Synthesis of Optimal Control for Some Nonlinear Systems with Small Parameter and Nonclassical Criterion

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Abstract

Optimal control synthesis is constructed for systems with perturbations in the form of small nonlinear terms of order \( \varepsilon \). If \( \varepsilon = 0 \), the initial system is bilinear, that is, linear on phase coordinates when the control is fixed, and linear on the control when the coordinates are fixed. The minimized functional differs from the quadratic one by some nonlinear term. When \( \varepsilon = 0 \), the optimal control synthesis is found in an exact analytic form. Successive approximations to the optimal control are constructed with the help of the perturbation method. Error estimates of the suggested method are presented.

1 Introduction

An important class of nonlinear control systems is bilinear systems. Such systems are linear on phase coordinates when the control is fixed, and linear on the control when the coordinates are fixed. The first point for the study of bilinear systems is to investigate the dynamic processes of nuclear reactors, kinetics of neutrons, and heat transfer, which started at the beginning of the 60s [1]. Further investigations show that many processes in engineering, biology, ecology and other areas can be described by the bilinear systems [2]. As an example, in [3] it is shown that bilinear systems may be applied to describe some chemical reactions and many physical processes in the growth of the human population. Then, in [4] we can see the theoretical and applied aspects of bilinear systems, and their structural properties. Also we can find some questions of identification and optimization. In the excellent surveys [5],[6] various results for deterministic and stochastic bilinear systems are systematized and the basis for the further development of the mathematical interest to such systems is created. In these works the questions of optimal control, stability, filtration and estimation are considered. We also mention paper [7] where the optimization method for bilinear systems is developed by means of periodic control functions, and where it is shown how this method can be used in cell division control processes and cancer therapy. In [8] a special representation of the trajectories of bilinear systems is obtained, and also, their continuous dependence on controls.

We consider control nonlinear systems that can be described in the form

\[
\dot{x}(t) = A(t)x(t) + B(t)x(t)u(t) + \varepsilon f_1(t,x),
\]

\[
x(0) = x_0, \quad 0 \leq t \leq T. \quad (1)
\]

Here the vector \( x(t) \) is from the Euclidean space \( E_n \), the control \( u(t) \) is in \( E_m \), the matrices \( A \) and \( B \) have continuous and bounded elements, \( \varepsilon \geq 0 \) is a small parameter, and the initial vector \( x_0 \) is in \( E_n \) and the constant \( T \geq 0 \) are given. The function \( f_1(t,x) \) is continuous in the totality of its arguments, and for all \( x \) and \( y \) satisfies

\[
|f_1(t,x) - f_1(t,y)| \leq C|x - y|, \\
|f_1(t,x)|^2 \leq C(1 + |x|^2), \quad (2)
\]

where \( | \cdot | \) is the Euclidean norm of \( x \). Note that if \( \varepsilon = 0 \), initial system (1) is bilinear, that is, it contains a nonlinearity of form \( x(t)u(t) \). It is known, that if \( u = 0 \), there exists only one solution of the Cauchy problem (1) [9]. The problem is to find a control \( u \) minimizing the functional \( J(0,u) \), where

\[
J(x,t) = x'(t)H_1x(t) + \int_0^T [x'(s)H_2(s)x(s) + u'(s)H_3(s)u(s) + f(s,x)] ds. \quad (3)
\]

Here \( H_i, \ i = 1,2,3 \) are given matrices, so that \( H_1, H_2(t) \) are non-negative defined, \( H_3(t) \) is positive defined in the interval \([0,T]\), and the matrices \( H_2(t) \) and \( H_3(t) \) are measurable and bounded. The vector \( f(t,x) \) is determined below, and a prime indicates the transpose. The functional (3) is differs from the quadratic cost criterion and is called nonclassical [10].

2 Algorithm of successive approximations

We denote by \( u_0(t,x) \) and \( V_0(t,x) \) the optimal control and Bellman function, respectively, in problem (1), (3) with \( \varepsilon = 0 \).
Suppose that $V_0(t,x)$ satisfies the Bellman equation [11]
\[
\min_{u} \left[ \frac{\partial V_0}{\partial t} + \left( \frac{\partial V_0}{\partial x} \right)^t (Ax + B xu) + x^{'} \ H_2 \ x + u^{'} \ H_3 \ u + f \right] = 0.
\]

Here $\frac{\partial V_0}{\partial t}$ is the partial derivative with respect to time, and $\frac{\partial V_0}{\partial x}$ - the vector of partial derivatives with respect to coordinates of the vector $x$.

From (4) follows that $u_0(t,x)$ is given by
\[
u_0(t,x) = - \frac{1}{2} H_3^{-1}(t) B^{'}(t) \frac{\partial V_0}{\partial x}.
\]

Now we require the Bellman function $V_0(t,x)$ in the following form
\[
V_0(t,x) = x^{'} P(t)x,
\] (6)
where the symmetric matrix $P$ will be defined.

If we substitute (5) and (6) in the Bellman equation (4), we obtain on the left-hand side the quadratic function of the phase vector $x$ with coefficients depending on time. Equating to zero the quadratic term coefficients gives the linear matrix differential equation for the matrix $P$, that is,
\[
\dot{P} + PA + A^{'} P + H_2 = 0, \quad P(T) = H_1. \tag{7}
\]

Moreover, the matrix $P$ is bounded and non-negative defined in the interval $0 \leq t \leq T$. According to (5)-(7), the optimal control $u_0(t,x)$ in problem (1), (3) with $e = 0$ is given by
\[
\dot{u}_0 = x^{'} H_3^{-1}(t) B^{'}(t) P(t)x^2.
\] (8)

If we set to zero the remaining terms (after finding $P$), we obtain for the function $f(t,x)$ from (3):
\[
f(t,x) = (x^{'} P^{2} B(t) H_3^{-1}(t) B(t) P(t)x^2,
\] (9)
which with regards to (8), may be written in the form
\[
f = u_0 H_3 u_0.
\] (10)

Now, let $v(t,x)$ and $V(t,x)$ be the optimal control and Bellman function, respectively in problem (1), (3), and suppose that $V(t,x)$ satisfies the Bellman equation
\[
\min_{u} \left[ \frac{\partial V}{\partial t} + \left( \frac{\partial V}{\partial x} \right)^t (Ax + B xu) + \epsilon f_1 + x^{'} \ H_2 \ x + u^{'} \ H_3 \ u + f \right] = 0.
\] (11)

From this equation follows, that
\[
v(t,x) = - \frac{1}{2} H_3^{-1}(t) B^{'}(t) \frac{\partial V}{\partial x} x.
\] (12)
We introduce the notation
\[
B_{i}(t) = B(t) H_3^{-1}(t) B^{'}(t).
\] (13)

We substitute (12) in (11) to obtain the Cauchy problem for the Bellman function $V(t,x)$
\[
\frac{\partial V}{\partial t} + \left( \frac{\partial V}{\partial x} \right)^t (Ax - \frac{1}{4} x^{'} \left( \frac{\partial V}{\partial x} \right) B_{1} \frac{\partial V}{\partial x} x + x^{'} H_2 x + f = 0,
\] (14)
\[
V(T,x) = x^{'} H_1 x.
\]

Now we describe the algorithm of the successive approximations to the optimal control $v(t,x)$. The Bellman function $V(t,x)$ is represented as a series of the parameter $\epsilon$ [13]
\[
V(t,x) = v_0(t,x) + \epsilon v_1(t,x) + \epsilon^2 v_2(t,x) + \ldots \tag{15}
\]

Here the function $V_0(t,x)$ is the Bellman function for problem (1), (3) with $\epsilon = 0$, and defined according to (6) and (7).

Substituting (15) into (14) and equating the coefficients of the same powers of $\epsilon$ gives the linear equations for determining the rest of the functions $V_j$, $j \geq 1$,
\[
\frac{\partial V_j}{\partial t} + f_{1} \frac{\partial V_{j-1}}{\partial x} + x^{'} A \frac{\partial V_{j}}{\partial x} - \frac{1}{2} x^{'} \left( \frac{\partial V_{j}}{\partial x} \right) B_{1} \frac{\partial V_{j}}{\partial x} x + \frac{1}{4} \sum_{k=0}^{j-1} x^{'} \left( \frac{\partial V_{k}}{\partial x} \right) B_{1} \frac{\partial V_{j-k}}{\partial x} x = 0,
\]
\[
V_j(T,x) = 0. \tag{16}
\]

According to (12) and (15), the optimal control will be given by the formula
\[
v(t,x) = - \frac{1}{2} H_3^{-1}(t) B^{'}(t) \left( \frac{\partial V_0}{\partial x} + \epsilon \frac{\partial V_1}{\partial x} + \ldots + \epsilon^3 \frac{\partial V_3}{\partial x} \right) x, \tag{17}
\]
and i-approximation, $u_i(t,x)$ to the optimal control - by (12) in which the partial sum of (15) instead of $V$ is substituted, that is, by
\[
u_i(t,x) = - \frac{1}{2} H_3^{-1}(t) B^{'}(t) \left( \frac{\partial V_0}{\partial x} + \epsilon \frac{\partial V_1}{\partial x} + \ldots + \epsilon^i \frac{\partial V_i}{\partial x} \right) x.
\] (18)

By virtue of (6), we can write the equations for $V_j$, $j \geq 1$, in the following way:
\[
\frac{\partial V_j}{\partial t} + f_{1} \frac{\partial V_{j-1}}{\partial x} + \left[ x^{'} A \frac{\partial V_{j}}{\partial x} - \frac{1}{2} x^{'} \left( \frac{\partial V_{j}}{\partial x} \right) B_{1} \frac{\partial V_{j}}{\partial x} x - \frac{1}{4} \sum_{k=1}^{j-1} x^{'} \left( \frac{\partial V_{k}}{\partial x} \right) B_{1} \frac{\partial V_{j-k}}{\partial x} x \right] = 0,
\]
\[
V_j(T,x) = 0. \tag{19}
\]
If $j = 1$, the last term on the left-hand side of (19) is equal to zero. Taking this into account we can produce the function $V_j(t,x)$ in a form depending only on $f_i(t,x)$ and on the preceding approximations $V_i(t,x)$, $i = 1, \ldots, j-1$:

$$V_j(t,x) = \int_t^T \left[ f_i(s,y) \frac{\partial V_{j-1}(s,y)}{\partial y} \right] ds,$$

$$-\frac{1}{4} \sum_{k=1}^{j-1} \gamma_k' \left( \frac{\partial V_k(s,y)}{\partial y} \right) B_1(s, \frac{\partial V_{j-2}(s,y)}{\partial y}) \right] ds,$$

$$V_j(T,x) = 0,$$

where $y$ is given by

$$\dot{y}(s) = \left[ A(s) - \frac{1}{2}(y(s))^2 P(s) + B_1(s) y(s) \right] ds,$$

and $y(t) = x$, $s \geq t$. Now we pass to accuracy estimation of this method.

3 Error estimation for the zero-approximation of the optimal control.

In this section we present an analysis of the zero-approximation of the optimal control. First, note that the initial system can be reduced to a form where the coefficient of $x(t)$ is equal to zero. Indeed, if we take

$$x(t) = z(t) w(t),$$

and the matrix $z(t)$ is defined by

$$\dot{z}(t) = A(t) z(t), \quad z(0) = I,$$

where $I$ is identical matrix, then the initial system (1) takes the form

$$\dot{w}(t) = D(t) y(t) u(t) + \varepsilon z^{-1}(t) f_i(t, [z(t),w(t)]),$$

$$D(t) = z^{-1} B(t) z(t).$$

Q.E.D. So, with no loss of generality, we may assume that the matrix $A(t)$ in (1) is equal to zero.

Sometimes we use the notation $X_{t,x}(t,u)$ for the solution of (1) in the interval $[\tau, T]$ with the control $u$ and the initial condition $X_{t,x}(\tau,u) = x$. Now we show that formula (8) gives the zero-approximation to the optimal control in problem (1), (3). To this end we must evaluate the difference

$$J(X_{t,x}(t,u)) - J(y_{t,x}(t,u_0)),$$

and show that it is of order $\varepsilon$. We introduce a new function $y_{t,x}(\tau,u)$ that is a solution of equation (1) with $\varepsilon = 0$ and control $u$ in the interval $t \leq \tau \leq T$ so that $y_{t,x}(t,u) = x$. In addition, let $u(\tau) = u_0(\tau, y_{t,x}(\tau,u_0)), t \leq \tau \leq T$. From the definition of the Bellman functions $V$ and $V_0$ we have

$$V(t,x) \leq J(X_{t,x}(\tau,u)) = V_0(t,x) + [J(X_{t,x}(\tau,u)) - J(y_{t,x}(\tau,u))].$$

With regard to the explicit form of the functional (3) we obtain

$$J(X_{t,x}(\tau,u)) - J(y_{t,x}(\tau,u)) =$$

$$X_t^{y_{t,x},x}(T,a) H_1 X_t,x(T,a) - y_{t,x}(T,a) H_1 y_{t,x}(T,a) +$$

$$+ \int_t^T (X_t^{y_{t,x},x}(\tau,a) H_2(\tau) X_{t,x}(\tau,a) - y_{t,x}(\tau,a) H_2(\tau) y_{t,x}(\tau,a)) d\tau$$

We denote

$$z = X_t^{y_{t,x},x}(T,a) H_1 X_{t,x}(T,a) - y_{t,x}(T,a) H_1 y_{t,x}(T,a) =$$

$$= [X_{t,x}(T,a) - y_{t,x}(T,a)]' H_1 [X_{t,x}(T,a) + y_{t,x}(T,a)].$$

By virtue of the motion equation (1)

$$X_{t,x}(\tau,a) - y_{t,x}(\tau,a) = e \int_t^\tau f(t, X_{t,x}(\tau,a)) d\tau +$$

$$+ \int_t^\tau B(\tau)(X_{t,x}(\tau,a) - y_{t,x}(\tau,a)) u(\tau) d\tau$$

Taking into account (2), we obtain

$$|X_{t,x}(\tau,a) - y_{t,x}(\tau,a)| \leq e \int_t^\tau |f_i(\tau, X_{t,x}(\tau,a))| d\tau +$$

$$+ \int_t^\tau |B(\tau)| |X_{t,x}(\tau,a) - y_{t,x}(\tau,a)| |u(\tau)| d\tau$$

(27)

From (27) and the Gronwall-Bellman lemma [12] it follows that

$$|X_{t,x}(\tau,a) - y_{t,x}(\tau,a)| \leq e\int_t^\tau |f_i(s, X_{t,x}(s,a))| ds \times$$

$$\times \exp \left( \int_t^\tau |B(s)| |u(s)| ds \right)$$

(28)

Since $B(t)$ and $u(t)$ are measurable and bounded, and the integral in (28) is taken in a finite interval, we can write (28) in the following form

$$|X_{t,x}(\tau,a) - y_{t,x}(\tau,a)|^2 \leq e^2 C +$$

$$+ e^2 C \int_t^\tau |X_{t,x}(s,a)|^2 ds.$$ (29)

Because $y_{t,x}(\tau,u_0)$ is the solution of (1) with $\varepsilon = 0$ and control (8), where $P(t)$ and $B(t)$ are measurable and bounded, we obtain

$$|X_{t,x}(\tau,a)|^2 \leq C(1 + |z|^2).$$ (30)
Now we use the Cauchy-Bunyakovskii inequality in (25) to obtain
\[
|z| \leq (|x| + |a|)^2. \tag{31}
\]
From (29) and (30) we conclude that
\[
|z| \leq \epsilon C(1 + |x|^2). \tag{32}
\]
The second difference in (24) is evaluated analogously, and as a result we have
\[
|J(X_{t,x}^*, \tau, a)) - J(y_{t,x}(\tau, a))| \leq \epsilon C(1 + |x|^2). \tag{33}
\]
With the help of (33) we get the important relation
\[
V(t, x) \leq V_0(t, x) + \epsilon C(1 + |x|^2). \tag{34}
\]
Now we introduce a new function \(b(\tau) = v(\tau, X_{t,x}(\tau, v))\), which is the optimal control value on the optimal trajectory \(X_{t,x}(\tau, v)\). Analogously to (23) we write
\[
V_0(t, x) \leq V(t, x) + J(y_{t,x}(\tau, b)) - J(X_{t,x}(\tau, b)). \tag{35}
\]
The difference \(J(y_{t,x}(\tau, b)) - J(X_{t,x}(\tau, b))\) is evaluated exactly as in (24). That is, first we write the relations (24), (25), (26) in which \(a(\tau)\) is replaced by \(b(\tau)\). Then, we obtain the inequalities (27)-(29) where again \(a(\tau)\) is replaced by \(b(\tau)\). Now, to obtain the inequality for \(b(\tau)\) as in (30) we must evaluate the integral
\[
\int_0^T |b(s)|^2 ds, \tag{36}
\]
since we have no explicit formula for \(b(\tau)\).

We denote by \(\mu > 0\) the lower bound of the minimal eigenvalue of \(H_2(t)\). Since \(v(t)\) is the optimal control, \(J(X_{t,x}(t,0)) \geq J(X_{t,x}(t, b))\). From here and (3) follows
\[
\mu \int_0^T |b(t)|^2 dt \leq J(X_{t,x}(t,0)). \tag{37}
\]
Next, similarly to (30) with \(u = 0\), we obtain
\[
|X_{t,x}(t,0)| \leq C(1 + |x|^2). \tag{38}
\]
Hence
\[
\int_0^T |b(t)|^2 dt \leq C(1 + |x|^2). \tag{39}
\]
Now we note that
\[
|V(t, x) - J(X_{t,x}(\tau, u_0))| \leq |V(t, x) - V_0(t, x)| + \|V_0(t, x) - J(X_{t,x}(\tau, u_0))| \tag{42}
\]
Thus, according to (41), we must evaluate \(|V_0(t, x) - J(X_{t,x}(\tau, u_0))|\). This difference may be written in the form
\[
|J(y_{t,x}(\tau, u_0)) - J(X_{t,x}(\tau, u_0))| \leq \epsilon C(1 + |x|^2). \tag{43}
\]
Comparing (24) with (43) shows that the difference (43) may be evaluated exactly as in the left-hand side of (24). So, we conclude that
\[
|J(y_{t,x}(\tau, u_0)) - J(X_{t,x}(\tau, u_0))| \leq \epsilon C(1 + |x|^2). \tag{44}
\]
Finally, (41) and (44) give
\[
0 \leq J(X_{t,x}(\tau, u_0)) - V(t, x) \leq \epsilon C(1 + |x|^2). \tag{45}
\]
We summarize the obtained result as a theorem.

**Theorem 1.** Let the coefficients of system (1), (3) satisfy the conditions of paragraph 1. So, if we use zero-approximation instead of the optimal control in the initial system, then the error with respect to the cost criterion (3) will not exceed the right-hand side of (45).

### 4 Error estimation for the first approximation to the optimal control.

Now we give an analysis of the first approximation, \(u_1(t, x)\), to the optimal control, which according to (18) has the form
\[
u_1(t, x) = -\frac{1}{2} H^{-1}_3(t) B^T(t) \left( \frac{\partial V_1(t,x)}{\partial x} + \epsilon \frac{\partial V_1}{\partial x} \right), \tag{46}\]
where the function \(V_1(t, x)\) is the solution of the boundary value problem
\[
\frac{\partial V_1}{\partial t} + 2x' P f_1 + \left( \frac{\partial V_1}{\partial x} \right)' A x - \frac{1}{2} (x')^2 B_1 \frac{\partial V_1}{\partial x} - \frac{1}{2} x' \frac{\partial V_1}{\partial x} B_1 P_0 = 0, \tag{47}\]
\[
V_1(T, x) = 0.
\]
First of all, we build a control system in which the control (46) is optimal and the function
\[
Q(t, x) = V_0(t, x) + \epsilon V_1(t, x)
\]
will be the Bellman function. To this end, we write (47) in the form
\[
\frac{\partial V_1}{\partial t} + f_1 \frac{\partial V_0}{\partial x} + \left( \frac{\partial V_1}{\partial x} \right)' B_1 \frac{\partial V_1}{\partial x} A x - \frac{1}{4} x' \left( \frac{\partial V_1}{\partial x} \right)' B_1 \frac{\partial V_1}{\partial x} x = 0, \tag{48}\]
\[
V_1(T, x) = 0.
\]
From eqs. (14) with \( \epsilon = 0 \) and (48) we obtain for the function \( Q(t,x) \):

\[
\frac{\partial Q}{\partial t} + \left( \frac{\partial Q}{\partial x} \right)' A x + \epsilon f_1 \frac{\partial V_0}{\partial x} - \frac{1}{4} x' \left( \left( \frac{\partial \dot{V}_0}{\partial x} \right)' B_1 \frac{\partial V_0}{\partial x} \right) x + x' H_2 x + f - \frac{1}{4} \epsilon x' \left( \left( \frac{\partial V_1}{\partial x} \right)' B_1 \frac{\partial V_1}{\partial x} \right) x - \frac{1}{4} \epsilon x' \frac{\partial V_1}{\partial x} = 0.
\]

(49)

After some non difficult simplifications we can write the equation (49) in following form

\[
\frac{\partial Q}{\partial t} + \left( \frac{\partial Q}{\partial x} \right)' A x + \epsilon f_1 \frac{\partial Q}{\partial x} - \frac{1}{4} x' \left( \left( \frac{\partial \dot{V}_1}{\partial x} \right)' B_1 \frac{\partial Q}{\partial x} \right) x + x' H_2 x + f = 0,
\]

(50)

\[
Q(T, x) = x' H_1 x.
\]

Comparing (50) with (14) gives important result, that the control \( u_1(t,x) \) is optimal for the following problem: find a control that on initial system (1) trajectories, minimizes the cost criterion \( J_1(X_{\tau,x}(t,u)) \), which is given by

\[
J_1(X_{\tau,x}(t,u)) = J(X_{\tau,x}(t,u)) + J^*(y),
\]

(51)

where \( J^*(y) \) defined by

\[
J^*(y) = \epsilon^2 \int_0^T \left[ \frac{1}{4} x' \left( \left( \frac{\partial \dot{V}_1(x,t)}{\partial y} \right)' B_1(x,t) \frac{\partial V_1(x,t)}{\partial y} \right) x + \epsilon f_1(x,t) \frac{\partial V_1(x,t)}{\partial y} \right] dt,
\]

(52)

with \( y = X_{\tau,x}(t,u) \). Note that in optimal control problem (1), (51), the Bellman function is exactly equals \( Q(t,x) \). This means, that the auxiliary optimal control problem is built. In this problem we see the same motion equations (1), and the cost functional (51) differs from (3) by the quantity \( J^*(y) \) of order \( \epsilon^2 \).

Now we note, that according to [14]

\[
\left| \frac{\partial V_1(t,x)}{\partial x} \right| \leq C(1 + |x|).
\]

(53)

With regards to the explicit form of \( V_0 \) (see (6)) and (52) we obtain that the control \( u_1 \) satisfies

\[
[u_1(t,x)] \leq C(1 + |x|).
\]

(54)

From here and [15] follows, that

\[
|X_{\tau,x}(t,u_1)|^2 \leq C(1 + |x|^2).
\]

(55)

Now note that according to (2), (52) and (54) we get

\[
|J^*(X_{\tau,x}(t,u_1))| \leq \epsilon^2 C(1 + |x|^2).
\]

(56)

Next, analogously to zero-approximation to the optimal control, we write

\[
V(t,x) \leq J(X_{\tau,x}(\tau,u_1)) = Q(t,x) - J^*(X_{\tau,x}(\tau,u_1)).
\]

Then, with the help of (55) we obtain

\[
V(t,x) \leq Q(t,x) + \epsilon^2 C(1 + |x|^2).
\]

(57)

Now we deduce analogously to (35)

\[
Q(t,x) \leq J_1(X_{\tau,x}(\tau,v)) = V(t,x) + J^*(X_{\tau,x}(\tau,v)).
\]

(58)

Earlier we obtained the relation

\[
|X_{\tau,x}(\tau,v)|^2 \leq C(1 + |x|^2).
\]

(59)

Hence

\[
|J^*(X_{\tau,x}(\tau,v))| \leq \epsilon^2 C(1 + |x|^2).
\]

(60)

From the last inequality and (58) follows, that

\[
Q(t,x) \leq V(t,x) + \epsilon^2 C(1 + |x|^2).
\]

(61)

From here and (57), we conclude

\[
|V(t,x) - Q(t,x)| \leq \epsilon^2 C(1 + |x|^2).
\]

(62)

If we use in initial system the control \( u_1(t,x) \), then the cost functional will be equal to

\[
J(X_{\tau,x}(\tau,u_1)) = Q(t,x) - J^*(X_{\tau,x}(\tau,u_1)).
\]

(63)

But for functional \( J^*(X_{\tau,x}(\tau,u_1)) \) we already have (55).

Then,

\[
|J(X_{\tau,x}(\tau,u_1)) - Q(t,x)| \leq \epsilon^2 C(1 + |x|^2).
\]

(64)

The last inequality with (62) finally give

\[
0 \leq J(X_{\tau,x}(\tau,u_1)) - V(t,x) \leq \epsilon^2 C(1 + |x|^2).
\]

(65)

Thus, we have proved the following theorem.

**Theorem 2.** Let the conditions of Theorem 1 are satisfied and the Cauchy problem (47) has a unique solution. So, if we use first approximation, \( u_1 \), instead of the optimal control in the initial system, then the error with respect to the cost criterion (3) will not exceed the right-hand side of (65).
5 Error estimation for the $i$-approximation to the optimal control

In this section we shall prove that if in the initial system we use the control of the $i$-approximation instead of the optimal control, then

$$J(X_{t,x}(\tau, u_i)) - V(t,x) = O(\epsilon^{i+1}). \quad (66)$$

Here, $O(\epsilon^{i+1})$ means the terms of order $\epsilon^{i+1}$ and $u_i$ are given by formula (18).

The validity of (66) is based on the following assertion [14]: there exist functional

$$J(X_{0,x}(t, u)) + O(\epsilon^{i+1}), \quad (67)$$

so that the control $u_i(t,x)$ will be optimal in the control problem (1), (67), and the Bellman function in this problem $W(t,x)$, will be given by

$$W(t,x) = V_0(t,x) + \epsilon V_1(t,x) + \ldots + \epsilon^i V_i(t,x). \quad (68)$$

Note that in (68), $V_i(t,x)$ is the solution of the boundary-value problem

$$\frac{\partial V_i}{\partial t} + f_1 \frac{\partial V_{i-1}}{\partial x} + x' A \frac{\partial V_i}{\partial x} - \frac{1}{4} \sum_{k=0}^i x' \left( B_1 \frac{\partial V_k}{\partial x} \right)' x = 0, \quad (69)$$

$$V_i(T,x) = 0.$$

Now we mention the basic steps in construction problem (1), (67). First, using (14), with $\epsilon = 0$, and equation (17) with $j = 1, \ldots, i$, we get an equation for the function $W(t,x)$. Then, making some simplifications, we write this equation as:

$$\frac{\partial W}{\partial t} + x' A \frac{\partial W}{\partial x} - \frac{1}{4} x' \left( \left( \frac{\partial W}{\partial x} \right)' B_1 \frac{\partial W}{\partial x} \right) x + \epsilon f_1^i \frac{\partial W}{\partial x} + x' H_2 x + f - \epsilon^{i+1} f_1^i \frac{\partial V_i}{\partial x} +$$

$$+ \frac{1}{4} \epsilon^{i+1} x' \left( \left( \frac{\partial V_i}{\partial x} \right)' B_1 \frac{\partial V_{i-1}}{\partial x} \right) x + \ldots + \ldots \quad (70)$$

$$+ \frac{1}{4} \epsilon^i x' \left( \left( \frac{\partial V_{i-1}}{\partial x} \right)' B_1 \frac{\partial V_i}{\partial x} \right) x = 0,$n

$W(T,x) = x' H_1 x.$

Now comparing (70) with (14), we conclude that the control $u_i(t,x)$ is optimal for the following problem: find a control that on system (1) trajectories, minimizes the cost functional $J_i(X_{t,u}(t,x))$, defined by

$$J_i(X_{t,u}(t,x)) = J(X_{t,u}(t,x)) + J_{i+1}(y). \quad (71)$$

Here $J_{i+1}$ is given by the formula

$$J_{i+1} = \epsilon^{i+1} \int_0^T \left[ \frac{1}{4} x' \left( \left( \frac{\partial V_i(t,y)}{\partial y} \right)' B_1(\tau) \frac{\partial V_i(t,y)}{\partial y} \right) x + \ldots + \right.$$

$$+ \left. \frac{1}{4} \epsilon^i x' \left( \left( \frac{\partial V_{i-1}(t,y)}{\partial y} \right)' B_1(\tau) \frac{\partial V_i(t,y)}{\partial y} \right) x - \right.$$

$$- f_1^i \frac{\partial V_i(t,y)}{\partial y} \right] d\tau,$$

where we use the notation $y = X_{t,u}(t,u)$.

According to this construction, the Bellman function of the problem (1), (71) will be exactly the same as $W(t,x)$, which was introduced in (68). This means, that the auxiliary optimal control is built. This problem has the same motion equations (1), and the cost criterion (71) differs from (3) by the quantity $J_{i+1}$ of order $\epsilon^{i+1}$.

References


