A lattice Boltzmann wave model applied to fracture phenomena

B.Chow and S.Maronci
CUI, University of Geneva
24, rue Général-Dufour
1211 Genève 4 Switzerland
{Bastien.Chopard,Stephane.Maronci}@cui.unige.ch

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Abstract: We propose to study generic features of fracture phenomena using a Lattice Boltzmann (LB) wave formalism. This formalism is numerically equivalent to the Transmission Line Matrix (TLM) and is very similar to a cellular automata dynamics. Fracture of a discrete body is obtained by allowing links with local energy higher than a given threshold to disappear. To validate our use of the LB-wave model, several phenomena have been looked at: stress profile at the tip of a crack, micro-crack formation, propagation of a crack and fragmentation.

1 Introduction

Fracture processes in solids and fragmentation are important phenomena that are not yet fully understood [1, 2]. There is therefore a clear interest to propose simple theoretical models that can account for the generic properties of these phenomena. In particular one would like to understand what governs the crack propagation speed and how the distribution of fragment size scales.

In this article, we address this problem within the so-called lattice Boltzmann paradigm and propose a simple dynamical model of a solid body at a mesoscopic scale.

Lattice Boltzmann (LB) models [3, 4] are dynamical systems, discrete in space and time, aimed at simulating the behavior of a real physical system in terms of the local density of fictitious particles moving and interacting on a regular lattice. LB methods have been largely used to simulate systems of point particles which interact locally, such as complex fluid dynamics [3, 5, 4, 6, 7], reaction-diffusion processes and wave propagation in a heterogeneous media [4].

However, modeling a solid body with this approach (i.e. modeling an object made of many particles that maintains its shape and coherence over distances much larger than the inter-particle spacing) has remained mostly unexplored. A first attempt to model a one-dimensional solid body as a cellular automata is described in [8]. Then a 2D model has been proposed in [4, 9]. The crucial ingredient of these models is the fact that collective motion is achieved because the “atoms” making up the solid vibrate in a coherent way and produce an overall displacement. This vibration propagates as a wave throughout the solid and reflects at the boundary.

In this paper we review the main aspects of this method (i.e. we explain what is the LB model for wave propagation and show why it provides a natural framework to describe a simple mesoscopic model of a solid body). Then, we propose a new generalized model in which the speed of sound is adjustable and which has a much more elegant mathematical formulation including various shape of boundaries. Finally, we apply our approach to simulate numerically some generic properties of real solids and, in particular, analyze complex phenomena such as fracture propagation, micro-cracks formation and fragmentation.

The paper is organized as follows. Section 2 reviews the LB formalism for the case of wave propagation. Section 3 introduces the simple and general versions of our solid body model and shows its relation to the LB-wave dynamics. Section 4 shows simulation results obtained for various fracture experiments.

2 The LB-wave model

The dynamics of a LB model is expressed in terms of variables $f_i(\vec{r}, t)$ which describe some mesoscopic properties of the system. This $\ell$ is the discrete counterpart of the distribution function used in the Boltzmann equation of statistical mechanics. The macroscopic quantities of interest (density, velocity field...) can be defined from the $f_i$s.

Here we shall interpret $f_i(\vec{r}, t)$ as a flux traveling on the lattice, or a population of pseudo-particles. In our notation, $\vec{r}$ refers to the lattice site, $t$ the iteration time and the subscript $i$ labels the admissible velocities $\vec{v}_i$ (e.g. along the main lattice directions). The value $i = 0$ corresponds to a population of rest particles with $\vec{v}_0 = 0$.

Lattice BGK models [10] are a particular type of LB models in which the interaction between the $f_i$s is written as a relaxation to a local equilibrium distribution. Lattice BGK models are the most common implementation of LB systems. They are characterized by the following dynamics

$$f_i(\vec{r} + \tau \vec{v}_i, t + \tau) - f_i(\vec{r}, t) = \frac{1}{\xi} \left( f_i^{(0)}(\vec{r}, t) - f_i(\vec{r}, t) \right)$$  (1)

where $\tau$ is the time step, $f_i^{(0)}(\vec{r}, t)$ the so-called local equi-
library distribution and $\xi$ is a parameter representing a relaxation time.

The functions $f_i^{(0)}$s are the key ingredients of the model for they actually contain the properties of the physical process under study: this is the distribution to which the dynamics spontaneously relaxes and which is, therefore, intimately related to the nature of the system.

Wave phenomena, whether mechanical or electromagnetic derive from two conserved quantities $\Psi$ and $\vec{J}$, together with time reversal invariance and a linear response of the media. The quantity $\Psi$ is a scalar field and $\vec{J}$ its associated current. The idea behind the LB approach is to "abstract" a physical process to a discrete space and time universe, so that it can be efficiently simulated on a (parallel) computer. For waves, this generalization is obtained by keeping the essential ingredients of real phenomena, namely conservation of $\Psi$ and $\vec{J}$, linearity and time reversal invariance. Thus, in a discrete space-time universe, a generic system leading to wave propagation is obtained from Eq. (1) by an appropriate choice of the local equilibrium distribution

$$f_i^{(0)} = a_i \Psi + b_i \vec{v}_i \cdot \vec{J} \quad \text{if } i \neq 0, \text{ and } f_0^{(0)} = a_0 \Psi \quad (2)$$

where $v$ is the ratio of the lattice spacing to the time step, $a_0$ and $b$ are parameters to be determined and $\Psi$ and $\vec{J}$ are related to the $f_i$s in the standard way:

$$\Psi = \sum_i m_i f_i \quad \vec{J} = \sum_i m_i f_i \vec{v}_i$$

The $m_i$ are weights associated with each direction.

Note that, here, we make no restriction on the sign of the $f_i$s which may well be negative in order to represent a wave.

As opposed to hydrodynamics [3], $f_i^{(0)}$ is a linear function of the conserved quantities, which ensures the superposition principle. The parameters $a$, $b$ and $a_0$ are chosen such that $\Psi = \sum_i m_i f_i^{(0)}$ and $\vec{J} = \sum_i m_i \vec{v}_i f_i^{(0)}$, which ensures the conservation of $\Psi$ and $\vec{J}$. For a two-dimensional square lattice (D2Q5 to use the terminology of [3]) we choose $m_i = 1$ and find $a_0 + 4a = 1$ and $b = 1$. The freedom on the value of $a_0$ can be used to adjust locally the wave propagation speed. Time reversal invariance is enforced by choosing $\xi = 1/2$ as can be easily checked from Eq. (1) with $\vec{J} \rightarrow -\vec{J}$ and $\Psi \rightarrow \Psi$ in relation (2). Note that the D2Q5 lattice is known for giving anisotropic contributions to the hydrodynamic equations. These terms are not present in our wave model because they appear with a vanishing coefficient when $\xi = 1/2$ [4].

The multiscale Chapman-Enskog expansion [4] can be used to derive the macroscopic behavior of $\Psi$ and $\vec{J}$ when the lattice spacing and time step go to zero. We obtain [4, 11]

$$\partial_t^2 \Psi - 2av^2 \nabla^2 \Psi = 0$$

which is a wave equation with propagation speed $c = v\sqrt{2a}$ (note that $v$ is the speed at which information travels).

The wave propagation speed $c$ can be adjusted from place to place by choosing the spatial dependency of $a$. Provided that $a_0 + 4a = 1$ and $a_0 \geq 0$ (required for numerical stability reasons), the largest possible value is $a = 1/4$ and corresponds to a maximum velocity $v_0 = \sqrt{2}$. Therefore media with different refraction indices $n = c_0/c = 1/(2\sqrt{a})$ can be modeled.

Perfect reflection on obstacles can be included by modifying the micro-dynamics to be $f_i(\vec{r} + \tau \vec{v}_i, t + \tau) = -f_i(\vec{r}, t)$ on mirror sites, where $\tau$ is defined so that $\vec{v}_i = -\vec{v}_i$, i.e. the fluxes bounce back to where they came from with a change of sign. Absorption on non-perfect transmitter sites can be obtained by modifying the conservation of $\Psi$ to $\sum_i m_i f_i^{(0)} = \mu \Psi$, where $0 \leq \mu \leq 1$ is an attenuation factor. This modifies $a \rightarrow \mu a$ and $a_0 \rightarrow \mu a_0$. Finally, by substituting (2) into (1) and using the expression of $a$ and $a_0$ in terms of $c$, free propagation with reflection index $n(\vec{r})$, and partial transmission and reflection can be expressed as

$$f_i(\vec{r} + \tau \vec{v}_i, t + \tau) = \frac{\mu}{2n^2} \Psi - f_{i+2}(\vec{r}, t)$$

$$f_0(\vec{r}, t + \tau) = \frac{n^2 - 1}{n^2} \Psi - f_0(\vec{r}, t) \quad (3)$$

In this equation, $\mu = 0$ corresponds to perfect reflection, $\mu = 1$ to perfect transmission and $0 < \mu < 1$ describes a situation where the wave is partially absorbed. A particular version of our LB-wave model has been successfully validated by the problem of radio wave propagation in a city [12].

In hydrodynamic models, $\xi = 1/2$ corresponds to the limit of zero viscosity, which is numerically unstable. In our case, this instability does not show up provided we use an appropriate lattice. Indeed, in the D2Q5 lattice, our dynamics is also unitary [13] which ensures that the quantity $E = \sum_i f_i^2$ is conserved. This extra property prevents the $f_i$s from becoming arbitrarily large (with positive and negative signs, since $\Psi$ is conserved). It also offers us the possibility to define an energy in the model. This will turn out useful in the following sections.

Note finally that the above LB formulation is yet another way to express the generalized Transmission Line Matrix (TLM) numerical scheme [14, 15, 16, 17].

### 3 A LB mesoscopic solid body model

#### 3.1 A model with a simple geometrical interpretation

A cellular automata model of a one-dimensional chain of particles has been discussed in [8]. Similarly, in 2D, a solid body can be thought of as an arrangement of particles linked to their four nearest neighbors with a spring-like interaction [4, 9]. An elementary cell of such a solid is displayed in Fig. 1 where a particle is shown with its four connected neighbors. The configuration shown in this figure corresponds to a situation with a local deformation: the four neighbors are not at the same distance from the central particle and the positions differ from that of a square lattice.
The complete dynamics is obtained by alternating the black and white motions. It turns out \[4, 9\] that the above rule to (5), but this time from the point of view of the white parti-

cles remain fixed. The adjacent particles. 

As shown in Fig. 1 we denote the location of the black par-

cle by \( \bar{r}_{i,j} = (x_{i,j}, y_{i,j}) \). The surrounding white particles 

\[
\begin{align*}
\tilde{f}_1(i,j) &= \bar{r}_{i,j} - (\bar{r}_{i-1,j} + \vec{h}) \\
\tilde{f}_2(i,j) &= \bar{r}_{i,j} - (\bar{r}_{i,j-1} + \vec{u}) \\
\tilde{f}_3(i,j) &= \bar{r}_{i,j} - (\bar{r}_{i+1,j} + \vec{h}) \\
\tilde{f}_4(i,j) &= \bar{r}_{i,j} - (\bar{r}_{i,j+1} + \vec{u})
\end{align*}
\]

where the \( \tilde{f}_i \) are now vector quantities, and \( \vec{h} = (ro, 0) \) and \( \vec{u} = (0, ro) \) can be thought of as representing some horizontal 

or vertical equilibrium length \( r_0 \) of the spring connecting adjacent particles. 

In this simplest version of our model, the motion of the particles can be defined by the following geometrical rule (we now assume that the black particles are moving): a bulk particle such as the black one shown in Fig. 1 moves to a symmetric location with respect to the center of mass of its four neighbors, namely \( \sum (1/4)[\bar{r}_{i-1,j} \bar{r}_{i+1,j} + \vec{h} + \vec{u} + \bar{r}_{i,j-1} + \bar{r}_{i,j+1}] \). 

Then, in a second step, the white particles move and the black ones remain fixed. The \( \tilde{f}_i \)'s are recomputed according to (5), but this time from the point of view of the white particles and their respective neighbors. 

The complete dynamics is obtained by alternating the black and white motions. It turns out \[4, 9\] that the above rule of motion is equivalent to updating the local deformation \( \tilde{f}_i \)'s by the lattice Boltzmann dynamics given in Eq. (3), provided that we choose \( n = 1 \) (which implies that \( \tilde{f}_0 = 0 \)). This is not surprising since deformation propagates like a wave in a solid.

It is also found that the value of \( \bar{\Psi} = \sum n_i \tilde{f}_i \) for particle with label \((k,l)\) in the 2D lattice corresponds to the momentum \( \vec{p} \) of that particle. Indeed, simple algebra \[4\] shows that the new location \( \bar{r}_{kl}(t+1) \) of particle \((k,l)\) is given by 

\[
\bar{r}_{kl}(t+1) = \bar{r}_{kl}(t) - (1/M_{kl}) \bar{\Psi}
\]

where \( M_{kl} \) is a mass (yet to be determined) associated with the particle. 

Note also that, in this interpretation, \( \vec{J}_x = \sum_i \vec{v}_{ix} \tilde{f}_i \) and \( \vec{J}_y = \sum_i \vec{v}_{iy} \tilde{f}_i \) respectively give, the separation between its left-right and up-down neighbors. Likewise, the quantity \( E \) defined in (4) is interpreted as the local energy of the particle. It clearly takes the form of an elastic energy of deformation.

Whereas mass, momentum and energy are naturally conserved in this dynamics which is not the case of angular momentum. Indeed, in the present model, the coupling between adjacent particles is not given by the Euclidean distance but is decoupled along each coordinate axis (however, a deformation along the \( x \)-direction will propagate along the \( y \)-direction and conversely). Although this technique breaks up the rotational invariance, it allows to work with a square lattice and is expected not to play a crucial role in the fracture processes we shall consider below.

### 3.2 Boundary Conditions

The updating rule explained above works well for particles having exactly four neighbors. However, this is not always the case when describing a 2D solid body which has some boundaries. For a square-shaped solid body, particles may have two or three neighbors. In what follows, we shall consider fracture processes which may produce particle with only one link.

From our geometrical interpretation of the dynamics, we obtain the rule of motion for the particles at the border of the system by applying the same rule, but with less neighbors: (1) we consider the locations \( \bar{r} \) of the \( k \) neighbors of the particle to be moved; (2) we add to these positions a vector representing the connecting “spring” at rest; (3) we compute the geometrical center of mass of these \( k \) quantities \( \bar{r} \) the new location of the jumping particle is given by its symmetrical position with respect to the center of mass.

An algebraic expression for this rule can be found. It takes a simple form provided that a different mass is defined for the bulk or the border particles. Some simple calculations show that the appropriate mass \( M \) of a particle should be 

\[
M = k \frac{1}{2}
\]

The precise formulation of the dynamics is given below, for the case of the generalized model.

### 3.3 Generalized model

As was discussed in the previous section, the above mesoscopnic model of a solid is described by a LB dynamics. How-

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Figure 1: Picture of a generic “atom” and its four connected neighbors in our 2D solid model. The \( \tilde{f}_i \)'s are defined as the local deformation along each direction. The cross indicates the location of the geometrical center of mass of the four white particles. At the next iteration, the black particle jumps to a symmetrical position with respect to this point.
ever, Eq. (3) allows for more freedom than what we have imposed geometrically on our model. In particular there is no need to partition the solid’s atoms in white and black subsets. Also, one can add to each atom \((k,l)\) a rest field \(f_0\) which allows us to adjust locally the speed of sound of the system.

If we relax the checkerboard updating rule and add a fifth field, the interpretation of the \(f_s\) is no longer given consistently by the local deformation as expressed in Eq. (5). However, the conserved quantities like the momentum \(\Psi\) and the energy \(E\) are still well defined within this extension and can be used to interpret the model.

It turns out that Eq. (3) can be rewritten in a quite elegant way to take into account the fact that some particles may have less than five links

\[
\begin{align*}
\vec{f}_i(t+1) &= \mu \frac{\Psi}{M} - \vec{f}_{i+2} \\
\vec{f}_0(t+1) &= \mu \sqrt{2M_0 \over M} \vec{\Psi} - \vec{f}_0
\end{align*}
\]

where the quantity \(M_0\) is the mass associated with the internal link \(f_0\) and is related to the refraction index \(n\) as

\[M_0 = 2(n^2 - 1)\]

The mass \(M\) of a particle is then given by \(M_0\) and the contributions of all \(k\) connected links, namely

\[M = M_0 + k \frac{1}{2}\]

since each spatial link counts for \(1/2\).

If \(\mu = 1\), it can be shown by direct substitution that the momentum

\[\vec{\Psi} = \sqrt{2M_0 \vec{f}_0} + \sum_{i>0} \vec{f}_i\]

and the energy

\[E = \sum_{i<0} f_i^2\]

are conserved by Eq. (6), for any value of \(k\).

In the above expression, the summation over \(i\) concerns the \(k\) existing links. Note also that in Eq. (6), it is assumed that, for missing links, the corresponding fields \(f_i\) are always zero.

Since the geometrical interpretation no longer holds with the dynamics given by (6) (also remember that it does not anymore requires a checkerboard updating rule), the position of each particle must be computed from its momentum according to

\[\vec{r}(t+1) = \vec{r}(t) - \Psi / M\]

Also, one must specify the initial values of the \(\vec{f}_i\) in a consistent way. A possibility is to set them so that the macroscopic deformation \(\Delta \vec{L} = \sum \vec{r}\), where \(\vec{J} = \sum \vec{f}_i \vec{v}_i\) is a tensor interpreted as the local spatial deformation. Note that \(\vec{J}\) is not conserved at the borders.

In this model, total energy \(E\) of each particle is thus a well defined quantity as well as its kinetic energy \(E_{\text{kin}} = (1/2M)\Psi^2\). From these two quantities an internal energy \(E_{\text{int}} \equiv E_{\text{tot}} - E_{\text{kin}}\) can be introduced. The internal energy \(E_{\text{int}}\) can be set proportional to a temperature \(T\) using the equipartition theorem. In the initial configuration, \(T\) is typically introduced by adding a noise of standard deviation \(\sqrt{T}\) to the rest position of each atom.

### 3.4 Fracture rule

We now show how our LB solid model can be used to describe a fracture process. Fracture is a phenomenon for which no definite theory is available [1, 2] and a simple model is certainly useful to help understanding generic properties.

Since our LB-wave model can easily accommodate border conditions, only one step is needed to introduce a dynamical change in these borders by allowing for the links between the particles of the lattice describing our solid to be removed. At the level of our description, a link may break locally if its deformation is too large. Precisely we consider the energy stored in a link as the quantity determining the breaking. The energy \(E(t,\vec{r},t)\) of each link \(l = 1,4\) at site \(\vec{r}\) and time \(t\) is defined as \(E_l = \vec{f}_l^2\). Since, as mentioned above, our dynamics is unitary, the total energy \(E_{\text{tot}} = \sum \vec{r}_l (\vec{f}_l^2 + \sum E_l(\vec{r}))\) is conserved until a link breaks.

Specifically, the breaking rule we impose is the following: a link \(l\) breaks if the corresponding \(E_l\) is larger than a given threshold \(\epsilon(\vec{r})\)

\[E_l \geq \epsilon(\vec{r})\]  

which may, in principle, depend on the position (local defects). Particles with one or more broken links then behave like particles at a free boundary.

### 4 Numerical experiments

In this section, we will show how the LB-wave model can describe the kinematics of an object, the propagation of stress in it and how the extra breaking rule can be applied to several studies in fracture phenomena. The strength of the model lies in the fact that only initial conditions mimicking experimental conditions are needed to apply the model to different...
applications. Our intention is not as much to go in full simulations of these phenomena as illustrate the versatility of the LB-wave model.

4.1 A large moving object

One of the objectives of the LB-wave has been to describe the kinematics of a solid body. The goal was to obtain a macroscopic behavior from local rules of motion. To illustrate this capacity of the model, Fig. 2 shows the trajectory of a square 100x100 particles object bouncing against walls (notice that the object has conserved its integrity).

Fig 3 illustrates the conservation of energy as defined in (4). The object has an initial temperature, i.e. noise on the initial positions of particles, which leads to a non-zero initial internal energy. At each bounce, kinetic energy is transformed into internal energy which corresponds to the deformation of the object. After the bounce, practically all of the kinetic energy is restored, though one can see that the internal energy is slowly increasing with successive bounces.

4.2 Stress at the tip of a crack

However useful, the study of a flawless, homogeneous medium helps little with the understanding of fracture whose origin depends deeply on the length scales where the heterogeneity of the chemistry and the texture of the medium cannot be disregarded. However, at an intermediate length scale, the texture of a medium can be seen as a continuous medium with two kind of randomly distributed heterogeneities called micro-cavities: "pores" and "micro-cracks". Pores have roughly the same length in all directions. Micro-cracks on the other hand are much longer than they are wide.

It is useful then to understand how a micro-cavity modifies the mechanical behavior of a medium. To do so we consider an experiment similar to Young’s experiments. We take a 2D solid body with a slit, which models a micro-cavity, and apply a mode-I load in the direction perpendicular to the slit. In our simulation, the solid is in fact preconstrained with the load and the borders in the direction of the load are kept fixed. We then measure the energy at each site of the lattice.

Fig. 4 shows how energy is distributed in the solid in such an experiment. For a 2D solid, analytical calculations are possible. It is thus known [18] that for a elliptical hole in an infinite, linearly elastic medium, the stress decays like $1/\sqrt{r}$ where $r$ is the distance to the tip, while far from the hole the decay is that of a dipole namely $\sim 1/r^2$.

Fig.5 shows the decay of energy as a function of the distance to the tip of the slit. The plot is taken at the x-coordinate corresponding to x position of the slit. It is then fitted to the dipole field with a logarithm scale in the vertical axis to have a better visualization. Fits to $1/\sqrt{r}$ do not happen to be as good. This shows that the LB-wave model easily gives a correct behavior at long distance. However the failure to capture the short distance behavior can be accounted for by the discrete nature of the lattice on which we model the stress.

4.3 Crack speed and profile

Mechanical considerations are necessarily of importance to understand how much a material is subject to fracture. Het-
Energy $= \alpha + \beta / r^2$

Figure 5: Stress profile at distance $r$ from the tip of a micro-cavity. The stress behaves like a dipole field with decay $\sim 1/r^2$ as shown by the fit.

Figure 6: Illustration of a smooth crack simulated using the LB-wave model on a 150x150 particles solid body with attenuation $\mu = 0.98$. The original slit in the solid is represented with a solid line.

Figure 7: Illustration of a branching crack simulated using the LB-wave model on a 150x150 particles solid body with attenuation $\mu = 0.98$. The original slit in the solid is represented with a solid line.

erogeneities such as micro-cavities can, as we have seen, give rise to high stress in a material which can in turn develop the cavity into a full fracture. However these considerations are static in nature and many observations have shown that the dynamics of fracture is in itself as important to understand. Intuitively speaking, any cavity inside a material gives rise to high stress which can lead to the growth of the cavity. Depending on how the growth behaves, we are potentially confronted with a self-entertaining mechanisms.

The study of the dynamics of fracture growth usually takes into consideration two aspects of a fracture: the cracks shape and its speed with regard to the speed of sound $c_s$ in the material under study. The shape of a crack can be roughly divided into two modes: smooth and branching. Phenomenological observations show that cracks travel at speeds which are lower than the speed of sound. The branching cracks travel at higher speeds than smooth cracks. This can be intuitively understood in the following way: if the tip of a crack travels faster than the energy released by it, the energy left over behind the tip will eventually produce a bifurcation. The transition between the two modes has usually been observed at about half the speed of sound.

Despite the simplicity of our model, the simulations do reproduce the two modes of fractures as can be seen on Fig. 6 and Fig. 7. The setup for the simulation of the propagation of a fracture is similar to the one described in the previous section (4.2) with the added breaking rule (7).

There are several parameters one can adjust to change the behavior of the crack such as the initial load on the body, noise or temperature added to this load, the threshold on energy before a link breaks, a noise on this threshold, an attenuation $\mu$ on the LB-wave dynamics and the initial length of the slit. However, the main ingredient to distinguish the two modes in our simulations is the attenuation $\mu$.

In addition we are able to adjust the speed of sound $c_s = c_0/n$ in the lattice. We observe than the rule of the thumb for the smooth/branching crack transition is conserved by varying $c_s$, see Fig. 8 for $n = 1.0$ and Fig. 9 for $n = 1.5$.

4.4 Micro-cavities

Fracture precursors or micro-cavities have been the subject of statistical study [19]. It has been suggested by [20] that the patterns of micro-cavities might be of fractal nature. The measurement suggested is to look at the evolution of the dis-
Figure 8: Relative speed of cracks in a solid body with refraction index $n = 1.0$, the smooth cracks speed is smaller and more regular than that of a branching crack. The solid line indicates the phenomenological transition between smooth and branching cracks.

Figure 9: Relative speed of cracks in a solid body with refraction index $n = 1.5$, the smooth cracks speed is smaller and more regular than that of a branching crack. The solid line indicates the phenomenological transition between smooth and branching cracks.

Figure 10: Micro-cavities pattern in solid bodies of various sizes with mode-I load: (a) 20x20, (b) 50x50, (c) 100x100 and (d) 200x200. For the sake of comparison, the samples have been scaled such that they all appear the same size. The simulation consists in applying a mode-I load in the x- and y-direction of our 2D solid. Again, the solid is pre-constrained with the appropriate load and all borders are kept fixed. The pattern of micro-cavities obtained in such a way for various size of solid can be seen in Fig.(10) for a solid of size 20x20 up to 200x200. The only measure taken into account is the number $N_L$ of broken links. In fact, this measure yields a distribution which does scale with the size of the solid i.e.

$$N_L \sim L^2$$

A more thorough investigation of the parameter phase space or some new ingredient might reveal some fractal behavior. Nonetheless, the next section will show that if the growth of these micro-cracks is left to its own, eventually leading to the destruction of the solid, there is a measurement which indicates a complex process at work.

4.5 Fragmentation

Fragmentation is a fracture phenomena pushed to its limits in the sense that it is actually a destruction process. To do so, a very large stress is applied on a body in a very short time. Dissipation will therefore not work to unload the solid and only the creation of a considerable amount of free surfaces i.e., fractures will do. Due to the destructive nature of fragmentation, one typically observes the distribution of fragment sizes which results from the breakup process. These breaking mechanisms will depend on the material properties, the geometry of the solid body and on the energy input to
\begin{align*}
\chi^2/\nu & = 347.3 \div 987 \\
\alpha & = -0.9556 \\
\beta & = 1017.
\end{align*}

\begin{equation}
M(r) = \beta r^{-\alpha}
\end{equation}

Figure 11: Cumulative mass distribution of the fragmentation of a 100x100 particles solid body. The distribution is fitted according to a power-law and gives a value of \( \sim 1.0 \) for the exponent.

The system which can be controlled to some extend with the LB-wave model.

In the case of destructive breaking, the distribution of cumulative mass of fragments \( M(r) \), i.e. the number of fragments with mass less than \( r \), are often found to follow a simple power-law

\begin{equation}
M(r) \sim r^\alpha
\end{equation}

with exponent values \( \alpha \) to vary between 0.5 and 1.0. This power-law behavior is usually understood to be the signature of a critical self-organized dynamics. At the least it points to the fact that the dynamics is taking place at all scales.

Our simulations show that the LB-wave model captures this feature of fragmentation, see Fig. 11 for the power-law distribution of cumulative mass of fragments and Fig. 12 for the actual fragmentation of the solid body.

It has been observed in numerical experiments that with very high stress the distribution starts being exponential [21]. This has also been observed in our simulations; we propose that this effect is due to the finite size of the sample.

5 Conclusion

We have shown that the LB-wave model can model various generic aspects of solid body physics such as kinematics, stress propagation, cracks and fragmentation. We have validated our model by numerical experiments with a solid body represented by a two-dimensional square lattice. However, the generalization to a three-dimensional model is straightforward and the use of other lattice topologies should be possible (e.g. hexagonal). The simplicity of the model allows for a richness of investigation which can still be augmented; for instance, plasticity could be introduced by allowing dynamical changes in the rest length of links between particles.

Some local properties of the solid body could also be easily related to the energy.

Finally, we would like to stress that this simplicity allows for easy and efficient implementation on any kind of computer, personal or parallel, which seems to lack in many current models [22].

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