small \( \hat{f}(\zeta) - f(\zeta) = t P(\zeta,\hat{f}(\zeta)) \) may be a large function at some \( \zeta \). Wild behavior of \( P \) off \( E \) disposes this function to be large.

Now we start to derive estimated on the improvement which an update makes.

The setting is the following. We assume that there exists a polynomial

\[
p(\zeta,z) = \sum_{n=0}^{N} c_n(\zeta)z^n \quad \text{with} \quad c_n \in C(T) \quad \text{such that} \quad \forall \gamma > \gamma_{OPT} \quad \text{there exists a} \nu > 0 \quad \text{and} \quad \alpha > 0 \quad \text{(depending on} \gamma) \quad \text{with}
\]

\[
\text{dist} \left( z + t p(\zeta,z), S_\zeta(\gamma) \right) > \nu t \quad \text{(IV.1)}
\]

for \( 0 \leq t \leq \alpha \) and \( z \in \partial S_\zeta(\gamma) \).

**Lemma IV.2.** Let \( f \in A \) with \( \gamma(f) > \gamma_{OPT} \) and set

\[
E = \{ \zeta \in T: \Gamma(\zeta,f(\zeta)) \geq \frac{1}{2} (\gamma(f) + \gamma_{OPT}) \}
\]

There exists \( P(\zeta,z) = \sum_{n=0}^{N} b_n(\zeta)z^n \) with \( b_n \in A \) and constants \( \nu_1 > 0, \alpha_1 > 0 \) such that \( \forall \zeta \in E \) we have

\[
\Gamma(\zeta,f(\zeta) + t P(\zeta,f(\zeta))) \leq \Gamma(\zeta,f(\zeta)) - \nu_1 t \quad \text{(IV.2)}
\]

for \( 0 \leq t \leq \alpha_1 \).
Proof. To construct \( P(\gamma, z) \) we need only choose \( b_n \in A \) such that (IV.1) holds on \( E \) with \( \nu \) replaced by \( \frac{\nu}{2} \). We can do this by Lemma 5 of [HH]. For each \( \gamma \) between \( \frac{1}{2}(\gamma(f) + \gamma_{OPT}) \) and \( \Gamma(f) \) we have for \( z \in \partial S(\gamma) = \{ z \in C : \Gamma(\gamma, z) = \gamma \} \) that \( \Gamma(\gamma, z + tP(\gamma, z)) = \Gamma(\gamma, z) + t\alpha(\gamma, z) + O(t^2) \) with \( \alpha(\gamma, z) = \frac{\partial \Gamma}{\partial z} (\gamma, z) \cdot P(\gamma, z) = \frac{\partial \Gamma}{\partial n}(\gamma, z)(n \cdot P(\gamma, z)) \). Here \( n \) is the exterior normal to the level set \( \partial S(\gamma) \) and \( \alpha(\gamma, z) < 0 \) on this set. (By our choice of \( b_n \), \( P(\gamma, z) \) continues to point inward on \( \partial S(\gamma) \) for each \( \gamma > \gamma_{OPT} \)). The set \( K \triangleq \{(\gamma, f(\gamma)) : \gamma \in E \} \) is compact in \( T \times C \) so \( \exists \mu < 0 \) such that \( \alpha(\gamma, z) < \mu \) on \( K \). (Here we are using the assumed smoothness of \( \Gamma \) to give us the continuity of \( \alpha(\gamma, z) \).) Since \( t^{-1}0(t^2) \to 0 \) as \( t \to 0 \) uniformly on \( K \) there exists an \( \alpha_1 > 0 \) such that \( \alpha(\gamma, z) + t^{-1}0(t^2) < \frac{\mu}{2} \Delta \nu_1 \) for \( 0 \leq t \leq \alpha_1 \).

Now we show how the set
\[
K' = \{(\gamma, f(\gamma)) : \gamma \in T \text{ and } \Gamma(\gamma, f(\gamma)) \leq \frac{1}{2}(\gamma(f) + \gamma_{OPT})\}
\]
constrains our choice of \( t \); we want
\[
\Gamma(\gamma, z + tP(\gamma, z)) < \gamma(f) - \frac{1}{4}(\gamma(f) - \gamma_{OPT})
\]
(IV.3)
on \( K' \). Our bound on \( t \) will come from the Taylor expansion

---

\[\text{Lemma 5 of [HH] guarantees that we can do this for } b_n \in H^\infty. \text{ A numerical procedure for producing } b_n \in A \text{ satisfying this condition utilizes the iteration of the solution of a disk problem (see [HS]). This is done by specifying a weight function } 0 \leq w(\gamma) \leq 1 \text{ on } T \text{ which is chosen to be near } 1 \text{ on } E \text{ and small off of } E. \text{ Then the } b_n \in A \text{ are computed to minimize}
\sup_{|\gamma| = 1} \text{ wrt } (c_n(\gamma) - b_n(\gamma)) \].
\[ \Gamma (\xi, x + t \mathbf{P}(\xi, x)) - \Gamma (\xi, x) \]
\[ = t \text{Real} \left( \Gamma_2 \cdot \mathbf{P} \right) + \frac{1}{2} t^2 \mathbf{P}^T \left[ \begin{array}{cc} \Gamma_{zz} & \Gamma_{zr} \\ \Gamma_{rz} & \Gamma_{rr} \end{array} \right] \mathbf{P} + \cdots \tag{IV.4} \]

To guarantee that (IV.3) holds we need only compute a bound on the right hand side of (IV.4) since we know that \( \Gamma (\xi, x) \leq \frac{1}{2} (\gamma (f) + \gamma_{OPT}) \) on \( K' \). A crude bound is
\[ t \sup_{K} |\Gamma_2 \mathbf{P}| + \frac{1}{2} t^2 \sup_{K} \{|\mathbf{P}|^2 : |2 \Gamma_{zz} + \Gamma_{zr} + \Gamma_{rr}| \} . \tag{IV.5} \]

A smooth \( \Gamma \) allows estimation of bounds on its partial derivatives on a compact set like \( K' \), but the variation in \( |\mathbf{P}| \) is harder to gauge. If we set \( B_n = \sup_{|\xi| = 1} |b_n(\xi)| \) then by the Schwarz inequality we obtain the crude bound
\[ |P(\xi, f(\xi))| = |\sum_n b_n(\xi)f(\xi)^n| \]
\[ \leq \left( \sum_n |B_n|^2 \right)^{1/2} \sup_{|\xi| = 1} \left( \sum_n |f(\xi)|^{2n} \right)^{1/2} . \tag{IV.6} \]

Insert the bound we obtain from (IV.6) and an estimate of the magnitude of the partials of \( \Gamma \) on \( K' \) into (IV.5) to obtain the conclusion:

**Lemma IV.3.** \( \exists \beta > 0 \) such that for \( 0 \leq t \leq \beta \) and \( z \in E' \Delta \{ \xi \in \mathbf{T} : \Gamma (\xi, f(\xi)) \leq \frac{1}{2} (\gamma (f) + \gamma_{OPT}) \} \) we have that
\[ \Gamma (\xi, f(\xi) + t \mathbf{P}(\xi, f(\xi))) < \Gamma (f) - \frac{1}{4} (\gamma (f) - \gamma_{OPT}) . \tag{IV.7} \]

Lemmas IV.2 and IV.3 together yield the following more quantitative version of Theorem IV.1.
Theorem IV.4. Choose $\alpha_1$ and $\nu_1$ as in Lemma IV.2 and $\beta$ from Lemma IV.3. Set $\tau^* = \min(\alpha_1, \beta)$. For $\hat{f}(\zeta) \triangleq f(\zeta) + \tau^* p(\zeta, f(\zeta))$ we obtain the improvement

$$\gamma(\hat{f}) \leq \gamma(f) - \min(\nu_1, 1) \left( \frac{1}{4} (\gamma(f) - \gamma_{OPT}) \right).$$

Proof. For $\zeta \in E$, $\Gamma(\zeta, \hat{f}(\zeta)) \leq \Gamma(\zeta, f(\zeta)) - \nu_1 \tau^* \leq \gamma(f) - \nu_1 \tau^*$ and for $\zeta \in E'$,

$$\Gamma(\zeta, \hat{f}(\zeta)) \leq \gamma(f) - \frac{1}{4} (\gamma(f) - \gamma_{OPT}).$$

Since $T \subset E \cup E'$ and $\gamma(\hat{f}) = \sup_{|\zeta|=1} \Gamma(\zeta, \hat{f}(\zeta))$ we have our result. \qed

References


[HSW] J. W. Helton, D. F. Schwartz and S. E. Warschawski, "Local optima in $H^\infty$ produce a constant objective function," accepted for publication by the Journal of Complex Variables.