On the strong stabilizability of time-delay linear systems

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Abstract

In this paper we consider some algebraic aspects of output feedback stabilization of a time-delay linear system by a stable finite dimensional system. A sufficient condition for the existence of a stable finite dimensional stabilizing compensator for a single-input-single-output time-delay system is given.

1 Introduction

Stability and stabilization of time-delay systems have been extensively studied using various approaches, such as the analytic and algebraic methods, and more recently the LMI (linear matrix inequalities) method, in both the frequency domain and the time domain. See, e.g., [1, 6, 5] and the literature cited there.

In the frequency domain formulation, a linear single-input-single-output (SISO) system without time-delay (conventionally referred to as a deterministic finite dimensional system) can be described by a rational transfer function (or a rational transfer matrix for a multiple-input-multiple-output (MIMO) system), and can be treated adequately by powerful polynomial algebraic methods. However, a time-delay system involves exponential function terms, and it is hence more difficult to apply classical polynomial algebraic methods.

Nevertheless, many researchers have made fruitful efforts at solving control problems in time-delay systems in the algebraic framework. There are two representative approaches within the algebraic framework. One of them is to explore loyally the abstract algebraic structure of the rings generated by certain kinds of rational and exponential functions in a single variable in the frequency domain. This approach aims at obtaining exact solutions for time-delay systems and is an attractive direction for future research. See, for example, [9].

Another approach is to assume first the exponential term as an independent variable and apply purely algebraic and algebraic geometric methods to obtain useful results which are subsequently interpreted back into the time-delay system formulation. Although the latter usually can only obtain weak results due to its strong assumption, it can still yield very useful results just because of its simplicity and elegance. See, e.g., [10] for an interesting result by such a method. Closely related to this approach is the study of multidimensional (n-D) systems which tries to establish a unified formulation of the analysis and control of linear systems possessing more than one independent variables [2, 3].

In this note we apply a recent result on strong stabilizability for linear n-D systems to time-delay linear systems. Specifically, we present a useful algebraic criterion for determining the existence of a stable finite dimensional stabilizing compensator for a SISO linear system with a commensurate time-delay. We note that in the literature, e.g., [10], conditions have only been given for the existence of a stabilizing compensator, which is not necessarily stable itself. The problem of stabilization of an unstable plant by a stable stabilizing compensator is known as strong stabilization, which was solved by Youla et al. [20] for finite dimensional systems. In [20] a well-known parity interlacing property was given as a criterion for testing strong stabilizability. The generalization of this criterion for an n-D linear system has been given in [17], which is applied here to the determination of the strong stabilizability of a time-delay system. The main result, along with an illustrative example, is described in the next section. A conclusion with some open problems is given in Section 3.
2 Main Result

Consider the following transfer function with a commensurate time-delay $h > 0$.

$$p(s, e^{-hs}) = n(s, e^{-hs})/d(s, e^{-hs}),$$  

(1)

where $n$ and $d$ are polynomials in $s$ and $e^{-hs}$.

The transfer function is stable by definition if

$$d(s, e^{-hs}) \neq 0, \ Re\{s\} \geq -\gamma, \ for \ some \ \gamma > 0.$$  

(2)

Note: For some system, e.g., a retarded system, the state will be asymptotically stable if the characteristic function, which is the denominator of the transfer function, is free from 0 in $Re\{s\} \geq 0$. However, for systems of more general type, the stronger condition (2) described above is necessary for stability [8].

When $p(s, e^{-hs})$ is not stable, we consider the problem of finding a proper stable rational function to achieve the closed-loop stability for the feedback system. With an output feedback control configuration, as shown in Figure 1, the closed-loop transfer function is

$$c(s)p(s, e^{-hs}) = \frac{c(s)n(s, e^{-hs})}{d(s, e^{-hs}) + c(s)n(s, e^{-hs})},$$

and the above problem is reduced to the problem of finding a proper rational function $c(s)$ such that

$$c(s) \ is \ analytic \ in \ Re\{s\} \geq -\gamma.$$  

and

$$d(s, e^{-hs}) + c(s)n(s, e^{-hs}) \neq 0, \ Re\{s\} \geq -\gamma.$$  

(3)

Let

$$\bar{U}^2 = \{(z_1, z_2) \in C^2| \ |z_1| \leq 1, |z_2| \leq 1\}$$

be the closed unit bidisc in $C^2$. The problem is attacked by first looking upon the transfer function as a bivariate rational function over the closed unit bidisc by the following transform of variables:

$$z_1 = \frac{1 - s}{1 + s}, \ z_2 = e^{-hs}.$$  

(4)

The numerator and denominator of $p(s)$ can then be written as

$$n(s, e^{-hs}) = n\left(\frac{1 - z_1}{1 + z_1}, z_2\right) = \frac{n_1(z_1, z_2)}{(1 + z_1)^r},$$

$$d(s, e^{-hs}) = d\left(\frac{1 - z_1}{1 + z_1}, z_2\right) = \frac{d_1(z_1, z_2)}{(1 + z_1)^t},$$

where $r$ an $t$ are nonnegative integers, $n_1$ and $d_1$ are polynomials in $z_1, z_2$ and are free from the factor $1 + z_1$. Let

$$v = \max\{r + 1, t + 1\},$$

Let

$$f(z_1, z_2) = n(s, e^{-hs})/(1 + s)^v$$

$$= n\left(\frac{1 - z_1}{1 + z_1}, z_2\right)\frac{1 + z_1}{2},$$

$$g(z_1, z_2) = d(s, e^{-hs})/(1 + s)^t$$

$$= \frac{d_1(z_1, z_2)}{2}\frac{1 + z_1}{2}.$$  

(5)

It is clear that $f$ and $g$ are polynomials in $z_1$ and $z_2$ and that $(1 + z_1)$ is a factor of $f$ of order $v - r > 0$, but not a factor of $g$.

**Theorem 1** If there exists a bivariate polynomial $x(z_1, z_2)$ such that

$$g(z_1, z_2) + x(z_1, z_2)f(z_1, z_2) \neq 0,$$  

(6)

$$|z_1| \leq 1, \ |z_2| \leq 1,$$

then the plant $p(s, e^{-hs})$ can be stabilized by a proper stable rational transfer function for each $h > 0$.

**Proof:**

Since the algebraic variety defined by $g + xf = 0$ is a closed set in the Euclidean topology and $\bar{U}^2$ is compact, the condition (6) implies that

$$g(z_1, z_2) + x(z_1, z_2)f(z_1, z_2) \neq 0,$$  

(7)

$$|z_1| \leq 1 + \epsilon, \ |z_2| \leq 1 + \epsilon, \ for \ some \ \epsilon > 0.$$  

Let $\epsilon$ be chosen such that $\epsilon < 1$. By elementary operations on complex numbers, it is easy to show that for an arbitrary but fixed real number $\gamma, 0 < \gamma < \frac{1}{4\epsilon}$, if $Re\{s\} \geq -\gamma$, then $|\frac{1}{1 + s}| \leq 1 + \epsilon$. Let

$$\tilde{H}_\gamma = \{s \in C|Re\{s\} \geq -\gamma\} \cup \{\infty\}.$$  

The condition (7) implies that for each $h > 0$,

$$d(s, e^{-hs})/(1 + s)^t + x\left(\frac{1 - s}{1 + s}, e^{-hs}\right)n(s, e^{-hs})/(1 + s)^v \neq 0,$$  

(8)

$$Re\{s\} \geq -\gamma.$$  

Let $\phi$ denote the transform

$$s \mapsto \xi = \frac{1 - s}{1 + s},$$

then we have

$$\phi(\tilde{H}_\gamma) \subset \{\xi \in C| |\xi| \leq 1 + \epsilon\}.$$
Since $\gamma < 1$, it is clear that $x(\frac{1-s}{1+s}, e^{-hs})/(1+s)^k$ ($k = v - t > 0$) is continuous in $\bar{H}_s$, and is analytic in the interior of $\bar{H}_s$. Under the transform of the variable $\phi : s \mapsto \xi$, we obtain a function $x(\xi)/(\frac{1+s}{2})^k = x(\xi, e^{-h \frac{1-s}{2}})/(\frac{1+s}{2})^k$, which is continuous on $\phi(\bar{H}_s)$, and is analytic in its interior.

By Mergelyan’s Theorem (see [11] or [12]) $x(\xi)/(\frac{1+s}{2})^k$ can be approximated by a polynomial $c(\xi)$ uniformly over $\phi(\bar{H}_s)$. This implies the existence of a rational function $c(\frac{1+s}{1+s})$ such that

$$d(s, e^{-hs})/(1+s)^t +
\begin{aligned}
\frac{1}{1+s}n(s, e^{-hs})/(1+s)^t \neq 0,
\end{aligned}$$

or equivalently,

$$d(s, e^{-hs}) + \frac{1}{1+s}n(s, e^{-hs}) \neq 0,$$

$$Re\{s\} \geq -\gamma,$$

It is therefore clear that $c(\frac{1+s}{1+s})$ is a proper rational transfer function that stabilizes $p(s, e^{-hs})$.

Let $V(f)$ be the complex variety defined by $f$. The function $g$ is by definition zero-homotopic on $V(f) \cap \mathbb{C}^2$ if it maps every closed curve in $V(f) \cap \mathbb{C}^2$ into a cycle in $\mathbb{C}^*$ which has zero winding number around the origin [7, 17]. This condition can be tested by an algorithm using computer algebra [19]. As $g$ has real coefficients, its values on $V(f) \cap \mathbb{C}^2 \cap \mathbb{R}^2$ are real.

**Proposition 1 ([17])**

There exists a bivariate polynomial $x(z_1, z_2)$ such that

$$g(z_1, z_2) + x(z_1, z_2)f(z_1, z_2) \neq 0,$$

$$|z_1| \leq 1, \ |z_2| \leq 1,$$

if and only if the following two conditions are satisfied

(i) $g$ is zero-homotopic on $V(f) \cap \mathbb{C}^2$ and

(ii) $g$ has constant sign on $V(f) \cap \mathbb{C}^2 \cap \mathbb{R}^2$.

**Example: (Schumacher [14], Kamen [10])**

$$p = \frac{1}{s(s + \frac{\pi}{2})}, \quad z = e^{-s}.$$

$$f = 1/(1+s)^3 = (1+z_1)^3/8,$$

$$g = s(s + \frac{\pi}{2})/(1+s)^2 =
\begin{aligned}
\frac{1}{4}(1-z_1)(1-z_1 + \frac{\pi}{2}z_2(1+z_1)).
\end{aligned}$$

As $V(f) \cap \mathbb{C}^2 = \{-z_2, |z_2| \leq 1\}$ is a contractible set, $g$ is zero-homotopic on it. Since $g(-1,z_2) = 1$, the second condition in Proposition 1 is also satisfied. Therefore $p$ can be stabilized by a stable rational function.

Unfortunately we could not use the method in the proof of Theorem 1 for finding a stabilizer. However, there does exist a stable finite dimensional stabilizing compensator

$$c(s) = \frac{(25 - (3\pi/2))s + (3 + (\pi/2))4}{s + 10},$$

as was present in [10].

### 3 Conclusion

In this note we have presented an algebraic criterion for the existence of a stable rational function $c(s)$ for stabilizing a SISO time-delay system represented by a rational transfer function $p(s, e^{-hs})$ for each specific $h > 0$.

Two interesting open problems are posed as follows.

- The development of an algebraic procedure for designing a stable rational function $c(s)$ when its existence is known.

- The strong stabilizability problem for a time-delay MIMO linear system remains open. For an MIMO finite dimensional system, this problem was completely solved in [20]. In the n-D case, the strong stabilizability problem for an MISO (multiple-input-single-output) or SIMO (single-input-multiple-output) system has been solved, but only a necessary condition is known for a general MIMO n-D system [18].

### References


