ANALYTIC FUNCTIONS OPTIMIZING COMPETING CONSTRAINTS

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Abstract. Optimization of sup-norm-type performance functions over the space of $H^\infty$ functions is an area of extensive research. In electrical engineering, it is central to the subject of $H^\infty$ design, while in several complex variables, it is often required to produce analytic discs with valuable properties.

It has been known for many years that an $H^\infty$-type optimum is frequency independent (flat). In this paper, we study simultaneous (Pareto) optimization of several competing performances $\Gamma_1, \ldots, \Gamma_l$.

We find under strong assumptions on the performance functions that if we are optimizing over $N$ functions $(f_1, \ldots, f_N)$ in $H^\infty$ and have $l$ performance measures with $l \leq N$, then at a nondegenerate Pareto optimum $(f_1^*, \ldots, f_N^*)$, every performance is flat.

Besides flatness, there are other gradient–alignment conditions which must hold at an optimum. The article presents these and thus gives the precise first-derivative test for a natural class of $H^\infty$ Pareto optima.

Such optimality conditions are valuable for assessing how iterations in a computer run are progressing. Also, in the traditional case, optimality conditions have been the base of highly successful computer algorithms; see [J. W. Helton, O. Merino, and T. Walker, Indiana U. Math. J., 42 (1993), pp. 839–874].

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1. Introduction. This paper analyzes a problem in which one optimizes objective functions over the space $H_N^\infty$ of vector-valued functions $f = (f_1, \ldots, f_N)$ defined on the unit circle, $T$, where each coordinate function $f_j$ belongs to $L^\infty(T)$ and extends to be analytic on the entire unit disk.

The objectives that we optimize are described in terms of nonnegative continuous functions $\Gamma$ defined on $T \times \mathbb{C}^N$. Given positive functions $\Gamma_j(e^{i\theta}, z) \in C^1(T, \mathbb{C}^N)$, $j = 1, \ldots, l$, and a function $f \in H_N^\infty$, we define the $l$ performances

$$\gamma_j(f) := \sup_{\theta \in T} \Gamma_j(e^{i\theta}, f), \quad j = 1, \ldots, l.$$ 

The goals of this paper are best illustrated by restricting our study to the case of two performance functions $\Gamma_1$ and $\Gamma_2$, even though our results hold for $l$-performance functions.

Definition. A function $f^* \in H_N^\infty$ is called a Pareto optimum for $\Gamma_1$ and $\Gamma_2$ if for each $f \in H_N^\infty$ one of the following two inequalities holds:

$$\gamma_1(f) \geq \gamma_1(f^*) \quad \text{or} \quad \gamma_2(f) \geq \gamma_2(f^*).$$

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The book of Boyd and Barratt [BB] gives a good discussion of Pareto optimality.

**1.1. Degenerate versus nondegenerate Pareto optima.** A function \( f^* \) can be a Pareto optimum for \( \Gamma_1 \) and \( \Gamma_2 \) in two basic ways.

**Degenerate optima.** The first case is where \( f^* \) can optimize \( \Gamma_1 \), that is,
\[
\gamma_1(f^* + h) \geq \gamma_1(f^*) \quad \forall h \in H_N^\infty, \quad h \neq 0,
\]
in which case \( \Gamma_2 \) is irrelevant. Similarly, \( f^* \) can optimize \( \Gamma_2 \), and then \( \Gamma_1 \) is irrelevant. This case has been seriously studied, and the main optimality result is stated in Theorem 1.1 below.

**Nondegenerate optima.** The second case is when both \( \Gamma_1 \) and \( \Gamma_2 \) are relevant. In this case, there is a pair of analytic functions \( h_1 \) and \( h_2 \) such that
\[
\gamma_1(f^* + h_1) < \gamma_1(f^*), \quad \gamma_2(f^* + h_2) < \gamma_2(f^*).
\]
In other words, we can improve each performance *separately* by adding \( h_1 \) or \( h_2 \) to \( f^* \), but we cannot improve both performances at the same time.

The case of nondegenerate optima is the subject of this paper.

**Example for \( N = 2 \).**
\[
\Gamma_1(e^{i\theta}, z) = |\psi_1(e^{i\theta}) - z_1|^2, \quad \Gamma_2(e^{i\theta}, z) = |\psi_2(e^{i\theta}) - z_2|^2,
\]
where \( \psi_j \) are rational. In this case, the problem is separable into two completely independent one-dimensional single-performance problems. Therefore, there exists no nondegenerate Pareto optimum for \( \Gamma_1 \) and \( \Gamma_2 \). An example of a degenerate Pareto optimum would be any pair of functions \( f = (f_1, f_2) \) such that either \( f_1 \) or \( f_2 \) is an optimum for \( \Gamma_1 \) alone or \( \Gamma_2 \) alone, respectively.

If, on the other hand, we consider the problem with
\[
\tilde{\Gamma}_1(e^{i\theta}, z) = |\psi_1(e^{i\theta}) - z_1|^2, \quad \tilde{\Gamma}_2(e^{i\theta}, z) = |\psi_2(e^{i\theta}) - z_1 - z_2|^2,
\]
then the problem cannot be separated into two independent single-performance problems, and for generic \( \psi_1 \) and \( \psi_2 \), almost all Pareto optima are nondegenerate.

**1.2. A characterization of Pareto optima.** The main result of this paper is that for a special class of \( \Gamma_j \)'s, namely the ones that are the norms of certain rational functions, a nondegenerate local pareto optimum \( f^* \in H_N^\infty \) for \( N > 1 \) satisfies
\[
\Gamma_1(e^{i\theta}, f^*(e^{i\theta})) = \text{const.}_1 \quad \text{and} \quad \Gamma_2(e^{i\theta}, f^*(e^{i\theta})) = \text{const.}_2
\]
for all \( \theta \). *The striking fact is that both performances are flat.* See Theorem 2.1 for the precise statement of this result in the general case of \( l \)-performance functions.

Note that if \( N = 1 \) then both performances are almost never flat. Also, we give a result that indicates that there is a large class of \( \Gamma_j \)'s for which flatness will not hold.

An earlier instance of the flatness condition (Theorem 2.1(I)) for Pareto optima was discovered by Young (see [PY] for proofs). It applies to jointly minimizing the first, second, third, etc. singular values of matrix-valued functions, which is quite a different context from the one in this paper.
1.3. **Engineering motivation.** This type of problem is central to frequency-domain-system design problems, where stability is a key constraint. In particular, it is important to the area of $H^\infty$ control [H], [Fr]. The basic physical idea is simple. Recall that a linear-time-invariant system has a frequency-response function $F$ defined on the imaginary axis and that the system is stable if $F$ has no poles in the closed right half-plane (RHP). The behavior of the system when excited with a pure sine wave of frequency $\omega$ is determined by $F(i\omega)$. The following often occurs in a design procedure. We are required to build a system $S$, but part of the system is given (we are stuck with it) and part of the system is designable (denote its frequency-response function by $f$); see Figure 1. The performance of the system $S$ at frequency $\omega$ is a function $\Gamma(\omega, f(i\omega))$ which depends on $\omega$ and on one’s choice of the designable subsystem $f$. Let us adopt the convention that large $\Gamma$ is bad while small $\Gamma$ is good. Then in a worst-case “broadband” design, we consider the worst performance over all frequencies

$$\sup_{\omega} \Gamma(\omega, f(i\omega))$$

and try to minimize it over all admissible $f$. If our main constraint is that the designable subsystem $f$ must be stable, then the design problem becomes the problem of finding a Pareto optimum with one $\Gamma$ after transforming the RHP to the unit disk. When $N > 1$, these problems usually pose serious difficulties since traditional graphical trial and error methods are inadequate.

A number of authors (Mayne, Polak, and Salucidean; Fan and Tits; Streit; Boyd; Daleh; Pearson; Doyle, Glover, and Packard; Helton, Merino, and Walker; and Sideris) have written computer programs to search for an optimal $f^*$ with certain kinds of $\Gamma$; see, e.g., [BB], [D], [FKTW], [HMW], [MNPW], [Si], and [St].

1.4. **The classical case: One performance function.** If we restrict our attention to the degenerate Pareto optimum, then we end up with a classical case of optimizing a single-performance function.

Now we state an earlier result for a single-performance function which this paper extends. Let $\Gamma(e^{i\theta}, z)$ be continuous nonnegative function. We are trying to find $f^* \in H^\infty_N$ which minimizes the following quantity:

$$\sup_{\theta \in \mathbb{R}} \Gamma(e^{i\theta}, f^*(e^{i\theta})).$$
Theorem 1.1 (see [H1]). Let $\Gamma$ be of class $C^3$ and $f^* \in H_N^\infty \cap C(T)$ be such that $(\partial \Gamma/\partial z)(e^{i\theta}, f^*(e^{i\theta}))$ never equals 0 on $T$. If $f^*$ is a local minimizer, then

(I) the function $\Gamma(\cdot, f^*(\cdot))$ is a constant on $T$ and

(II) there exist functions $F \in H_N^1$ and $\lambda : T \to \mathbb{R}^+$ measurable and positive almost everywhere on $T$ such that

$$
\lambda(e^{i\theta}) \frac{\partial \Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta})) = e^{i\theta} F(e^{i\theta}) \quad \text{a.e. on } T.
$$

When one adds a condition (III) asserting that $\Gamma$ is convex in some directions, then one obtains a necessary and sufficient condition [HM1]. Indeed, [HM1] is the best reference for this result. Theorem 1.1 is extremely useful in that conditions (I) and (II) are a basis for computer diagnostics and software developed by Helton, Merino, and Walker; see [HMW].

1.5. Smooth performances versus multiple performances. In engineering applications, a single-performance function $\Gamma$ can be used to incorporate several performance criteria. In particular, a Pareto optimum $f^*$ for performance functions $\Gamma_1, \ldots, \Gamma_l$ can be viewed as a solution to the optimization problem with only one performance function defined by

$$
\tilde{\Gamma}(e^{i\theta}, z) := \max \left\{ \frac{\Gamma_1(e^{i\theta}, z)}{\gamma_1(f^*)}, \ldots, \frac{\Gamma_l(e^{i\theta}, z)}{\gamma_l(f^*)} \right\}.
$$

Namely, it is easy to check that $f^*$ is a Pareto optimum for $\Gamma_1, \ldots, \Gamma_l$ if it is an optimum for $\tilde{\Gamma}$. The main disadvantage of introducing $\tilde{\Gamma}$ is that it is almost never differentiable, even though the $\Gamma_j$'s are. As a consequence, the results proved for a single-performance-function optimization, reproduced in the theorem above, cannot be applied to $\tilde{\Gamma}$.

1.6. Outline of the paper. This paper has the following structure. In section 2, we state and prove the main result of this paper. In section 3, we give two auxiliary results on uniqueness and existence of the Pareto optimum. In section 4, we reproduce the proofs of several lemmas which were proved by Trepreau in [T] and which have not been published. Section 4 is completely independent of the rest of the paper. In section 5, we discuss the connection between the M-OPT problem which had its origin in engineering mathematics and the analytic-disc techniques used in the popular several-complex-variables problem of extending a function defined on a manifold $M$ in $\mathbb{C}^N$ to a function analytic in a neighborhood of a given point on $M$.

2. First-order conditions. In this section, we give a precise statement of our results in the general case of $l$-performance functions.

Definition. A function $f^* \in H_N^\infty$ is called a local Pareto optimum for $\Gamma_1, \ldots, \Gamma_l$ if there exists $\epsilon > 0$ such that

for all $f \in H_N^\infty$ and $\|f - f^*\| < \epsilon$, there exists $j$, $\gamma_j(f) \geq \gamma_j(f^*)$.

For $l = 1$, this definition means that $f^*$ minimizes $\sup_\theta \Gamma(e^{i\theta}, f(e^{i\theta}))$.

2.1. Main results. We introduce the notation

$$
\frac{\partial \Gamma}{\partial z} = \begin{pmatrix}
\frac{\partial}{\partial z_1} \Gamma_1 & \cdots & \frac{\partial}{\partial z_N} \Gamma_1 \\
\cdots & \cdots & \cdots \\
\frac{\partial}{\partial z_1} \Gamma_l & \cdots & \frac{\partial}{\partial z_N} \Gamma_l
\end{pmatrix}.
$$

(2)
Denote the unit disc in \( \mathbb{C} \) by \( \Delta \).

We will impose the following assumption on performance functions.

**Assumption 1.** Suppose \( N \geq l \) and suppose \( \Gamma_j = |P_j(e^{i\theta}, z)/Q_j(e^{i\theta}, z)|^2 \), where \( P_j \) and \( Q_j \) are holomorphic polynomials in \( z \) with coefficients which are rational functions in \( e^{i\theta} \). We assume, in addition, that the coefficients do not have poles on \( T \).

We will consider a candidate for Pareto optimum \( f^* \) for which the following assumption holds.

**Assumption 2.** The function \( f^* \in H_N^{\infty} \cap C^\alpha \), \( \alpha > 1/2 \), with performances \( \gamma_1 \ldots, \gamma_l \) satisfies the following condition:

There exists an analytic direction \( h_j \in H_N^{\infty} \cap C^\alpha \), \( \alpha > 1/2 \), that improves all performances except for \( \gamma_j \). Namely, there exist \( C > 0 \) and \( t_0 > 0 \) such that for every \( t < t_0 \),

\[
\sup_{\theta} \Gamma_k(e^{i\theta}, f^*(e^{i\theta}) + th_j(e^{i\theta})) - \sup_{\theta} \Gamma_k(e^{i\theta}, f^*(e^{i\theta})) < -Ct \quad \text{for } k = 1, \ldots, l, \quad k \neq j.
\]

Assumption 2 means that \( f^* \) is a nondegenerate Pareto optimum as discussed earlier, i.e., all \( l \) performances \( \Gamma_1 \ldots, \Gamma_l \) play an active role. If not all of them matter, we have a smaller \( l \).

Now we state the main result of this paper.

**Theorem 2.1.** Suppose the performance functions \( \Gamma_j \) satisfy Assumption 1. Suppose that \( f^* \in H_N^{\infty} \cap C^\alpha \) is a local Pareto optimum that satisfies Assumption 2. Suppose further that \( \partial_{z_j}(e^{i\theta}, f^*(e^{i\theta})) \in C^\alpha \) with \( \alpha > 1/2 \) and that

\[
\text{rank } \partial_{\Gamma}(e^{i\theta}, f^*(e^{i\theta})) = l
\]

for every \( e^{i\theta} \in T \).

Then the following hold:

(I) Flatness:

\[
\Gamma_j(e^{i\theta}, f^*(e^{i\theta})) = \text{const.}, \quad j = 1, \ldots, l.
\]

(II) Gradient alignment: There exists a row-vector-valued function \( \lambda \in C^\alpha(T, \mathbb{R}) \), \( \lambda \neq 0 \), with nonnegative entries, such that

\[
\lambda(e^{i\theta}) \partial_{\Gamma}(e^{i\theta}, f^*(e^{i\theta})) = e^{i\theta}F, \quad F \in H_N^{\infty}.
\]

Here \( \partial_{\Gamma}(e^{i\theta}, f^*(e^{i\theta})) \) denotes the \( l \times N \) derivative matrix (2) evaluated at \( z = f(e^{i\theta}) \).

Remark: We will, in fact, show that for (II) to hold, it is enough to assume only that \( f^* \) is an optimum such that \( f^* \in C^\alpha \) with \( \alpha > 1/2 \) and the rank condition (3) holds. In other words, Assumptions 1 and 2 are not needed for (II).

### 2.2. The classical Riemann–Hilbert problem.

The main step in the proof of the flatness condition is solving the following version of Riemann–Hilbert problem:

Given an \( l \times N \) matrix-valued function \( A \in C^\alpha(T) \) with invertible values, given a closed interval \( I \subset T, I \neq T \), find \( h \in H_N^{\infty} \) such that

\[
(\Re(Ah))_j > 0 \quad \text{for } e^{i\theta} \in I, \quad (\Re(Ah))_j > 0 \quad \text{for all } e^{i\theta}, \quad j = 2, \ldots, l.
\]

Here \( (\cdot)_j \) stands for taking the \( j \)th entry of a vector. (Later the matrix \( A \) will be taken to be equal to \( \partial_{\Gamma}(e^{i\theta}, f^*(e^{i\theta})) \).
(4) is, in fact, a problem about the range of the Riemann–Hilbert map

\[ w \rightarrow 2 \text{Re} (Aw). \]

Questions about the range of the Riemann–Hilbert map arise in several aspects. For example, the gradient-alignment condition (Theorem 2.1(II)) will be shown to be equivalent to the fact that the range of the derivative map

\[ w \rightarrow 2 \text{Re} \left( \frac{\partial \Gamma(e^{i\theta}, f^*(e^{i\theta}))}{\partial z} w \right) \]

does not contain strictly positive functions, i.e., that all performances cannot be improved to first order at the same time.

Questions about the range of the Riemann–Hilbert map also arise in the analytic-disc techniques in one theoretical complex-variables problem; see section 6 for more details. See \cite{Ve} as a standard reference on the theory of the Riemann–Hilbert problem.

In this paper, we give conditions on \( A \) which insure that problem (4) always has a solution.

To state our main condition, we need the following definition.

**Definition.** A function \( u(e^{i\theta}) \in L^\infty(T,\mathbb{C}) \) has a pseudomeromorphic continuation inside \( \Delta \) if there exists a function \( \tilde{u} \), meromorphic in \( \Delta \) and with finitely many poles in \( \Delta \), such that

\[
\lim_{r \to 1} \tilde{u}(re^{i\theta}) = u(e^{i\theta}) \quad \text{a.e. } T.
\]

**Theorem 2.2.** Suppose that \( l \leq N \). Suppose that an \( l \times N \) matrix-valued function \( A \in C^\alpha(T,\mathbb{C}) \), \( \alpha > 1/2 \), takes values of rank \( l \) on \( T \). If

(i) the entries of \( A \) have pseudomeromorphic continuation inside \( \Delta \)

or, more generally,

(ii) the matrix \( A \) can be written as \( D\tilde{A} \), where \( D \) is an invertible diagonal matrix function and \( \tilde{A} \) has a pseudomeromorphic continuation inside \( \Delta \),

then problem (4) has a solution \( h \).

3. Proofs.

3.1. Proof of the flatness condition (Theorem 2.1(I)). First, we observe that every \( \Gamma_j \) has a Taylor expansion

\[
\Gamma_j(e^{i\theta}, z + tw) = \Gamma_j(e^{i\theta}, z) + t2\text{Re} \left( \frac{\partial \Gamma_j(e^{i\theta}, z)}{\partial z} \cdot w \right) + O(t^2).
\]

Now suppose that \( f^* \in H_N^\infty \cap C^\alpha \) is a minimizer which produces performances \( \gamma_1^*, \ldots, \gamma_l^* \) and that the performance function \( \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) \) is not constant. Then we can find \( \epsilon_0 > 0 \) and \( I \subset T \) so that \( T \setminus I \) is an open, nonempty interval and

\[
\Gamma_1(e^{i\theta}, f^*(e^{i\theta}))|_{T \setminus I} \leq \gamma_1^* - \epsilon_0 = \sup_{\theta} \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) - \epsilon_0.
\]

We first want to find a vector-valued function \( w \in H_N^\infty \cap C^\alpha \) that satisfies

\[
\begin{pmatrix}
\frac{\partial \Gamma_1(e^{i\theta}, f^*(e^{i\theta}))}{\partial z_1} & \ldots & \frac{\partial \Gamma_1(e^{i\theta}, f^*(e^{i\theta}))}{\partial z_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial \Gamma_l(e^{i\theta}, f^*(e^{i\theta}))}{\partial z_1} & \ldots & \frac{\partial \Gamma_l(e^{i\theta}, f^*(e^{i\theta}))}{\partial z_N}
\end{pmatrix}
\begin{pmatrix}
w_1 \\
\vdots \\
w_N
\end{pmatrix}
= \begin{pmatrix}
\varphi \\
0 \\
0
\end{pmatrix},
\]

where \( \varphi \) is a constant.
where \( \varphi \) is an arbitrary function that never equals 0 on \( I \).

Recall that \( \Gamma_j(e^{i\theta}, z) = (P_j/Q_j)(P_j/Q_j) \), where \( P_j(e^{i\theta}, z) \) and \( Q(e^{i\theta}, z) \) are holomorphic polynomials in \( z \) with rational coefficients depending on \( e^{i\theta} \). The derivative is given by

\[
\frac{\partial \Gamma_j(e^{i\theta}, f^*)}{\partial z_k}(e^{i\theta}, f^*) = \frac{\partial P_j(e^{i\theta}, f^*)Q_j(e^{i\theta}, f^*) - P_j(e^{i\theta}, f^*)\partial Q_j(e^{i\theta}, f^*)}{\partial z_k} \cdot \frac{P_j(e^{i\theta}, f^*)Q_j(e^{i\theta}, f^*)^{-1}}{[Q_j(e^{i\theta}, f^*)]^2}.
\]

We introduce the notation

\[
\Psi_j(e^{i\theta}) := \frac{P_j(e^{i\theta}, f^*(e^{i\theta}))^{-1}Q_j(e^{i\theta}, f^*(e^{i\theta}))}{[Q_j(e^{i\theta}, f^*(e^{i\theta})]^2}.
\]

Note that \( P_j \neq 0 \) because \( \frac{\partial \Gamma_j}{\partial z_k} \) has maximal rank. Then \( \frac{\partial \Gamma_j}{\partial z_k}(e^{i\theta}, f^*(e^{i\theta}))\Psi_j(e^{i\theta}) \) can be extended meromorphically inside the unit disc. By Assumption 1, the meromorphic functions \( \frac{\partial \Gamma_j}{\partial z_k}(z, f^*(z))\Psi_j(z) \) do not have poles on the boundary of the unit disc or accumulating to the boundary. Let \( \beta(z) \) be the finite Blaschke product such that \( \beta(z)\frac{\partial \Gamma_j}{\partial z_k}(z, f^*(z))\Psi_j(z) \) is holomorphic in the unit disc for \( j = 1, \ldots, l \) and \( k = 1, \ldots, N \).

Multiply both sides of (7) on the left by the \( l \times l \) diagonal matrix \( D \), which has the diagonal entries \( \Psi_1(e^{i\theta}), \ldots, \Psi_l(e^{i\theta}) \). Note that each \( \Psi_j(e^{i\theta}) \) does not vanish anywhere on \( T \).

By our assumptions, the matrix \( D \frac{\partial \Gamma}{\partial z} \) has rank \( l \) everywhere. Since it is also of class \( C^\alpha \) with \( \alpha > 1/2 \), by Proposition 5.1, there exists the \( N \times N \) constant matrix \( H \) such that the first \( l \) columns of the product \( D \frac{\partial \Gamma}{\partial z} H \) are linearly independent everywhere on \( T \).

Denote by \( B \) the the first \( l \) columns of the holomorphic matrix \( D \frac{\partial \Gamma}{\partial z} H \beta \). Let \( \tilde{B} \) be an \( l \times l \) holomorphic matrix-valued function such that

\[
BB = (\det B)I_{l \times l}.
\]

Then the vector

\[
\left( \begin{array}{c} \underbrace{w_1} \\ \underbrace{w_2} \\ \ldots \\ \underbrace{w_N} \end{array} \right) = \beta H \left( \begin{array}{c} \tilde{B} \\ 0 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \\ \ldots \\ 0 \end{array} \right)
\]

satisfies (7) with \( \varphi(e^{i\theta}) \neq 0 \) for all \( \theta \). Here \( \left( \begin{array}{c} \tilde{B} \\ 0 \end{array} \right) \) is an \( N \times l \) matrix with the last \( N - l \) rows equal to zero.

Since \( \varphi \) is nonzero on \( I \), its argument \( \arg \varphi|_I \) is well defined. Extend \( \varphi|_I \) to the whole \( T \) in such a way that the extension \( \tilde{\varphi} \) has winding number zero. Let \( h \) be a holomorphic function such that \( \arg h = -\arg \tilde{\varphi} \). Then \( h\varphi|_I \) is real valued and positive, and therefore the vector \( -hw \) has the property that

\[
2\text{Re} \left( \frac{\partial}{\partial z} \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) \cdot [-hw] \right) < -\epsilon_1, \quad e^{i\theta} \in I.
\]

Choose \( \epsilon_2 \) small enough so that

\[
2\text{Re} \left( \frac{\partial}{\partial z} \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) \cdot [-\epsilon_2 hw] \right) \leq \epsilon_0/2, \quad e^{i\theta} \in T \setminus I.
\]
Then there exist positive constants \( C \) and \( t_0 \) such that for any \( t \leq t_0 \), the following holds:

\[
\sup_{\theta} \left[ \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) - t \text{Re} \left( \frac{\partial}{\partial z} \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) \right) \right] - \sup_{\theta} \Gamma_1(e^{i\theta}, f^*(e^{i\theta})) < -Ct,
\]

\[
2\text{Re} \left( \frac{\partial}{\partial z} \Gamma_1(f^*) \right) = 0, \quad j = 2, \ldots, l.
\]

In other words, we have produced the analytic direction \(-hw\) that “improves” the performance \( \gamma_1 \) and leaves the performances \( \gamma_2, \ldots, \gamma_l \) “unchanged” up to the first order.

By Assumption 2, there exists a direction \( v \in H_{\infty}N \cap C^\alpha \) that “improves” the performances \( \gamma_2, \ldots, \gamma_l \):

\[
\sup_{\theta} \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) + tv(e^{i\theta}) - \sup_{\theta} \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) < -C't, \quad j = 2, \ldots, l.
\]

Now consider \( \tilde{w} = -hw + \epsilon v \) for \( \epsilon \) small enough. Then the analytic direction \( \tilde{w} \) “improves” all \( \Gamma_j \)’s. Therefore, by (5), for small \( t \), the function \( f^* + t\tilde{w} \) has better performances:

\[
\gamma_j(f^* + t\tilde{w}) < \gamma_j(f^*), \quad j = 1, \ldots, l.
\]

We have reached a contradiction. \( \square \)

**Proof of Theorem 2.2.** The proof is a line-by-line repetition of a part of the proof of Theorem 2.1(I). \( \square \)

### 3.2. Counterexample

Now we give some indications that we need Assumption 1 on \( \Gamma \)’s for flatness to hold. The flatness result, if it holds, would imply that if \( \Gamma_2(e^{i\theta}, f(e^{i\theta})), \ldots, \Gamma_l(e^{i\theta}, f(e^{i\theta})) \) are constants and \( \Gamma_1(e^{i\theta}, f(e^{i\theta})) \) satisfies (6), then we can find an analytic vector \( h \) such that

\[
\text{Re} \left( \frac{\partial \Gamma_1}{\partial \bar{z}}(e^{i\theta}, f(e^{i\theta})) \cdot h \right) > 0 \quad \text{for } e^{i\theta} \in I,
\]

\[
\text{Re} \left( \frac{\partial \Gamma_j}{\partial \bar{z}}(e^{i\theta}, f(e^{i\theta})) \cdot h \right) > 0 \quad \text{for all } e^{i\theta}, \quad j = 2, \ldots, l.
\]

Considering the question of existence of such an \( h \) in a little more general setting leads to problem (4).

Proposition 3.2 below shows that there exists an \( A \) such that problem (4) is not solvable. While we have not done so, we suspect that one can construct a simple \( A \) for which (4) is not solvable and which can be written as \( \frac{\partial \Gamma}{\partial \bar{z}}(e^{i\theta}, f^*(e^{i\theta})) \) for some \( \Gamma_j \)’s and \( f^* \).

We start with the following lemma which gives a characterization of the derivative map (see [BRT] and [T]).

**Lemma 3.1.** Assume \( N \geq 1 \) and suppose that \( A \) is an \( l \times N \) complex matrix-valued function on \( T \) of class \( C^\alpha \) with \( 1/2 < \alpha < 1 \) and which has maximal rank \( \ell \) at every point on \( T \). Consider the following Riemann–Hilbert operator:

\[
\mathcal{F} : H^2_N \longrightarrow L^2(T, \mathbb{R}), \quad \mathcal{F}(w) = 2\text{Re}(Aw).
\]

Then \( (\text{range}\mathcal{F})^\perp \subset L^2(T, \mathbb{R}) \) consists of all such real-valued \( \ell \)-vectors \( g \in L^2 \) so that

\[
A^i g \in zH^2_N.
\]
Proof. Suppose that $g \in L^2$ belongs to $(\text{range} \mathcal{F})^\perp$. Then
\begin{equation}
\int g \cdot \text{Re} Aw = 0 \quad \forall w \in H^2.
\end{equation}
Since this is true for $iw$, we have
\begin{equation}
\int g \cdot Aw = 0 \quad \forall w \in H^2.
\end{equation}
Therefore, $(A^T g, w) = 0$, where $(\cdot, \cdot)$ is the inner product in $L^2(T, C)$. Then (9) follows.

Remark. If we denote the restriction of $\mathcal{F}$ to $H^2 \cap C^\alpha$ by $\mathcal{F}^\alpha$, then the continuity of $\mathcal{F}$ implies that
\begin{equation}
(\text{range} \mathcal{F}^\alpha)^\perp = (\text{range} \mathcal{F})^\perp.
\end{equation}

Proposition 3.2 below states that problem (4) cannot be solved for every $A$.

Proposition 3.2. Suppose that $N = l = 2$. Given any closed $I \subset T$ with $I \neq T$, there exists $A$ such that for any $v \in C^\alpha(I)$ with $\alpha > 1/2$, $v > 0$, on $I$ and any $\psi \in C^\alpha(T)$ with $\psi > 0$, there exists no solution $h \in H^\infty \cap C^\alpha$ to the following Riemann–Hilbert problem:
\begin{equation}
2\text{Re}(Ah) = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad \varphi|_I = v.
\end{equation}
Proof. Take $g_0 = (p(e^{i\theta}), 1)$ with $p = 0$ on $T \setminus I$ and $p > 0$ on int$(I)$. Then $(g_0, (\varphi, \psi))_{L^2} > 0$ for any $\psi > 0$ and any $\varphi$ with $\varphi|_{\text{int}(I)} > 0$.
Now we take
\begin{equation}
A = \begin{pmatrix} e^{i\theta} \\ -p(e^{i\theta})e^{i\theta} \\ 0 \end{pmatrix}.
\end{equation}
Obviously, $g_0 A \in zH^2$, and therefore $g_0$ belongs to $(\text{range} \mathcal{F})^\perp$. □

3.3. Proof of Theorem 2.1(II). We first need the following theorem.

Theorem 3.3. Suppose that a subspace $\mathcal{R} \subset C^\alpha \subset L^2(T, \mathbb{C})$ has the property that its complement $(\mathcal{R})^\perp$ is finite dimensional and a subset of $C^\alpha$. Suppose also that its closure in the $L^2$ topology $\mathcal{R}$ satisfies $\mathcal{R} \cap C^\alpha = \mathcal{R}$. If $\mathcal{R}$ does not contain vector functions with every component strictly positive, then there exists a function $\lambda \neq 0$ in $(\mathcal{R})^\perp$ with every component positive (but not necessarily strictly positive).

The proof requires the following.

Lemma 3.4. If the vectors $v_1, \ldots, v_m$ satisfy
\begin{equation}
\{x : \exists j, 1 \leq j \leq m, \quad x \cdot v_j > 0\} = \mathbb{R}^n \setminus \{0\},
\end{equation}
then the set
\begin{equation}
\text{hull}(v_1, \ldots, v_m) := \{t_1 v_1 + \cdots + t_m v_m : t_j \geq 0\}
\end{equation}
is the whole of $\mathbb{R}^n$.

Proof. Abbreviate $\text{hull}(v_1, \ldots, v_m)$ by $V$. Suppose that $V \neq \mathbb{R}^n$. Take any point $x_0$ which is not in $V$. Then there exists a plane $\{x : x \cdot \xi = r\}$ which separates $x_0$ and $V$, i.e., $x_0 \cdot \xi > r$ and $x \cdot \xi < r$ for every $x \in V$. 

Since $V$ is a cone, the latter implies that $x \cdot \xi \leq 0$ for every $x \in V$. In particular, it means that

$$v_j \cdot \xi \leq 0, \quad j = 1, \ldots, m.$$ 

We have reached a contradiction. □

Proof of Theorem 3.3. Suppose that the functions $g_1, \ldots, g_n$ form a basis of $(R)^\perp$. We want to prove that there exists a linear combination $a_1g_1(\xi) + \cdots + a_ng_n(\xi)$ in $(R)^\perp$ that is positive on all of $T$. Here $a_j \in R$. Denote the $n \times l$ matrix with rows $g_1(\xi), \ldots, g_n(\xi)$ by $g(\xi)$. We claim that it is enough to show the inequality

$$\inf_{a \in R^n, \|a\| = 1} \max_{\xi \in T} a^t g(\xi) \leq 0. \quad (13)$$

Here we use the notation

$$\max_{\xi \in T} (b_1(\xi), \ldots, b_l(\xi)) = \max\left(\max_{\xi \in T} b_1(\xi), \ldots, \max_{\xi \in T} b_l(\xi)\right).$$

Note that we do not take the absolute values of functions.

The quantity $\max_{\xi \in T} a^t g(\xi)$ depends continuously on $a$, which varies over a compact set in $R^n$. Therefore, if (13) holds, then there exists $a_0 \in R^n$ such that

$$\max_{\xi \in T} a_0^t g(\xi) \leq 0.$$

Hence all components of the vector function $-a_0^t g(\xi)$ are positive and the claim follows.

Now we use a dual extension argument. This uses the fact that for any function $\mu \in C_0^\infty (T)$,

$$\max_{\xi \in T} \mu(\xi) = \sup_{u \in B_1^+} \int_{T} \mu(\xi) \cdot u(\xi) d\xi,$$

where we use the notation $B_1^+ = \{ u \in C_0^\infty (T, R) : \|u\|_{L^1} = 1, u_j(\xi) > 0, \ j = 1, \ldots, l\}$. Therefore, we need to prove the inequality

$$\inf_{a \in R^n, \|a\| = 1} \sup_{u \in B_1^+} \int_{T} (a^t g(\xi)) \cdot u(\xi) d\xi \leq 0. \quad (14)$$

Given any $u \in B_1^+$, we define the vector $c(u) \in R^n$ as follows:

$$c(u) := \left( \int g_1 \cdot u, \ldots, \int g_n \cdot u \right).$$

Lemma 3.5. The set $\{x : \exists u \in B_1^+, \ x \cdot c(u) > 0\}$ is not the whole of $R^n \setminus \{0\}$.

Proof. Suppose the contrary: $\{x : \exists u \in B_1^+, \ x \cdot c(u) > 0\} = R^n \setminus \{0\}$. Since, in fact, we have an open covering of the unit sphere $S^{n-1}$, which is compact, there exist $u_1, \ldots, u_m$ such that $\{x : \exists j, 1 \leq j \leq m, x \cdot c(u_j) > 0\} = R^n \setminus \{0\}$. Then by Lemma 3.4, the convex hull $hull(c(u_1), \ldots, c(u_m)) = R^n$.

Therefore, there exists a positive, nontrivial linear combination of $c(u_j)$’s which is zero:

$$b \in R^m, \quad b \neq 0, \quad b_j \geq 0, \quad \sum b_j c(u_j) = 0.$$
We consider \( \tilde{u}(\zeta) = \sum b_j u_j(\zeta) \). Then \( \tilde{u}(\zeta) \in C^\alpha \) would be a vector function with every component strictly positive. However, on the other hand,

\[
\int g\tilde{u} = \int \sum b_j (gu_j) = \sum b_j (u_j) = 0
\]

which implies that \( \tilde{u} \in \mathbb{R} \cap C^\alpha = \mathcal{R} \), i.e., we have reached a contradiction. \( \blacksquare \)

Now we continue with the proof of Theorem 3.3. Take a point \( a_0 \) not in \( \{ x : \exists u \in B_1^+ \mid x \cdot c(u) > 0 \} \) with \( \| a_0 \| = 1 \). Then

\[
\int (a_0 g(\zeta)) \cdot u(\zeta) d\zeta = a_0^* \int g(\zeta) u(\zeta) d\zeta = a_0^* c(u) \leq 0 \quad \forall u \in B_1^+,
\]

and therefore inequality (14) holds. \( \blacksquare \)

Now we continue with the proof of Theorem 2.1(II). Since \( f^* \) is an optimum, there is no strictly positive function in the range of the derivative map

\[
\mathcal{F}^\alpha : H_N^2 \cap C^\alpha \rightarrow C^\alpha, \quad \mathcal{F}^\alpha(w)(\zeta) = 2\text{Re} \left( \frac{\partial \Gamma(\zeta, f^*(\zeta))}{\partial z} w(\zeta) \right).
\]

We observe that Lemma 5.5 and (12) imply that \( (\text{range} \mathcal{F}^\alpha)^\perp \) is a finite-dimensional subset of \( C^\alpha \). We claim that range \( \mathcal{F}^\alpha \cap C^\alpha = \text{range} \mathcal{F}^\alpha \). To show the claim, we first note that range \( \mathcal{F} \), where \( \mathcal{F} \) is defined as in Lemma 3.1, is closed in \( L^2 \). Therefore, range \( \mathcal{F}^\alpha = \text{range} \mathcal{F} \). The \( C^\alpha \)-hypoellipticity of \( \mathcal{F} \) (see Lemma 5.5) implies that range \( \mathcal{F} \cap C^\alpha = \text{range} \mathcal{F}^\alpha \), and the claim follows.

The argument above shows that we can apply Theorem 3.3 with \( \mathcal{R} = \text{range} \mathcal{F}^\alpha \). Therefore, there is a vector-valued function \( \lambda \) in \( (\text{range} \mathcal{F})^\perp \) with each component positive. By Lemma 3.1, \( \lambda \) must satisfy (9), which implies Theorem 2.1(II). \( \blacksquare \)

4. Additional results. In this section, we state the results on uniqueness and existence of the M-OPT problem which are, in fact, easy corollaries of the results proved in [HMar], [HM1], and [V].

4.1. Uniqueness. Consider the sublevel sets

\[
\mathcal{S}_0^\alpha(\gamma) := \{ z \in \mathbb{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma \}.
\]

**Theorem 4.1.** Suppose that the performance functions \( \Gamma_1, \ldots, \Gamma_l \) satisfy Assumption 1. Suppose that the sublevel sets \( \mathcal{S}_0^\alpha(\gamma_j) \) are strictly convex for every \( j = 1, \ldots, l \), every \( \theta \in \mathbb{T} \), and every \( \gamma_j \).

Assume that the real numbers \( \gamma_1^a, \ldots, \gamma_l^a \) have the property that for every function \( f \) with performances \( \gamma_1(f), \ldots, \gamma_l(f) \) satisfying \( \gamma_j(f) \leq \gamma_j^a \), Assumption 2 holds and the matrix \( \partial \mathcal{F}^\alpha(e^{i\theta}, f)(e^{i\theta})/\partial z \) has rank 1.

Then if \( f^* \in H_N^\infty \cap C^\alpha \) with \( \alpha > 1/2 \) is a Pareto optimum with performances

\[
\sup_{\theta} \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) = \gamma_j(f^*),
\]

this Pareto optimum is unique, namely, there is no other function \( f \in H_N^\infty \cap C^\alpha \), \( \alpha > 1/2 \), with the property that \( \gamma_j(f) = \gamma_j(f^*) \) for all \( j = 1, \ldots, l \).

**Proof.** Suppose that such an \( f \) does exist. Then

\[
f(e^{i\theta}) \in \partial \mathcal{S}_0^\alpha(\gamma_1^a) \cap \cdots \cap \partial \mathcal{S}_0^\alpha(\gamma_l^a) \quad \forall e^{i\theta} \in \mathbb{T}
\]
by the flatness result. Here $\gamma^*_j = \gamma_j(f^*)$. Since (15) is true for $f^*$ as well, the function $h = 1/f + 1/f^*$ has the property that

$$h(e^{i\theta}) \in S^1_0(\gamma^*_1) \cap \cdots \cap S^1_l(\gamma^*_l) \quad \forall e^{i\theta} \in T.$$ 

Strict convexity implies that there exists $\theta_0$ satisfying

$$h(e^{i\theta_0}) \in \text{int} S^1_0(\gamma^*_1) \cap \cdots \cap \text{int} S^1_0(\gamma^*_l).$$

If the performance functions evaluated at $h$ are flat (i.e., the flatness result (Theorem 2.1(1)) holds for $h$), then (16) should hold for every $\theta$, and therefore $f^*$ is not an optimum. If the performance functions of $h$ are not flat, then by Theorem 2.1, they all can be improved simultaneously, and therefore $f^*$ is not an optimum in this case either.

4.2. Existence. The following theorem was proved in [HMar].

**Theorem 4.2.** Suppose that $\Gamma(e^{i\theta}, z)$ is a positive continuous function and suppose that the sublevel sets of $\Gamma$ satisfy $\partial S_\theta(\gamma) = \{e^{i\theta} : \Gamma(e^{i\theta}, z) = \gamma\}$. Suppose further that the sublevel sets $S_\theta(\gamma)$ are uniformly bounded and polynomially convex. Suppose that $f^n \in H_N^\infty$ satisfy $\lim_{n \to \infty} \|\Gamma(e^{i\theta}, f^n(e^{i\theta}))\|_\infty = \gamma$. Let $f$ be a normal limit of $f^n$. Then $\|\Gamma(e^{i\theta}, f(e^{i\theta}))\|_\infty \leq \gamma$.

As a corollary, we can state the following existence result.

**Theorem 4.3.** Suppose that $\Gamma_j$, $j = 1, \ldots, l$, are positive $C^1$ functions. Suppose that the sublevel sets satisfy $\partial S^{l}_\theta(\gamma) = \{e^{i\theta} : \Gamma_j(e^{i\theta}, z) = \gamma\}$ for $j = 1, \ldots, l$ and for every $\gamma$ and $\theta$. Suppose further that the sets $S^{1}_\theta(\gamma) \cap \cdots \cap S^{l}_\theta(\gamma)$ are uniformly bounded and polynomially convex for every $\gamma$.

Then there exists a Pareto optimum for $\Gamma_1, \ldots, \Gamma_l$.

**Remark.** Note that we do not impose Assumption 1 in this theorem. In particular, the flatness property of the optimum needs not hold.

**Proof.** First, we reduce the problem to the case of one performance function by introducing

$$\tilde{\Gamma}(e^{i\theta}, z) := \max \{\Gamma_1(e^{i\theta}, z), \ldots, \Gamma_l(e^{i\theta}, z)\}.$$ 

We start with an initial guess $f^1 = 1$. We then make further guesses, $f^n$'s, improving $\tilde{\Gamma}$ if possible. Since the $S_\theta(\gamma)$'s are uniformly bounded, the set $\{f^n\}$ is uniformly bounded in $H_N^\infty$ and therefore has a subsequence, converging locally uniformly in the open unit disc to some limit $f$. Applying Theorem 4.2, we conclude that $f$ is an optimum for $\tilde{\Gamma}$ with $\sup_\theta \tilde{\Gamma}(e^{i\theta}, f(e^{i\theta})) = \gamma$. \qed

5. Technical lemmas. In this section, we reproduce the proofs of several lemmas that were proved by Trepeau and which have not been published. They can all be found in the preprint [T]. Lemma 5.5 is very close to the results proved in [BG].

**Proposition 5.1.** Suppose that $A \in C^n(T)$ with $n > 1/2$ is an $l \times N$ matrix-valued function of maximum rank $l$ at every point on $T$. Then there exists an $N \times N$ constant complex matrix $H$ such that the first $l$ columns of $AH$ are linearly independent at every point on $T$.

To prove this proposition, we will need the following lemma, in which the role of the assumption $n > 1/2$ becomes clear.

**Lemma 5.2.** Given $u(x) \in C^n([0, 1], C)$ with $n > 1/2$, the range of $u$ has zero Lebesgue measure in $C$. 

Proof. Divide the interval \([0,1]\) into \(n\) equal parts by points \(j/n, j = 0, \ldots, n\). Then the range of \(u\) can be covered by the \(n\) discs of radius \(C(1/n)^\alpha\) centered at \(u(j/n), j = 0, 1, \ldots, n\), where \(C\) is the Hölder constant of \(u\). The total measure of these discs does not exceed 
\[
n\pi(C(1/n)^\alpha)^2 = C^2\pi n^{1-2\alpha}.
\]
Since \(\alpha > 1/2\), this quantity tends to zero as \(n\) approaches infinity. \(\Box\)

To prove Proposition 5.1, we begin by proving a special case.

Lemma 5.3. Let \(a \in C^\alpha(\mathbf{T}, \mathbf{C})\), \(\alpha > 1/2\), be the \(N\)-vector such that \(a \neq 0\) anywhere on \(\mathbf{T}\). Then there exists \(\beta \in \mathbf{C}^N\) such that \(a(a^\alpha) \cdot \beta\) vanishes nowhere on \(\mathbf{T}\).

Proof. We will prove the statement for any open subset \(I\) of \(\mathbf{T}\) by induction on \(N\). (We will assume that \(a\) is of class \(C^\alpha\) uniformly on \(I\).) Since the case where \(N = 1\) is trivial, we assume that the lemma holds for \(N-1\). Let \(Z \subset I\) be the zero set of \(a_N\) and \(Z_0 \subset I\) be an open neighborhood of \(Z\) on which \((a_1, \ldots, a_{N-1}) \neq 0\). By the induction hypothesis, there exists \((\beta_1, \ldots, \beta_{N-1})\) such that \(a_1 \beta_1 + \cdots + a_{N-1} \beta_{N-1}\) does not vanish on \(Z_0\). Also, the function \((a_1 \beta_1 + \cdots + a_{N-1} \beta_{N-1})/a_N \in C^\alpha(I \setminus Z)\) cannot be onto \(\mathbf{C}\) by Lemma 5.4 below. (Note that \(a/a_N\) is bounded on \(I \setminus Z_0\).)

Therefore, we can find a number \(-\beta_N\) that is not in the range of \(\sigma/a_N\), and so \(a_1 \beta_1 + \cdots + a_{N-1} \beta_{N-1} + a_N \beta_N\) vanishes nowhere on \(I\). \(\Box\)

Lemma 5.4. Let \(\sigma := a_1 \beta_1 + \cdots + a_{N-1} \beta_{N-1}\). The range of the function \(\sigma/a_N \in C^\alpha(Z_0 \setminus Z)\) is not the whole of \(\mathbf{C}\).

Proof. Note that \(Z_0 \setminus Z\) is a countable union of open intervals \((t_j, t_{j+1})\). We will show that the range of each of the restrictions \(\sigma/a_N \in C^\alpha((t_j, t_{j+1}))\) is of measure zero in \(\mathbf{C}\). Note that \(\sigma \neq 0\) on \(Z_0\) and \(a_N\) can vanish only at the endpoints \(t_j\). If it vanishes at \(t_j\), then it satisfies
\[
|a_N(t) - 0| \leq c|t - t_j|^{\alpha},
\]
which implies
\[
|\sigma(t)/a_N(t)| \geq c'|t - t_j|^{-\alpha}.
\]

Divide \(\mathbf{C}\) into the annuli \(L_k = \{z : k \leq |z| \leq k + 1\}\).

Away from points where \(a_N\) vanishes, we use the fact that both \(\sigma\) and \(a_N\) are of class \(C^\alpha\) uniformly on \(\mathbf{C} \setminus Z_0\) to conclude that the restriction of the range of \(\sigma/a_N\) \(|(t_j, t_{j+1})\) onto \(L_k\) is a curve which is uniformly \(C^\alpha\). In the neighborhood of points where \(a_N\) vanishes, we use estimate (18) to reach the same conclusion. Therefore, the range of \(\sigma/a_N\) \(|(t_j, t_{j+1})\) has zero measure in \(\mathbf{C}\) by Lemma 5.2. \(\Box\)

Proof of Proposition 5.1. We write \(A = (A_1, \ldots, A_N)\), where the \(A_j\)'s are the columns of \(A\). For \(p = 1, \ldots, l\), we denote the vector formed by the first \(p\) components of \(A_j\) by \(A_p^j\). By induction on \(p\), we will show that there exists an invertible constant matrix \(H\) such that the matrix \((AH_p^1, \ldots, AH_p^p)\) has rank \(p\) at every point.

Since \(A\) has rank \(l\) at every point, \((A_1, \ldots, A_N)\) has rank 1 and by Lemma 5.3 there exists \((\beta_1, \ldots, \beta_N)\) such that \(\beta_1 A_1 + \cdots + \beta_N A_N\) does not vanish. Let \(H_1\) be an invertible matrix with the first column \((\beta_1, \ldots, \beta_N)^t\). Then \((AH_1)^1\) has rank 1 everywhere.

Now we assume that the claim is true for \(p\). Replacing \(A\) by \(AH_p\), we can assume that \((A_p^1, \ldots, A_p^p)\) has rank \(p\) everywhere. This implies that at every point on \(\mathbf{T}\), one of the determinants
\[
d_j = \det(A_p^{j+1}, A_p^{j+1}, A_p^{j+1}), \quad j = p + 1, \ldots, N,
\]
is nonzero. By Lemma 5.3, we can find $\beta_j, j = p+1, \ldots, N$, such that $d_{p+1}\beta_{p+1} + \cdots + d_N\beta_N$ is nonzero on $T$. Let $\tilde{H}$ be a $(N-p) \times (N-p)$ invertible matrix whose first column is $\beta_{p+1}, \ldots, \beta_N$ and set

$$H_{p+1} = \begin{pmatrix} I_{p \times p} & 0 \\ 0 & \tilde{H} \end{pmatrix}$$

(20)

to obtain an $N$-dimensional matrix. Then the matrix whose columns are $(AH_{p+1})_{p+1}^1, \ldots, (AH_{p+1})_{p+1}^N$ has the determinant equal to

$$\det \left( A_{1}^{p+1}, \ldots, A_{p}^{p+1}, \sum_{j=p+1}^{N} \beta_j A_j^{p+1} \right) = d_{p+1}\beta_{p+1} + \cdots + d_N\beta_N$$

and therefore has rank $p+1$ everywhere. $\square$

Here is one more technical lemma that we used in the proofs.

**Lemma 5.5.** Let the map

$$F_A : H_N^2 \rightarrow L_1^2(T, \mathbb{R})$$

be defined by $F_A(w) = 2\text{Re}(Aw)$, where the $l \times N$ matrix $A \in C^\alpha$ has rank $l$ everywhere on $T$. Then we have

$$\text{(range } F_A)^\perp \subset C^\alpha(T, \mathbb{R}).$$

(21)

**Proof.** First, we prove (21) for the case where $N = l$. We will use the following result.

**Theorem 5.6** (see [Ve]). Suppose that $A$ is a square $l \times l$ matrix-valued function on $T$ with invertible values such that $A \in C^\alpha(T)$. Then there exists a holomorphic matrix-valued function $S \in C^\alpha(T)$ such that $S^{-1} \in H_{l \times l}^2$ and such that

$$A^{-1} = S^{-1}DS.$$

(22)

Here $D$ is the diagonal matrix with entries $e^{ik_1\theta}, \ldots, e^{ik_N\theta}$, where $k_j$ are the integers.

Vekua proved a slightly stronger result in his book [Ve, section 13, p. 97] by constructing the fundamental matrix of solutions for the Hilbert problem. However, he did not state his result in the form above since the Gohberg–Krein factorization, in which form the result is presented here, was discovered much later. The reduction is easy and can be found in, for example, [G].

To prove (21), consider the real-linear map

$$\mathcal{G} : q \in L^2(T, \mathbb{R}) \rightarrow q + i\mathcal{H}(q) = w \rightarrow \text{Re}(Aw) \in L^2(T, \mathbb{R}).$$

(23)

Here $\mathcal{H}$ is the Hilbert transform with any particular fixed choice of normalization.

Suppose that $u \in L^2$ belongs to $(\text{range } F_A)^\perp$. Then $u \in \ker \mathcal{G}^\perp$, where $\mathcal{G}^\perp$ is the operator adjoint to $\mathcal{G}$ in $L^2(T, \mathbb{R})$, i.e.,

$$\mathcal{G}^\perp(u) = \text{Re}A^t u + \mathcal{H}(\text{Im}A^t u).$$

(24)

Note that $\text{range } \mathcal{G}$ may be smaller than $\text{range } F_A$ because of the normalization of $\mathcal{H}$. Therefore, $u \in L^2(T, \mathbb{R})$ satisfies the following equation:

$$\text{Re}A^t u + \mathcal{H}(\text{Im}A^t u) = 0.$$

(25)
Let us introduce the holomorphic function $\psi \in H^2_N$:

$$\psi := \text{Im}A^t u + i\bar{H}($$

Then (25) and (26) imply that

$$\text{Re}A^t u = -\text{Im}\psi, \quad \text{Im}A^t u = \text{Re}\psi.$$  

Therefore, we have

$$A^t u = \text{Re}A^t u + i\text{Im}A^t u = -\text{Im}\psi + i\text{Re}\psi = i\psi,$$

$$\bar{A}^t u = \text{Re}A^t u - i\text{Im}A^t u = -\text{Im}\psi - i\text{Re}\psi = -i\bar{\psi},$$

which can be rewritten as

$$u = (A^t)^{-1}i\psi,$$

$$u = -(A^t)^{-1}i\bar{\psi},$$

which implies

$$(A^t)^{-1}i\psi = -(A^t)^{-1}i\bar{\psi}.$$  

Thus $\psi$ is a solution of the following Riemann–Hilbert problem:

$$\text{Re}((A^t)^{-1}\psi) = 0.$$  

By the results of Vekua, $\psi \in C^\alpha(T)$. Then (30) implies that the same is true for $u$. Inclusion (21) is proved for $N = l$.

For the case where $N > l$, we multiply the $l \times N$ matrix $A$ by the $N \times N$ holomorphic matrix $H$ from Proposition 5.1. Then we consider the operator $F_{A,l}$ defined by the first $l$ columns of $AH$. Thus we have

$$(\text{range}F_A)^\perp \subset (\text{range}F_{A,l})^\perp \subset C^\alpha(T,R).$$  

6. Connection with the analytic-disc technique used in the problem of extending Cauchy–Riemann functions. We discuss some connections between the M-OPT problem and the analytic-disc techniques used for studying the theoretical several-complex-variables problem of extending a function defined on a manifold $M$ in $C^N$ to a function analytic in a neighborhood of a given point on $M$.

6.1. Attached analytic discs correspond to the flatness condition of M-OPT. First, we give some definitions. A smooth manifold $M = \{z \in C^N : \rho_1(z) = \cdots = \rho_t(z) = 1\}$ is called generic if its defining functions $\rho_1, \ldots, \rho_t$ satisfy $\partial \rho_1 \wedge \cdots \wedge \partial \rho_t \neq 0$, where $\partial \rho_j$ are defined as in (2). We say that a continuous function $u$ on $M$ is a CR function if it is annihilated by all tangential antiholomorphic vector fields:

$$\sum_j a_j \frac{\partial}{\partial z_j} u = 0 \quad \text{if} \quad \sum_j a_j \frac{\partial}{\partial z_j} \rho_k = 0 \quad \forall k.$$  

We study the question of local extendibility of CR functions on $M$ holomorphically to some neighborhood of a given point $p$ on $M$. The basic theorem for the analytic-disc method is the approximation theorem by Baouendi and Treves.
THEOREM 6.1 (see [BT]). Suppose that \( M \) is a generic manifold, \( p \) is a point on \( M \), and \( u \) is a CR function on \( M \cap \Omega \), where \( \Omega \) is a neighborhood of \( p \). Then there exists \( \Omega' \subset \Omega \), a smaller neighborhood of \( p \), such that \( u \) is a (uniform on \( M \cap \Omega' \)) limit of holomorphic polynomials.

It follows from the maximum-modulus theorem that if we can fill some neighborhood of \( p \) with the interiors of analytic discs \( f \) attached to \( M \) (i.e., the boundary \( f(\partial \Delta) \subset M \)), then the function \( u \) can be extended holomorphically to (a part of) that neighborhood.

The construction of analytic discs attached to a manifold was the main tool used by many authors to prove the extendibility results; see [Tu1], [Tu2], and [BRT] as well as [Bog] and the references therein.

6.2. The notion of defect versus the gradient-alignment condition. Since CR functions analytically continue to the set that is the union of “small” analytic discs attached to \( M \), it is clear that this set is important. With this in mind, we let \( R_M \) denote the subset of \( \mathbb{C}^N \) which is swept out by analytic discs attached to \( M \). Be aware that not every CR function on \( M \) extends to \( R_M \) since the “small”-disc requirement has been dropped. Basic information about extension is provided by the notion of defect.

**Definition.** Given an analytic disc \( f \) attached to a manifold \( M = \{ z \in \mathbb{C}^N : \rho_1(z) = \cdots = \rho_l(z) = 0 \} \), we define the defect of the disc \( f \) as the real dimension of the space

\[
\left\{ \lambda \in \mathbb{C}^l(T, \mathbb{R}) : \lambda(e^{i\theta}) \frac{\partial \rho(f(e^{i\theta}))}{\partial z} \in H^2_N \right\}.
\]

It can be shown (see, e.g., [BRT]) that for any \( p \) on \( M \), there exists \( \varepsilon(p) \) such that if \( \| f \|_{C^{1,\alpha}} < \varepsilon(p) \) then the defect is an integer between 1 and \( l \).

The notion of defect was introduced for small discs by Tumanov in [Tu1]. The definition that we give was first introduced in [BRT]. The advantage of the latter is that it makes sense for any disc (not just small discs).

It was proved by Tumanov that if for any \( \epsilon > 0 \) there exists an analytic disc \( f \in H^\infty_N \cap C^{1,\alpha} \) of defect zero and such that \( \| f(e^{i\theta}) - p \|_{C^{1,\alpha}} < \epsilon \), then the interiors of small analytic discs attached to \( M \) sweep out a wedge with edge \( M \) near \( p \), and therefore any CR function on \( M \) extends to that wedge.

With a given disc \( f \) attached to \( M \) with \( f(0) = q \), we associate an M-OPT problem:

*Find \( h \in H^\infty_N \), which is a Pareto optimum for \( l \)-performance functions \( \Gamma_{q_1}, \ldots, \Gamma_{q_l} \) defined by

\[
(34) \quad \Gamma_{q_j}(e^{i\theta}, z) = \rho_j(e^{i\theta} z + q), \quad j = 1, \ldots, l.
\]

Note that

\[
\frac{\partial \Gamma_q(e^{i\theta}, h(e^{i\theta}))}{\partial z} = e^{i\theta} \frac{\partial \rho(f(e^{i\theta}))}{\partial z}.
\]
Then the condition that \( f \) is of defect zero is equivalent to

\[
(35) \quad \lambda(e^{i\theta}) \frac{\partial \Gamma(e^{i\theta}, f(e^{i\theta}))}{\partial z} \xi e^{i\theta} H_N^2 \quad \forall \lambda \in C^\alpha_{\Gamma}(T, R),
\]

which means that Theorem 2.1(II) is violated.

In other words, if disc \( f \) is of defect zero, then it cannot be a Pareto optimum for \( \Gamma_q, \ldots, \Gamma_q \), where \( q = f(0) \).

We use this observation to prove the following proposition, which shows that we can fill out an open set in \( C^N \) with discs attached to a hypersurface and close to a disc \( f \) of defect zero. Similar results of Tumanov require \( f \) to be small.

**Proposition 6.2.** Suppose that the disc \( f \in H^\infty_N \cap C^\alpha \) is attached to a smooth hypersurface \( M = \{ z \in C^N : \rho(z) = 1 \} \). Suppose that the defect of \( f \) is equal to zero. Then the set \( \{ g(0) : g \in H^\infty_N, g(\partial \Delta) \subset M \} \) contains a neighborhood of \( f(0) \) in \( C^N \).

Note that we use \( H^\infty \) discs rather than \( H^\infty \cap C^\alpha \) discs to fill a neighborhood in \( C^N \).

A similar result for the special case where \( M \) is a generic manifold of real codimension \( N \) can be found in [G]. The theorem in [G] is stated in terms of factorization indices of the matrix \( \frac{\partial \rho(f)}{\partial z} \), but it can be restated as above.

**Proof of Proposition 6.2.** We can assume that \( \rho(f(0)) > 1 \). Consider the performance function \( \Gamma \) given by (34). Then (35) implies that Theorem 2.1(II) is violated and the quantity

\[
\gamma(q) = \inf_{h \in H^\infty} \sup_{\theta} \Gamma_q(e^{i\theta}, h(e^{i\theta}))
\]

is less than 1.

We need the following lemma.

**Lemma 6.3.** The function \( q \to \gamma(q) \) is continuous on \( C^N \).

**Proof.** For any given \( \epsilon \), take \( \delta > 0 \) such that

\[
\|h_1 - h_2\|_\infty \leq \delta \implies \|\rho(h_1(e^{i\theta})) - \rho(h_2(e^{i\theta}))\|_\infty \leq \epsilon.
\]

Then by the definition of \( \gamma(q) \), the inequality \( \|q - \tilde{q}\| < \delta \) implies \( \gamma(\tilde{q}) \geq \gamma(q) - \epsilon \). At the same time, it implies \( \gamma(q) \geq \gamma(\tilde{q}) - \epsilon \). Therefore, \( \|\gamma(q) - \gamma(\tilde{q})\| \leq \epsilon \).

Note that the function \( q \to f^*_q \), where \( f^*_q \) is the optimum for \( \Gamma_q \), needs not be continuous.

Now we continue with the proof of Proposition 6.2. By Lemma 6.3, there exists \( U \), an open neighborhood of \( q \) in \( C^N \), such that \( \gamma(\tilde{q}) < 1 \) for all \( \tilde{q} \in U \). Take a point \( \tilde{q} \in U \). Let \( g \in H_N^\infty \) be a solution to the OPT problem with \( \Gamma_{\tilde{q}} \). Then \( \rho(e^{i\theta} g(e^{i\theta}) + \tilde{q}) < 1 \) for every \( \theta \). Denote by \( \Omega \) the connected component of the set

\[
\{ z \in \Delta : \rho(g(z)) > 1 \}
\]

containing zero. Construct \( \Omega^* \), the set containing \( \Omega \), according to the following procedure. Take a point \( \xi \in \Delta \setminus \Omega \). If there exists a closed Jordan curve in \( \Omega \) which encircles \( \xi \), then we set \( \xi \in \Omega^* \). If there is no such curve, we set \( \xi \notin \Omega^* \). Then \( \Omega^* \) is an open simply connected region such that \( \partial \Omega^* \subset \partial \Omega \), and therefore \( g|_{\partial \Omega^*} \equiv 0 \).

Let \( \Psi : \Omega \to \Delta \) be the Riemann map such that \( \Psi(0) = 0 \). Then \( g \circ \Psi^{-1} \) is the analytic disc which is attached to \( M \) and which passes through \( \tilde{q} \).
6.3. Manifolds of analytic discs. Motivated by the successful application mentioned above, the set of analytic discs attached to a manifold was studied in general. It was shown in [BRT] that the set of analytic discs, infinitely close to a point, forms an infinite-dimensional manifold. Namely, it was shown that the derivative

\[ F : H_N^\infty \cap C^\alpha \to C^\alpha(T, R), \quad F(w) = 2\text{Re} \left( \frac{\partial \rho(f(e^{i\theta}))}{\partial z} w \right) \tag{36} \]

is onto \( C^\alpha(T, R) \) for \( f \) close to a constant disc. Then application of the local-submersion theorem shows that we have a manifold.

In [F], [T], and [O], the restriction on the size of the disc \( f \) was removed and it was shown that under certain conditions the map (36) is onto.

In the engineering setup, if the set of discs close to \( f^* \) forms a manifold, then \( f^* \) is not an optimum. More precisely, suppose we are given a disc \( f^* \) attached to a loop of manifolds \( M_\theta = \{ \Gamma(e^{i\theta}, z) = c \} \). Then if the map (36) is onto, then there exists a holomorphic direction \( h \) such that

\[ 2\text{Re} \left( \frac{\partial \Gamma}{\partial z} h \right) < 0, \]

and therefore \( \gamma(f^* + th) < \gamma(f^*) \) for small \( t \), which means that \( f^* \) is not an optimum.

The following proposition shows how the defect is connected with the “manifold question” discussed above.

**Proposition 6.4** (see [BRT] and [T]). Suppose that the disc \( f \in H_N^\infty \cap C^\alpha \) is attached to a smooth manifold \( M = \{ z \in C^N : \rho_1(z) = \cdots = \rho_l(z) = 0 \} \). Let \( E \) be the set of discs passing through point \( f(0) \), i.e., \( E = \{ g \in H_N^\infty \cap C^\alpha : g(0) = f(0) \} \). If the defect of \( f \) is equal to zero, then the derivative map

\[ F_0 : T_E \to C^\alpha(T, R), \quad F_0(h) = 2\text{Re} \left( \frac{\partial \rho(f)}{\partial z} h \right) \]

is onto. Here \( T_E \) denotes the tangent space to \( E \) at \( f \), i.e., \( T_E = \{ h \in H_N^\infty \cap C^\alpha : h(0) = 0 \} \).

The proof of Proposition 6.4 is a line-by-line repetition of the proof of Lemma 3.1.

One can easily verify that Proposition 6.4 implies that the set of discs attached to \( M \) and passing through \( f(0) \) forms a Banach manifold near \( f \).

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