Activation Policy of Smart Controllers for Flexible Structures with Multiple Actuator/Sensor Pairs

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Abstract

This paper is concerned with the implementation of an activation policy and consequent control policy of actuators and sensors for flexible structures. It is assumed that the system under consideration has multiple collocated actuator/sensor pairs and it is desired to activate only one such actuator/sensor pair during a given time interval while the other actuator/sensor pairs remain dormant. The decision policy is based on minimizing a given performance index over a finite time interval. The minimum of the cost criterion over the given time interval is parametrized by the actuator/sensor candidate locations in order to derive the activation policy over the time interval in question. An example with numerical results of a flexible beam with piezoceramic patches is included to demonstrate the applicability of the activation algorithm.

1 Introduction

While many researchers focused on developing optimal control signals for control systems, less attention has been paid on the location/placement of the actuating and sensory devices. Some research has dealt with the positioning of fixed (in space) actuators and sensors in dynamical systems. The correct positioning of actuators and sensors in a control system will at least improve performance and even reduce the energy input required to meet a certain control objective.

In the past, many investigators have addressed this issue of (stationary or motionless) sensor and actuator placement, especially for flexible space structures, in the context of controllability and observability indices [10]. The algorithms for this placement along with references on prior work can be found in the books [11, 14, 20] in the context of flexible structures and in [5] for the general case of finite dimensional systems. In the case of moving actuators which is similar to the proposed work, fewer results can be found, see for example [6, 19]. Whether one has many actuators and activates only one during a given time interval, or has a single actuator that is mobile, it has little effect on the proposed algorithm as the two cases can be shown to be equivalent. The treatment here is somewhat different from earlier efforts [11, 19], in the sense that it is based on minimizing the optimal value of a performance index over a time interval in order to find the optimal pair for that interval, as opposed to considering observability and controllability measures.

The remainder of the manuscript is as follows. The mathematical formulation of the system under consideration is presented in section 2 and the activation policy along with the control policy for both the stationary and moving actuator and actuator/sensor pairs is described in section 3. Numerical examples with discussion for future research follow, in sections 4 and 5, respectively.

2 Mathematical Formulation

Typically the dynamics of flexible structures are described by a series of (coupled) partial differential equations defined in some domain \( \Omega \subset \mathbb{R}^2 \), which, when written in an abstract setting [13], take the form

\[
\mathcal{M}\dddot{\zeta}(t) + \mathcal{D}\dot{\zeta}(t) + \mathcal{K}\zeta(t) = \mathcal{B}_0u(t).
\]

The operators \( \mathcal{M}, \mathcal{D}, \mathcal{K} \) are the mass, damping and stiffness operators, respectively and are defined in the appropriate Hilbert/Sobolev spaces [13]. The input operator \( \mathcal{B}_0 \) is a function of the spatial variable \( \xi \) and incorporates information on the location and shape (and possible dynamics) of the actuating devices. The second order abstract system (1) can be placed in first order form [2]

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

where \( A \) and \( B \) are the appropriate operators corresponding to the first order formulation of (1), [2, 4, 13]. When the infinite dimensional system (1) is approximated via an exponential stabilizability-and-detectability-preserving scheme [2, 4], it results in the \( n \)-dimensional (vector) second order system

\[
\mathcal{M}\dddot{\zeta}(t) + \mathcal{D}\dot{\zeta}(t) + \mathcal{K}\zeta(t) = \mathcal{B}_0(\xi)u(t)
\]
where $z(t)$ is the finite dimensional representation of the state $\zeta(t)$. \cite{13,18}, $M = M^T$ is the $n \times n$ mass matrix, $K = K^T > 0$ the $n \times n$ stiffness matrix, $D = D^T$ (i.e. consider non-gyroscopic systems. \cite{20}) the $n \times n$ damping matrix, and $B_0(\xi)$ is the $n \times 1 \xi$-parametrized input vector (assuming a single actuator). The dependence of the input vector on the spatial variable is written explicitly to emphasize this dependence of the actuating device on the actuator location(s). It is assumed that there are $q$ available actuators and it is desired to activate only one during a given time interval of length $\Delta t$ while the remaining $(q - 1)$ actuators remain dormant.

To show how one can go about finding the optimal (with a performance based criterion) actuator activation sequence, the above system is first placed in a first order vector form ($2n$-dimensional)

$$
\frac{d}{dt} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -M^{-1}B_0(\xi) \end{bmatrix} u(t),
$$

or in a more compact form

$$
\dot{x}(t) = Ax(t) + B(\xi)u(t),
$$

along with the expression for the collocated rate \cite{20} output

$$
y(t) = B^T(\xi)x(t).
$$

This first order formulation will be used to minimize a quadratic performance criterion, and its optimal value will be parametrized by the actuator/sensor location. The minimum of the location parametrized optimal value of the performance index will then be minimized with respect to locations to arrive at the location that yields the smallest optimal cost.

First, the case of a single stationary actuator over a larger time interval $[t_0, t_f]$ is treated, as this would provide the basis for the activation policy of 1-active($q - 1$)-dormant actuators.

### 3 Activation Policy

As a first step, an algorithm for the optimal placement of actuator and/or actuator/sensor collocated pair over a fixed time interval will be presented. This procedure will then be modified to apply to multiple time subintervals, which will provide either the location of a moving actuator or the activation of a given actuator over that time subinterval. Similar work for flexible structures using an infinite horizon LQR cost function to place a stationary actuator can be found in \cite{9}.

#### 3.1 Optimal placement of stationary actuator/sensor pair over the interval $[t_0, t_f]$

It is assumed that one has the freedom to place an actuator at some a priori selected locations that render the system controllable. The criterion for this placement will be given in the context of performance improvement. For a fixed actuator location $\xi_k$, one can obtain the optimal control law $u(\cdot) \in L^2(t_0, t_f, \mathbb{R}^1)$ that minimizes the cost functional

$$
J(u) = \int_{t_0}^{t_f} x^T(t)Qx(t) + u^T Ru(t) \, dt + x^T(t_f)Mx(t_f),
$$

or in the $H^\infty$ setting, the disturbance-augmented cost

$$
J(u) = \int_{t_0}^{t_f} x^T(t)Qx(t) + u^T Ru(t) - \gamma^2 |w(t)|^2 \, dt + x^T(t_f)Mx(t_f).
$$

For the LQR case, the optimal control is given by

$$
u(t) = -K(t, \xi_k)x(t) = -R^{-1}B^T(\xi_k)P(t, \xi_k)x(t)
$$

where $P(t, \xi_k)$ solves the $2n \times 2n \xi$-parametrized matrix Differential Riccati Equation (DRE)

$$
-\dot{P}(t, \xi_k) = P(t, \xi_k)A + A^T P(t, \xi_k) + Q - P(t, \xi_k)B(\xi_k)R^{-1} B^T(\xi_k) P(t, \xi_k),
$$

with terminal condition $P(t_f, \xi_k) = M$. The optimal value of the cost functional over $[t_0, t_f]$ is then given \cite{7,15} by

$$
J^{opt}(u, \xi_k) = x^T(t_0)P(t_0, \xi_k)x(t_0).
$$

In the $H^\infty$ setting the optimal value of the cost functional is the same as (8) above, with the exception that $P(t, \xi_k)$ now solves

$$
-\dot{P}(t, \xi_k) = P(t, \xi_k)A + A^T P(t, \xi_k) + Q - P(t, \xi_k) \left[ B(\xi_k)R^{-1} B^T(\xi_k) - \frac{1}{\gamma^2} DD^T \right] P(t, \xi_k),
$$

with $P(t_f, \xi_k) = M$, where now equation (4) is re-written as

$$
\dot{x}(t) = Ax(t) + B(\xi_k)u(t) + D w(t).
$$

It therefore make sense to minimize the optimal cost (8) with respect to the actuator locations $\xi_k$, $k = 1, 2, \ldots, q$ as this was similarly proposed in \cite{3} in the context of filtering of distributed parameter systems, in \cite{12} for finite dimensional systems and in \cite{9,8} for structural systems.

Thus we have the following theorem.

**Theorem 3.1** Let $\Omega_\infty = \{\xi_1, \xi_2, \ldots, \xi_q\}$ denote the set in the spatial domain of definition $\Omega$ for the infinite dimensional system (1) where the pair $(A, B(\xi))$ is controllable (stabilizable) \cite{4}. Then, the optimal actuator location is found as the one that minimizes the optimal value $J^{opt} = x^T(t_0)P(t_0, \xi_k)x(t_0)$ of the performance index (8) with respect to all $\xi_k \in \Omega_\infty$.

**Remark 3.1** Since only $q$ candidate locations are considered for actuator locations, i.e. it is assumed the $q$ pairs $(A, B(\xi_k))$ are stabilizable, then one has to solve $q$-DRE's, one for each $\xi_k$, $k = 1, 2, \ldots, q$, and then find the minimum of $x^T(t_0)P(t_0, \xi_k)x(t_0)$ with respect to all $\xi_k$’s.
Remark 3.2 The degree of controllability is not utilized here as was treated in flexible structures [11], but it may be used to identify the q-locations that yield the “most” controllable pairs; in other words find the q locations that render the largest controllability indices and then choose from those q locations where to place the actuator in each time subinterval using a performance-based criterion.

The above method only provides the LQR (or $H^\infty$) performance-based stationary actuator location over $[t_0, t_f]$ given access to full state $x(t)$. Since the initial condition $x(t_0)$ needed for minimization of (8) is not available, let alone the full state $x(t)$ required for the implementation of (6), one must include the sensor location in order to implement either an optimal dynamic or static compensator. While lack of knowledge of $x(t_0)$ can be circumvented by minimizing the trace of the $P(t_0, \xi)$ as this was treated in [16], see also [17], absence of full state information must be dealt with in a different approach. In this case, the sensor location is also parametrized by the q possible locations and then the optimal filter for each location can be found.

One way to address the issue of the sensor location that reduces the implementational complexity and computational burden, is to collocate the sensors with the actuators, a practice often applied by engineers working with flexible structures. The advantage of such a choice is the avoidance of transmission zeros. Another attribute of collocation is that the system dissipative, assuming that $A + AT < 0$. In this case one could simply choose a static output feedback $u(t) = -Gy(t)$ with $G > 0$, to guarantee asymptotic stability of the closed loop system, [11, 21]. More relevant to this work is the performance-based, a way to find the collocated actuator/sensor location is to employ an LQG cost with a control law

$$u(t) = -K(t, \xi_k)\hat{x}(t),$$

where $\hat{x}(t)$ is the state estimate of $x(t)$ and whose dynamics are given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B(\xi_k)u(t)$$

$$+ L(t, \xi_k) [y(t) - B^T(\xi_k)\hat{x}(t)]$$

$$\hat{y}(t) = B^T(\xi_k)\hat{x}(t).$$

The observer gain $L(t, \xi_k) = \Sigma(t, \xi_k)B(\xi_k)N^{-1}$ can be found by solving the (collocated actuator/sensor) location-parametrized (Kalman) filter Riccati equation

$$\ddot{\Sigma}(t, \xi_k) = A\Sigma(t, \xi_k) + \Sigma(t, \xi_k)A^T$$

$$- \Sigma(t, \xi_k)B(\xi_k)N^{-1}B^T(\xi_k)\Sigma(t, \xi_k) + \tilde{Q},$$

with $\Sigma(t_0, \xi_k) = E\{[x(t_0) - \pi_0][x(t_0) - \pi_0]^T\}$ and $E\{x(t_0)\} = \pi_0$. In this case the optimal actuator/sensor location is found by minimizing the minimal(optimal) value [15] (either expression):

$$J^{\text{opt}}(\xi_k) = \pi^T_0 P(t_0, \xi_k)\pi_0 + \text{tr}\left\{ \int_{t_0}^{t_f} \Sigma(t, \xi_k)Q dt + \right.$$}

$$\int_{t_0}^{t_f} P(t, \xi_k)\Sigma(t, \xi_k)B(\xi_k)N^{-1}B^T(\xi_k)\Sigma(t, \xi_k) dt$$

$$+ P(t_f)\Sigma(t_f) \right\} = \pi^T_0 P(t_0, \xi_k)\pi_0 + \text{tr}\left\{ \int_{t_0}^{t_f} P(t, \xi_k)\tilde{Q}dt + \right.$$}

$$\int_{t_0}^{t_f} \Sigma(t, \xi_k)P(t, \xi_k)B(\xi_k)R^{-1}B^T(\xi_k)P(t, \xi_k) dt$$

$$+ P(t_0)\Sigma(t_0) \right\},$$

of the criterion [15]

$$J(u) = E\left\{ \int_{t_0}^{t_f} [x^T(t)Qx(t) + u^T Ru(t)] dt + x^T(t_f)Mx(t_f) \right\}$$

with respect to the variable $\xi_k \in \Omega_n$.

3.2 Optimally activated actuator/sensor pair over the subinterval $[t_k, t_i + T]$

The above algorithm provides the optimal actuator location for the interval $[t_0, t_f]$. To find what the activation sequence should be at different time instances, or alternatively what the location should be in the case of a single actuator capable of moving at q preselected positions across the domain $\Omega$, the interval $[t_0, t_f]$ is divided into n intervals each of length $\Delta t = \frac{t_f - t_0}{n}$. In the case of a single moving actuator, the length of the subinterval $\Delta t$ is a function of the natural frequency of the structure and the velocity at which it can move [19]. The activation of the $k^{th}$ actuator (or $k^{th}$ location at which the single moving actuator is residing during the length of the subinterval $[t_i, t_i + \Delta t]$) will be changing at the beginning of every time subinterval but will be remaining fixed for the duration $\Delta t$ of each of the subintervals; thus we now solve the optimal control problem over the $i^{th}$ subinterval $[t_i, t_i + \Delta t]$, $i = 0, 1, \ldots, n - 1$, in order to find which actuator should be activated in this subinterval (or what the actuator location should be in this subinterval in the case of a moving actuator). In the case of full state availability and hence the sequencing of actuator only, the LQR cost in the $i^{th}$ subinterval $[t_i, t_i + \Delta t]$ is given by

$$J_i(u) = \int_{t_i}^{t_i + \Delta t} (x^T(t)Qx(t) + u^T(t)Ru(t)) dt$$

$$+ x^T(t_i + \Delta t)M_i x(t_i + \Delta t).$$

The optimal value of $J_i$ in the $i^{th}$ subinterval $[t_i, t_i + \Delta t]$ is $J_i^{\text{opt}} = x^T(t_i)P_i(t_i)x(t_i)$, where now each $P_i(t)$ solves (7) in the subinterval $[t_i, t_i + \Delta t]$ having terminal condition
$P_i(t_i + \Delta t) = M_i$ where $M_i = P_{i+1}(t_i + \Delta t)$, $M_n = P_{n+1}(t_f)$ = $M$. If the above is parametrized by the location $\xi_k$, then one must solve the DRE (7) for each $k = 1, 2, \ldots, q$ that correspond to the $q$ possible actuator locations $\xi_k$ with terminal condition $P_i(t_i + \Delta t, \xi_k) = P_{i+1}(t_i + \Delta t, \xi_k)$. The minimum of the optimal value $x^T(t_i)P_i(t_i, \xi_k)x(t_i)$ over all $k = 1, 2, \ldots, q$ will then yield the optimal location $\xi_{opt}$ for $[t_i, t_i + \Delta t]$. This of course requires solving $q$-DRE’s in each subinterval backwards in time with one requiring a terminal condition for the current subinterval being the initial condition from the next subinterval. The solution of $\sum_{i=1}^{n} q^i$ DRE’s (due to the terminal conditions) can be circumvented as suggested in [6], which in essence requires the solution of $n \times q$-DRE’s in the interval $[t_0, t_f]$. It amounts to solving $n$ independent optimal control problems (one for each time subinterval). This while it yields a suboptimal solution, it reduces the number of DRE’s solved by a factor of $\frac{1}{n} \sum_{i=1}^{n} q^i$ or by $\sum_{i=1}^{n} q^i$ if adjusted for each time subinterval.

Now, when the collocated sensor is taken into consideration, then one has to repeat many times the procedure described in the case of a stationary actuator/sensor pair over $[t_0, t_f]$. This is given as follows: in the interval $[t_i, t_i + \Delta t]$, solve

$$-\dot{P}_i(t, \xi_k) = A^T P_i(t, \xi_k) + P_i(t, \xi_k)A - P_i(t, \xi_k)B(\xi_k) R^{-1} B(\xi_k) P_i(t, \xi_k) + Q,$$

with terminal condition $P_i(t_i + \Delta t, \xi_k) = M_i$, and

$$\dot{\Sigma}_i(t, \xi_k) = A\Sigma_i(t, \xi_k) + \Sigma_i(t, \xi_k)A^T - \Sigma_i(t, \xi_k)B(\xi_k) N^{-1} B^T(\xi_k) \Sigma_i(t, \xi_k) + Q_i,$$

with $\Sigma_i(t, \xi_k) = E \left\{ \{x(t_i) - \mathcal{X}_i\}[x(t_i) - \mathcal{X}_i]^T \right\}$, where $\mathcal{X}_i = E \{ x(t_i) \}$. The actuator/sensor pair that should be activated in the interval $[t_i, t_i + \Delta t]$ is chosen as the one that minimizes

$$J^{opt}_i(\xi_k) = \| \mathcal{X}_i \| P_i(t_i, \xi_k) \mathcal{X}_i + t \{ \int_{t_i}^{t_i + \Delta t} \Sigma_i(t, \xi_k)Q dt + \int_{t_i}^{t_i + \Delta t} P_i(t, \xi_k) \Sigma_i(t, \xi_k)B(\xi_k) N^{-1} B^T(\xi_k) \Sigma_i(t, \xi_k) dt + P_i(t_i + \Delta t, \xi_k) \Sigma_i(t_i + \Delta t, \xi_k) \}$$

over all $\xi_k \in \Omega_a$.

Continuing with other simplifications of the above algorithm due to numerical considerations, we propose the following sub-optimal activation policy.

**Algorithm for a suboptimal activation policy**

1. Solve the control DRE (7) with terminal condition $P(t_f) = M$ over the interval $[t_0, t_f]$ for each location $\xi_i, i = 1, 2, \ldots, q$. This requires the solution of only $q$ Control Riccati equations.

ii) Solve the filter DRE (10) over $[t_0, t_f]$ for each location $\xi_i, i = 1, 2, \ldots, q$. This also requires the solution of $q$ Filter Riccati Equations.

iii) In each time subinterval, minimize the cost (14) with $P_i(t, \xi_k)$ replaced by $P(t, \xi_k)$ and $\Sigma_i(t, \xi_k)$ replaced by $\Sigma(t, \xi_k)$.

iv) The optimal control law is then given by

$$u(t) = -K(t, \xi_k) \hat{v}(t).$$

Another simplification is to either approximate the derivative of the two DRE’s in (12) and (13) by

$$\dot{P}_i(t, \xi_k) \approx \frac{P_i(t + \Delta t, \xi_k) - P_i(t, \xi_k)}{\Delta t}$$

and

$$\dot{\Sigma}_i(t, \xi_k) \approx \frac{\Sigma_i(t + \Delta t, \xi_k) - \Sigma_i(t, \xi_k)}{\Delta t}$$

to arrive at approximated Riccati equations or set $t_f = \infty$. This now requires the solution of algebraic Riccati equations as opposed to differential Riccati equations, with the obvious computational benefits. Specifically, if $t_f = \infty$, then one has to solve $q$ control algebraic Riccati equations (CARE) and $q$ filter algebraic Riccati equations (FARE).

### 4 Example and Numerical Results

As an example, we consider a flexible beam fixed at both ends with 5 piezoceramic patches that act as both sensors and actuators. Only one piezoceramic pair will be activated at a given time interval as actuator/sensor while the other four pairs will remain dormant. The proposed algorithm above was utilized to find the activation sequence of the 5 patches.

The beam midspan displacement and velocities are presented in Figures 1 and 2, respectively. Both depict the signal for the spatially activated (solid) and spatially fixed (dashed) patch pairs. The advantage of the 1-active/4-dormant piezoceramic patches vs the fixed (stationary) patch on the controller performance is evident. The activation is presented in Figure 3 where it is observed that the third (middle) pair is initially activated, then the fifth one for 5 time steps (in the time interval [0.002,0.010]), then the first one for 15 time steps (in the time interval [0.012,0.041]), and lastly the fifth again for the remainder of the time (in the time interval [0.04,0.5]). For this set of simulations the length of the time subinterval was 0.002 seconds.

The energy norm of the system, given by

$$E(t) = \sqrt{z^T(t)Kz(t) + \dot{z}^T(t)M \dot{z}(t)}$$

is plotted against time for both the case of fixed (dashed) and moving (solid) actuator/sensor as seen in Figure 4. The $L_2(0,t)$ norm of the displacement for the three cases is given in the table below.
Figure 1: Beam midspan transverse displacement for moving and fixed actuator/sensor pair.

| case              | $\int_0^T ||z(t)||^2 dt$ | $\int_0^T ||u(t)||^2 dt$ |
|-------------------|--------------------------|--------------------------|
| open loop         | 1.0829                   | 0                        |
| closed loop-fixed | 0.5370                   | 1.7529 $\times 10^3$    |
| closed loop-moving| 0.2606                   | 0.9570 $\times 10^3$    |

Table 1: $L_2$ norms of displacement and control signals.

The evolution of the control signal for the two cases of fixed and moving actuator/sensor pair is depicted in Figure 5 where once again, it is observed that the case of moving actuator/sensor required less control effort than then case of a fixed stationary one.

5 Conclusions

A scheme for the optimal activation policy of collocated actuators and sensors in flexible structures was presented. The policy was based on performance criteria as opposed to previously considered controllability/observability criteria. While the proposed scheme yields the optimal activation policy, it does not include the optimal residence (i.e. time duration) of the actuator/sensor pair. This is the topic of future research.

More immediate extensions would concentrate on reducing the computational burden of the activation policy which requires the solution of a large number of differential Riccati equations in each time subinterval. A suboptimal method would opt for the reduction of the total number of Riccati equations solved resulting in a reduced computational and implementational complexity at the expense of optimality.

References


Figure 3: Locus of the activated actuator/sensor pair.

Figure 4: Norm $E(t)$ for moving and fixed actuator/sensor.


